Topological Recursion Revised

(on a series of joint papers with A.Alexandrov, B.Bychkov, P.Dunin-Barkowski, and S.Shadrin)

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Noncommutative Geometry Meets Topological Recursion

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Topological recursion: an overview

Chekhov-Eynard-Orantin '06
Eynard-Orantin '07

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Goal: to compute a system of quantities (*correlators*)

$$\left\{ f^{(g)}_{k_1,\dots,k_n} \right\}, \quad g \geq 0, \quad (k_1,\dots,k_n) \vdash d = \sum k_i$$

Examples:

- Hurwitz numbers (simple, double, monotone, weighted etc.);
- enumeration of maps (hypermaps, fully simple, weighted etc.);
- correlators of matrix models;
- correlators of CohFT's (GW invariants);
- WP volumes, MV volumes, etc.

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Potential (free energy) F; partition function (tau function) $Z=e^F$:

$$F(p_1,p_2,\dots;\hbar)=\sum_{g\ge 0,\ n\ge 1}\frac{\hbar^{2g-2+n}}{n!}\sum_{k_1,\dots,k_n\ge 1} f^{(g)}_{k_1,\dots,k_n}p_{k_1}\dots p_{k_n}$$

n-point function:

$$H^{(g)}_n(w_1,\dots,w_n)=\sum_{k_1,\dots,k_n} f^{(g)}_{k_1,\dots,k_n}\, w_1^{k_1}\dots w_n^{k_n},\quad n=1,2,\dots.$$

Topological recursion computes $H^{(g)}_n$ in a closed form inductively in g and n

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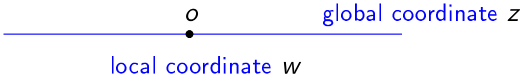
Spectral curve: $\Sigma = \mathbb{C}P^1$



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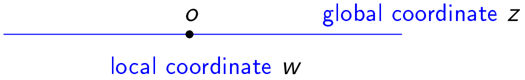

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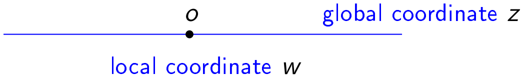

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$$H_n^{(g)} = \frac{\begin{pmatrix}\text{symmetric polynomial}\\ \text{in } z_1,\dots,z_n\end{pmatrix}}{\prod_{i=1}^{n}\prod_{j=1}^{N}(z_i - q_j)^{2(3g-3+n)+1}}, \quad 2g - 2 + n > 0.$$

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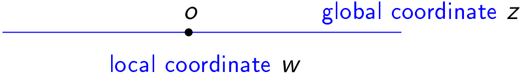

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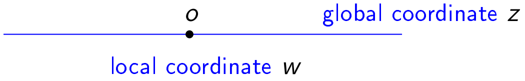

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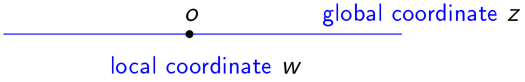

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$$\omega_n^{(g)} = d_1 \ldots d_n H_n^{(g)} + \delta_{g,0}\delta_{n,2}\frac{dw_1\,dw_2}{(w_1 - w_2)^2}$$

Example

Hurwitz numbers: $d=\sum_{i=1}^n k_i,\ m=2g-2+n+d,$

$$f^{(g)}_{k_1,\dots,k_n}=\frac{|\mathrm{Aut}(k_1,\dots,k_n)|}{m!d!}\#\left\{(\tau_1,\dots,\tau_m)\ \middle|\ \begin{array}{l}\text{1) } \tau_i{\in}S(d) \text{ a transposition}\\ \text{2) } \tau_1{\circ}\cdots{\circ}\tau_m \text{ has cyclic type } (k_1,\dots,k_n)\\ \text{3) connectness condition}\end{array}\right\}$$

$$\omega^{(g)}_n=\sum_{k_1,\dots,k_n} f^{(g)}_{k_1,\dots,k_n}\prod_{i=1}^n k_i w_i^{k_i-1}dw_i+\delta_{g,0}\delta_{n,2}\frac{dw_1dw_2}{(w_1-w_2)^2}$$

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$$\begin{aligned} w(z) &= z\,e^{-z} \\ &= z - z^2 + \frac{z^3}{2} - \dots \end{aligned} \quad \Rightarrow \quad \begin{aligned} \omega^{(0)}_3 &= \frac{dz_1\,dz_2\,dz_3}{(1-z_1)^2(1-z_2)^2(1-z_3)^2}, \\ \omega^{(1)}_1 &= \frac{(4-z_1)\,z_1}{24\,(1-z_1)^4}\,dz_1, \end{aligned} \qquad \mathcal{P} = \{z=1\},$$

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$$y(x)=z,\qquad x(z)=\log(w(z))=\log z-z,$$

$$\omega^{(0)}_1=y_1\ dx_1=(1-z_1^2)\,dz_1,\qquad \omega^{(0)}_2=\frac{dz_1\,dz_2}{(z_1-z_2)^2}.$$

Topological recursion: initial data

Initial data: $(\Sigma, dx, dy, B, \mathcal{P})$ $\overset{\text{CEO TR}}{\rightsquigarrow}$ $\{\omega_n^{(g)}\}_{g\geq 0, n\geq 1}$

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- $\Sigma = \mathbb{C}P^1$ (generalization: a smooth algebraic complex curve);
- $B(z_1, z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}$ (generalization: a bidifferential on Σ^2 with similar singularity on the diagonal)
- dx, dy meromorphic differentials on Σ
- $\mathcal{P} = \{q_1, \dots, q_N\}$ a set of *simple* zeroes of dx such that $dy|_{q_j} \neq 0$

Initial differentials:

$$\omega_1^{(0)}(z_1) = y(z_1)\,dx(z_1), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

The higher ω-differentials are computed by a recursive procedure inductively in g and n

Topological recursion: two step induction

2g − 2 + n > 0: $\quad K=\{2,\dots,n\},\ z_K=(z_2,\dots,z_n),$

First Step: $z\approx q_j\in\mathcal{P}$, $x(z)=x(\sigma(z))$

$$\tilde\omega^{(g)}_n(z,z_K)=\frac{\omega^{(g-1)}_{n+1}(z,\sigma(z),z_K)+\sum\limits_{\substack{g_1+g_2=g,\ J_1\sqcup J_2=K\\(g_i,|J_i|+1)\neq(0,1)}}\omega^{(g_1)}_{|J_1|+1}(z,z_{J_1})\omega^{(g_2)}_{|J_2|+1}(\sigma(z),z_{J_2})}{(y(z)-y(\sigma(z)))\,dx(z)}.$$

Second Step:

$$\omega^{(g)}_n(z,z_K)=\tilde\omega^{(g)}_n(z,z_K)+(\text{holomorphic in } z),\quad z\to q_j,\quad j=1,\dots,N.$$

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Equivalently,

$$\omega_n^{(g)}(z_1, z_K) = \sum_{j=1}^{N} \operatorname*{res}_{z=q_j} \tilde\omega_n^{(g)}(z, z_K) \int^{z} B(\cdot, z_1).$$

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Requirement: compatibility with limits under degenerations of the spectral curve data

Two examples

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Example (2)

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$$x_\epsilon(z) = z^5 + \epsilon\, z, \qquad \begin{array}{l} \text{(a) } y_\epsilon = \frac{1}{z^3} \\ \text{(b) } y_\epsilon = \frac{1}{z^3+\epsilon} \\ \text{(c) } y_\epsilon = \frac{z^2}{x_\epsilon} = \frac{z}{z^4+\epsilon} \end{array}$$

How CEO TR differentials of these families behave as $\epsilon \to 0$?

Two examples

Answer:

- (1), $k \leq 3$; (2b): NO LIMIT!
- (1), $k \geq 4$; (2a):
  - the limit does exist and is govern by GenTR
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- (2c):
  - the limit does exist but it is different from that one of the case (2a)
  - is govern by BE recursion
  - a closed formula for the TR differentials of this family is also available

Closed expression for GetTR differentials

$$(1): \quad x = z^2 + \frac{\epsilon}{z^{k-2}}, \quad y = z^2, \quad k \geq 4$$

$$(2a): \quad x = z^5 + \epsilon z, \quad y = z^{-3}$$

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Input data: two functions $x(z)$, $y(z)$ such that dx and dy are meromorphic

$$\hat{z}(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \quad \hat{z}_i^{\pm} = \hat{z}(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2} - e^{-u/2}}{u},$$

$$\mathbb{W}_n^{\vee}(z_1, v_1, \dots, z_n, v_n) = \sum_{g \geq 0} \hbar^{2g-2+n} \mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^{n} \left(e^{v_i(\mathcal{S}(v_i \hbar \partial_{y_i}) - 1) x_i} \sqrt{\frac{d\hat{z}_i^+}{dz_i} \frac{d\hat{z}_i^-}{dz_i}} \; \frac{dz_i}{dx_i} \right) (-1)^{n-1} \sum_{\sigma \in \mathrm{cycl}(n)} \prod_{i=1}^{n} \frac{1}{\hat{z}_i^+ - \hat{z}_{\sigma(i)}^-}$$

$$\frac{(-1)^n \omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1, \dots, k_n \geq 0} (-\partial_{x_1})^{k_1} \dots (-\partial_{x_n})^{k_n} \, [v_1^{k_1} \dots v_n^{k_n}] \, \mathbb{W}_n^{\vee,(g)}$$

Generalized TR: overview

CEO TR: an explicit formula for $\tilde{\omega}_n^{(g)}(z, z_2, \dots, z_n)$

Then, $\omega_n^{(g)}(z_1, z_2, \dots, z_n) = \sum_{q_j \in \mathcal{P}} \operatorname*{res}_{z=q_j} \tilde{\omega}_n^{(g)}(z, z_2, \dots, z_n) \int^{z} B(\cdot, z_1).$

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Example: $(g, n) = (0, 3)$

CEO TR: $$\tilde{\omega}_3^{(0)}(z, z_2, z_3) = \frac{B(z, z_2)B(\sigma(z), z_3) + B(z, z_3)B(\sigma(z), z_2)}{(y(z) - y(\sigma(z)))\, dx(z)}$$

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GenTR: $$\tilde{\omega}_3^{(0)}(z, z_2, z_3) = d_z \frac{B(z, z_2)B(z, z_3)}{dx(z)\, dy(z)}$$

Generalized TR: overview

$o \in \Sigma$, z local coordinate $\qquad \begin{array}{l} x = a\,z^r + \text{h.o.t}, \\ y = b\,z^s + \text{h.o.t}, \end{array} \qquad a\,b \neq 0,\ r, s \in \mathbb{Z}$

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Definition

The point $o \in \Sigma$ is called
special, if $r + s > 0$ and $(r, s) \neq (1, 1)$,
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Initial data of GenTR: $(\Sigma, dx, dy, B, \mathcal{P})$;

- Σ, B the same as for CEO TR
- dx, dy arbitrary meromorphic differentials (with no restriction on zeroes and poles)
- $\mathcal{P}$ is an arbitrary subset in the set of special points

Generalized TR: basic properties

- $\omega_1^{(0)} = y\,dx$, $\omega_2^{(0)} = B$
- $2g-2+n>0$: $\omega_n^{(g)}$ is global meromorphic, symmetris, and has poles at $z_i = q_j$, $q_j \in \mathcal{P}$.
- Two-step recursion for $\omega_n^{(g)}$:
  - $\tilde\omega_n^{(g)}(z, z_2, \dots, z_n)$ is given by an *explicit formula* (below)
  - It is global meromorphic in z
  - its poles in z are at special points and also at $z_2, \dots, z_n$
  - $\omega_n^{(g)}$ selects those poles of $\tilde\omega_n^{(g)}$ which are *key-special*

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Theorem (Compatibility with known versions of TR)

- $(r,s) = (2,1) \Leftrightarrow$ *CEO*
- $(r,s) = (2,-1) \Leftrightarrow$ *Chekhov-Norbury irregular recursion*
- $r>0$, $s=\pm1 \Leftrightarrow$ *BE recursion*
- $(r,s) = (1,0) \Leftrightarrow$ *LogTR of* [ABDKS23]

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- $r > 0$, $s = \pm 1 \Leftrightarrow$ *BE recursion*
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Remark. GenTR is not compatible with BE TR if $s \neq \pm 1$

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

GenTR is compatible with limits of the spectral curve data as long as key-special points and key$^{\vee}$-special points do not collapse together

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See e.g. Example (1), $k \geq 4$, Exampe (2a)

Example

$$x = z^2, \quad y = \frac{1}{z+s}, \quad \mathcal{P} = \{0\}$$

This TR is compatible with the limit as $s \to 0$

$s \neq 0$: CEO TR $\rightsquigarrow$ KW potential (with properly rescaled times)

$s = 0$: CN irregular TR $\rightsquigarrow$ BGW potential

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

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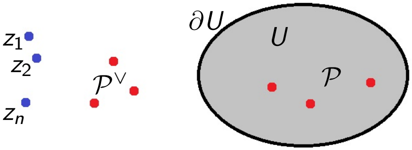
Proof



$$\omega_n^{(g)}(z_1, z_K) = \frac{1}{2\pi} \int_{z \in \partial U} \left(\tilde{\omega}_n^{(g)}(z, z_K) \int^z B(\cdot, z_1) \right), \qquad z_1, \dots, z_n \in \Sigma \setminus U.$$

Generalized TR: xy swap duality

xy duality transformation: an explicit closed formula $\quad \{\omega_n^{(g)}\} \quad \longleftrightarrow \quad \{\omega_n^{\vee,(g)}\}$

Theorem (Compatibility with xy swap)

$$\begin{array}{ccc} (\Sigma, dx, dy, B, \mathcal{P}) & & (\Sigma, dy, dx, B, \mathcal{P}^{\vee}) \\ \Big\downarrow {\scriptstyle GenTR} & & \Big\downarrow {\scriptstyle GenTR} \\ \{\omega_n^{(g)}\} & \xleftrightarrow{\ xy\ swap\ } & \{\omega_n^{\vee,(g)}\} \end{array}$$

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Remark.

GenTR relation for $\omega_n^{(g)}$ $\Leftrightarrow$ $\omega_n^{\vee,(g)}$ is regular at $\mathcal{P}$

Generalized TR: xy swap duality

Corollary

$\Sigma = \mathbb{C}P^1$, $\mathcal{P}^\vee = \emptyset$ (*all special points are key-special*). *Then,* $\omega_n^{\vee,(g)} = 0$ *for* $2g-2+n>0$ *and an explicit formula* $(*)$ *for* $\omega_n^{(g)}$ *holds.*

Generalized TR: xy swap duality

Corollary

$\Sigma = \mathbb{C}P^1$, $\mathcal{P}^\vee = \emptyset$ (all special points are key-special). Then, $\omega_n^{\vee,(g)} = 0$ for $2g-2+n>0$ and an explicit formula $(*)$ for $\omega_n^{(g)}$ holds.

$$\hat z(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \quad \hat z_i^{\pm} = \hat z(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2}-e^{-u/2}}{u},$$

$$\mathbb{W}_n^\vee(z_1,v_1,\dots,z_n,v_n) = \sum_{g\geq 0} \hbar^{2g-2+n}\mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^n \left(e^{v_i(\mathcal{S}(v_i\hbar\partial_{y_i})-1)x_i} \sqrt{\frac{d\hat z_i^+}{dz_i}\frac{d\hat z_i^-}{dz_i}}\,\frac{dz_i}{dx_i}\right)(-1)^{n-1}\sum_{\sigma\in\mathrm{cycl}(n)}\prod_{i=1}^n \frac{1}{\hat z_i^+ - \hat z_{\sigma(i)}^-}$$

$$\frac{(-1)^n\omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1,\dots,k_n\geq 0} (-\partial_{x_1})^{k_1}\dots(-\partial_{x_n})^{k_n}\,[v_1^{k_1}\dots v_n^{k_n}]\,\mathbb{W}_n^{\vee,(g)}$$

Generalized TR: KP integrability

Theorem (KP integrability)

If $\Sigma = \mathbb{C}P^1$, then GenTR differentials are KP integrable

(see the talk of Sasha Alexandrov for details)

Corollary

GenTR potential for $\begin{cases} dx = z^{r-1}dz \\ dy = z^{s-1}dz \end{cases}$ *is a solution of KP hierarchy for any* (r, s), $r + s > 0$

Example: $(r, s) = (1, 2)$

Example

$$\begin{cases} x = z, \\ y = z^2. \end{cases}$$

Special points $= \{0\}$

$\mathcal{P}$	$\mathcal{P}^\vee$	GenTR	GenTR$^\vee$
$\emptyset$	$\{0\}$	trivial	KW
$\{0\}$	$\emptyset$	new!	trivial

Expansion point: $z = \infty$, expansion local coordinate: $1/z$

$$F = -\tfrac{1}{48}p_2\hbar + (\tfrac{1}{96}p_1^4 - \tfrac{1}{96}p_2^2)\hbar^2 + (\tfrac{1}{48}p_2p_1^4 + \tfrac{1}{24}p_4p_1^2 - \tfrac{1}{144}p_2^3 - \tfrac{9}{1280}p_6)\hbar^3 + (\tfrac{9}{640}p_3p_1^5 + \tfrac{1}{32}p_2^2p_1^4 + \tfrac{125}{1152}p_5p_1^3 + \tfrac{9}{256}p_3^2p_1^2 + \tfrac{1}{8}p_2p_4p_1^2 + \tfrac{343}{2880}p_7p_1 + \tfrac{29}{2880}p_4^2 - \tfrac{1}{192}p_2^4 - \tfrac{27}{1280}p_2p_6)\hbar^4 + O(\hbar^5)$$

This potential is a solution of KP hierarchy

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CEO TR deformation: $\begin{cases} x = z + \frac{\epsilon}{z^2}, \\ y = z^2 \end{cases}$

To be explained

Still missing:

- GenTR recursion formula for $\tilde{\omega}_n^{(g)}$ (L2)
- compatibility with CEO and other versions of TR (L3)
- xy swap formula (L2)
- symplectic duality as a generalization of xy duality (L3)
- closed formulas for BE TR differentials as a special case of symplectic duality (L3)
- definition of KP integrability (the talk of A. Alexandrov)

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \qquad \longleftrightarrow \qquad \{\omega_n^{\vee,(g)}\}$$

$$\omega_1^{(0)}(z) = y\,dx \qquad\qquad \omega_1^{\vee,(0)}(z) = x\,dy$$

xy swap transformation at a nice point

$$\begin{matrix} \{\omega_n^{(g)}\} & & \{\omega_n^{\vee,(g)}\} \\ \omega_1^{(0)}(z) = y\,dx & \longleftrightarrow & \omega_1^{\vee,(0)}(z) = x\,dy \end{matrix}$$

Definition

A point $o \in \Sigma$ is called *nice* if $x = \log z + O(z)$, $y = \log z + O(z)$

$$dx = \frac{dz}{z} + (\text{holomorphic}), \quad dy = \frac{dz}{z} + (\text{holomorphic})$$

Then, $X = e^x$ and $Y = e^y$ can serve as local coordinates

$$dx = \frac{dX}{X}, \quad dy = \frac{dY}{Y}, \quad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y.$$

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

$$Q = \frac{1}{2}\sum_{i,j}\Big((i+j)p_i p_j \tfrac{\partial}{\partial p_{i+j}} + i\,j\,p_{i+j}\tfrac{\partial^2}{\partial p_i \partial p_j}\Big)$$

$$\omega_n = \sum_{g\geq 0} \hbar^{2g-2+n}\omega_n^{(g)} \qquad \omega_n^\vee = \sum_{g\geq 0} \hbar^{2g-2+n}\omega_n^{(g)}$$

$$\omega_n - \delta_{n,2}\tfrac{dX_1 dX_2}{(X_1-X_2)^2} - \delta_{n,1}\hbar^{-1}x_1 dx_1 = \sum_{k_1,\dots,k_n\geq 1} \frac{\partial^n F}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \textstyle\prod_{i=1}^n d(X_i^{k_i})$$

$$(-1)^n\omega_n^\vee - \delta_{n,2}\tfrac{dY_1 dY_2}{(Y_1-Y_2)^2} + \delta_{n,1}\hbar^{-1}y_1 dy_1 = \sum_{k_1,\dots,k_n\geq 1} \frac{\partial^n F^\vee}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \textstyle\prod_{i=1}^n d(Y_i^{k_i})$$

xy swap transformation at a nice point

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Theorem

The composition $\{\omega_n^{(g)}\} \longmapsto \{\omega_n^{\vee,(g)}\}$ is given by a closed finite expression that extends to a transformation of global meromorphic differentials and does not involve any information on a chosen expansion point o (and neither requires a very existence of a nice point).

The obtained transformation is called the *xy swap duality*

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \overset{e^{-\hbar Q}}{\longrightarrow} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

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Remark. The potentials F and $F^\vee$ do depend on the expansion point an a choice of local coordinates. The treatment of xy duality as the action of e^{-uQ} on the corresponding tau function is valid for a *nice point* only

xy swap: the formula

$$W_n(z_1,u_1,\dots,z_n,u_n) = \left(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\right)\frac{\omega_n}{\prod_{i=1}^n dx_i}, \qquad \mathcal{S}(u) = \frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \prod_{i=1}^n \frac{dx_i}{u_i\hbar}\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\ge 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\left(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\right)\mathbb{W}_n^{(g)}$$

Γ_n is the set of *hypergraphs* (graphs with hyperedges) with n marked vertices

xy swap: the formula

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Corrections and details. 1. The dependence of $\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n^{(g)}=[\hbar^{2g-2+n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n$ in u-variables is polynomial.

2. If $|e|=2$ and $e(1)=e(2)$, use the regularized differential $\omega_2(\tilde z_1,\tilde z_2)-\frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}$ instead in the definition of the edge contribution $W_{|e|}$.

xy swap: basic properties

1. $\omega_1^{\vee,(0)} = x\,dy$, $\omega_2^{\vee,(0)} = \omega_2^{(0)}$
2. $2g-2+n>0$: $\omega_n^{\vee,(g)}$ is globally defined and meromorphic
3. Moreover, it is *regular on diagonals*
4. The *inverse transformation* is given by the same formulas with x and y swapped

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Remark. We aware of no direct combinatorial proof of the last two properties. The arguments we are using involve computation in the space of power expansions at a (nice) point

Example

$$\omega_3^{(0)} + \omega_3^{\vee,(0)} = d_1 \frac{B(z_1,z_2)B(z_1,z_3)}{dx_1\,dy_1} + d_2 \frac{B(z_2,z_3)B(z_2,z_1)}{dx_2\,dy_2} + d_3 \frac{B(z_3,z_1)B(z_3,z_2)}{dx_3\,dy_3}$$

Example: intersection numbers with r-spin Chiodo classes

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases} , \quad dx = (1 - r\,z)\frac{dz}{z}$$

Example: intersection numbers with r-spin Chiodo classes

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases}, \quad dx = (1 - r\,z)\frac{dz}{z}, \qquad \begin{cases} x^\vee = z^s \\ y^\vee = \log z + O(z) \end{cases}$$

n-point functions of the dual (generalized) TR for $r = 3$, $s = 2$:

$$H_3^{\vee,(0)} = 0, \quad H_1^{\vee,(1)} = -\frac{1}{48} z_1^{-2}, \quad H_2^{\vee,(1)} = -\frac{1}{48}(z_1^{-1} z_2^{-3} + z_1^{-3} z_2^{-1})$$

n-point functions of the original (CEO) TR, by xy duality:

$$H_3^{(0)} = \tfrac{3}{2} \frac{(z_1 + z_2 + z_3)(3 z_1 z_2 z_3 + 1) + 3(z_1^2 z_2^2 + z_3^2 z_2^2 + z_1^2 z_3^2)}{(1 - 3z_1^3)(1 - 3z_2^3)(1 - 3z_3^3)}$$

$$H_1^{(1)} = \tfrac{1}{16} (3\,\partial_{x_1} - 1) \frac{z_1}{1 - 3z_1^3}$$

$$H_2^{(1)} = \tfrac{3}{32} \sum_{k_1, k_2, k_3} (-\partial_{y_1})^{k_1} (-\partial_{y_2})^{k_2} (-\partial_{y_3})^{k_3} [u_1^{k_1} u_2^{k_2} u_3^{k_3}]$$

$$\frac{(u_1 + u_2) z_1 z_2 + 3(u_1^2 + u_2 u_1 + u_2^2)(z_1^2 + z_2 z_1 + z_2^2) - 2(z_1^2 + z_2^2)}{(1 - 3z_1^3)(1 - 3z_2^3)}$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \xleftrightarrow{(o,X)} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \xleftrightarrow{(o,Y)} \{(-1)^n \omega_n^{\vee,(g)}\}$$

Notation: $\langle F \rangle = F \big|_{p=0}$, 'taking the free term of a series',

$$J^+(X) = \sum_{k=1}^{\infty} k X^k \partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2} \frac{X_1 X_2}{(X_1 - X_2)^2} = \sum_{k_1,\dots,k_n} \frac{\partial^n F}{\partial p_{k_1} \dots \partial p_{k_n}} \Big|_{p=0} \prod_{i=1}^n k_i X_i^{k_i}$$

$$= \big\langle J^+(X_1) \dots J^+(X_n) F \big\rangle$$

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Derivation of xy swap formula, Step 1: inclusion/exclusion

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$$J^+(X) = \sum_{k=1}^{\infty} k X^k \partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2}\frac{X_1X_2}{(X_1-X_2)^2} = \big\langle J^+(X_1)\dots J^+(X_n)e^F\big\rangle^\circ$$

where the 'connected' correlators are defined through inclusion/exclusion

$$\big\langle J^+(X)e^F\big\rangle = \big\langle J^+(X)e^F\big\rangle^\circ$$

$$\big\langle J^+(X_1)J^+(X_2)e^F\big\rangle = \big\langle J^+(X_1)J^+(X_2)e^F\big\rangle^\circ + \big\langle J^+(X_1)e^F\big\rangle^\circ \big\langle J^+(X_2)e^F\big\rangle^\circ$$

$$\dots$$

$$\big\langle J^+(X_1)\dots J^+(X_n)e^F\big\rangle = \sum_{\sqcup I_\alpha = \{1,\dots,n\}} \prod_\alpha \Big\langle \prod_{i\in I_\alpha} J^+(X_i)e^F\Big\rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

Notation: $\langle F \rangle = F\big|_{p=0}$, 'taking the free term of a series',

$$J^+(X) = \sum_{k=1}^{\infty} k X^k \partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2} \frac{X_1 X_2}{(X_1 - X_2)^2} = \big\langle J^+(X_1) \dots J^+(X_n) e^F \big\rangle^\circ$$

Moreover, define

$$J(X) = \sum_{k=-\infty}^{\infty} X^k J_k, \qquad J_k = \begin{cases} k\,\partial_{p_k}, & k > 0, \\ 0, & k = 0, \\ p_{-k}, & k < 0 \end{cases}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} = \big\langle J(X_1) \dots J(X_n) e^F \big\rangle^\circ$$

(with the singular $(0,2)$ correction taken into account automatically)

Derivation of xy swap formula, Step 1: inclusion/exclusion

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$$Q = \frac{1}{2} \sum_{i,j} \Big((i+j) p_i p_j \tfrac{\partial}{\partial p_{i+j}} + i\, j\, p_{i+j} \tfrac{\partial^2}{\partial p_i \partial p_j} \Big)$$

Similarly,

$$\frac{(-1)^n \omega_n^\vee}{\prod_{i=1}^n dy_i} = \big\langle J(Y_1) \dots J(Y_n)\, e^{F^\vee} \big\rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

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Similarly,

$$\begin{aligned}
\frac{(-1)^n \omega_n^\vee}{\prod_{i=1}^n dy_i} &= \big\langle J(Y_1)\dots J(Y_n)\; e^{F^\vee}\big\rangle^\circ \\
&= \big\langle J(Y_1)\dots J(Y_n)\; e^{-\hbar Q} e^F\big\rangle^\circ \\
&= \big\langle \mathbb{J}(Y_1)\dots \mathbb{J}(Y_n)\; e^F\big\rangle^\circ, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q}
\end{aligned}$$

The next step: to compute the operator $\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q}$ acting on $\mathbb{C}[[p_1, p_2, \dots]]$

Main tool: *bosonic representation of $\widehat{\mathfrak{gl}}(\infty)$* on $\mathbb{C}[[p_1, p_2, \dots]]$

$$\sum_{i,j\in\mathbb{Z}} z_1^j z_2^{-i-1} E_{i,j} = \frac{e^{\sum_{i<0}\frac{z_1^i - z_2^i}{i}J_i}\, e^{\sum_{i>0}\frac{z_1^i - z_2^i}{i}J_i} - 1}{z_1 - z_2}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q} J(Y) e^{-\hbar Q}$

Main tool: *bosonic representation of $\widehat{\mathfrak{gl}}(\infty)$* on $\mathbb{C}[[p_1, p_2, \dots]]$

$$\sum_{i,j\in\mathbb{Z}} z_1^j z_2^{-i-1} E_{i,j} = \frac{e^{\sum_{i<0} \frac{z_1^i - z_2^i}{i} J_i} e^{\sum_{i>0} \frac{z_1^i - z_2^i}{i} J_i} - 1}{z_1 - z_2}$$

$z_1 = Xe^{u/2}$, $z_2 = Xe^{-u/2}$, $\partial_x = X\partial_X$:

$$\mathcal{E}(X,u) = \sum_{k,m\in\mathbb{Z}} X^m e^{u(k+\frac{1-m}{2})} E_{k-m,m} + \frac{1}{u\mathcal{S}(u)} = \frac{e^{\sum_{i<0} u\mathcal{S}(ui)X^i J_i} e^{\sum_{i>0} u\mathcal{S}(ui)X^i J_i}}{u\mathcal{S}(u)}$$

$$= \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i} e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q}J(Y)e^{-\hbar Q}$

Main tool: *bosonic representation of* $\widehat{\mathfrak{gl}}(\infty)$ on $\mathbb{C}[[p_1, p_2, \dots]]$

$$\mathcal{E}(X,u) = \sum_{k,m\in\mathbb{Z}} X^m e^{u(k+\frac{1-m}{2})} E_{k-m,m} + \frac{1}{u\mathcal{S}(u)} = \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i} e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}$$

All operators involved belong to $\widehat{\mathfrak{gl}}(\infty)$:

$$J_m = [X^m u^0]\mathcal{E}(X,u) = \sum_{k\in\mathbb{Z}} E_{k-m,k}, \qquad J(X) = [u^0]\mathcal{E}(X,u),$$

$$Q = [X^0 u^2]\mathcal{E}(X,u) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(k+\tfrac{1}{2}\right)^2 E_{k,k}$$

$$\mathbb{J}(Y) = e^{\hbar Q}J(Y)e^{-\hbar Q} = \sum_{k,m\in\mathbb{Z}} \frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}} Y^m E_{k-m,k}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q} J(Y) e^{-\hbar Q}$

Lemma

$$\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q} = \sum_{j=0}^{\infty} (-\partial_y)^j [u^j] e^{-u(y-x)} \frac{dx}{dy} \mathcal{E}(X, u\hbar)$$

Or, taking the coefficient of $E_{k-m,k}$,

$$\frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}} Y^m = \sum_{j=0}^{\infty} (-Y\partial_Y)^j [u^j] \left(\tfrac{X}{Y}\right)^u e^{u\hbar\left(k+\frac{1-m}{2}\right)} \frac{dX}{X} \frac{Y}{dY} X^m$$

Substituting, we obtain

$$\frac{(-1)^n \omega_n^{\vee}}{\prod_{i=1}^n dy_i} = \Big\langle \prod_{i=1}^n \mathbb{J}(Y_i)\, e^F \Big\rangle^{\circ} = \sum_{k_1,\dots,k_n \geq 0} (-\partial_{y_1})^{k_1} \dots (-\partial_{y_n})^{k_n} [u_1^{k_1} \dots u_n^{k_n}] \left(\prod_{i=1}^n \frac{e^{-u_i y_i}}{dy_i} \right) \mathbb{W}_n$$

where $\quad \mathbb{W}_n = \left(\prod_{i=1}^n e^{u_i x_i} dx_i \right) \left\langle \prod_{i=1}^n \mathcal{E}(X_i, u_i\hbar)\, e^F \right\rangle^{\circ}$

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

$$\mathbb{W}_n = \Big(\prod_{i=1}^{n} e^{u_i x_i} dx_i\Big) \; \big\langle \mathcal{E}(X_n, u_n\hbar) \dots \mathcal{E}(X_n, u_n\hbar) \; e^F \big\rangle^{\circ}$$

- Insert $\mathcal{E}(X, u\hbar) = \frac{e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i<0} X^i J_i} e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i>0} X^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

$$\mathbb{W}_n = \Big(\prod_{i=1}^{n} e^{u_i x_i} dx_i\Big) \left\langle \mathcal{E}(X_n, u_n\hbar) \dots \mathcal{E}(X_n, u_n\hbar)\, e^{F} \right\rangle^{\circ}$$

- Insert $\mathcal{E}(X, u\hbar) = \frac{e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i<0} x^i J_i}\, e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i>0} x^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

$$W_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big) \frac{\omega_n}{\prod_{i=1}^{n} dx_i},$$

$$\mathbb{W}_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} \frac{dx_i}{u_i\hbar}\Big) \sum_{\gamma\in\Gamma_n} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{e\in E(\gamma)} W_{|e|}(z_{e_1}, u_{e_1}, \dots, z_{e_{|e|}}, u_{e_{|e|}})$$

xy swap formula: summary of computations

$$\omega_n \sim \Big\langle \prod J(X_i)\, e^F \Big\rangle^\circ \xrightarrow{\;xy \text{ swap}\;} \omega_n^\vee \sim \Big\langle \prod J(Y_i)\, e^{F^\vee} \Big\rangle^\circ = \Big\langle \prod \mathbb{J}(Y_i)\, e^{F} \Big\rangle^\circ$$

summation over graphs ↘ ↗ $\prod_{i=1}^{n} \sum_{k\geq 0} \partial_{y_i}^k [u_i^k]$

$$\mathbb{W}_n \sim \Big\langle \prod \mathcal{E}(X_i, u_i\hbar)\, e^F \Big\rangle^\circ$$

$$J(X) = \sum_{k=-\infty}^{\infty} J_k X^k, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q},$$

$$\mathcal{E}(X,u) = \frac{e^{u\mathcal{S}(u\partial_x)\sum\limits_{i<0} X^i J_i}\; e^{u\mathcal{S}(u\partial_x)\sum\limits_{i>0} X^i J_i}}{u\mathcal{S}(u)}, \qquad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y$$

xy swap: the formula (reminding)

$$W_n(z_1,u_1,\dots,z_n,u_n)=\Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\frac{\omega_n}{\prod_{i=1}^n dx_i},\qquad \mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n)=\Big(\prod_{i=1}^n\frac{dx_i}{u_i\hbar}\Big)\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)}W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

(If $|e|=2$ and $e(1)=e(2)$, use the regularized differential $\omega_2(\tilde z_1,\tilde z_2)-\frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}$ instead in the definition of the edge contribution $W_{|e|}$)

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i}=\sum_{k_1,\dots,k_n\ge 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n^{(g)}$$

More properties of xy swap

- The xy swap transformation produces no singularities apart from the special points: if $\omega_n^{(g)}$ is regular at some non-special point for all (g,n) with $2g-2+n>0$, then the same holds for $\omega_n^{\vee,(g)}$
- This property motivates the definition of GenTR: it is *defined* by the requirement that all xy dual differentials are holomorphic at the key-special points. Then the compatibility GenTR with xy swap becomes a reformulation of the definition:

$$\begin{array}{ccc} (\Sigma, dx, dy, B, \mathcal{P}) & & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \\ \Big\downarrow{\scriptstyle \text{GenTR}} & & \Big\downarrow{\scriptstyle \text{GenTR}} \\ \{\omega_n^{(g)}\} & \xleftrightarrow{\;xy \text{ swap}\;} & \{\omega_n^{\vee,(g)}\} \end{array}$$

More concretely, this idea is realized below

To be explained

Still missing:

- the definition of GenTR relation
- compatibility with CEO and other versions of TR
- symplectic duality as a generalization of xy duality
- closed formulas for BE TR differentials as a special case of symplectic duality

xy swap transformation

Nice point: $x = \log z + O(z)$, $y = \log z + O(z)$

The action on the *partition functions* associated with the local coordinates $X = e^x$, $Y = e^y$:

by the operator $\quad e^{-\hbar Q}, \qquad Q = \frac{1}{2}\sum_{i,j}\Big((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + i\,j\,p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\Big)$

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

xy swap transformation

Nice point: $x = \log z + O(z)$, $y = \log z + O(z)$

The action on the *power expansions of the differentials* at a nice point:

$$\omega_n \sim \Big\langle \prod J(X_i)\, e^F \Big\rangle^\circ \xrightarrow{\;xy \text{ swap}\;} \omega_n^\vee \sim \Big\langle \prod J(Y_i)\, e^{F^\vee} \Big\rangle^\circ = \Big\langle \prod \mathbb{J}(Y_i)\, e^F \Big\rangle^\circ$$

summation over graphs $\searrow$

$$\mathbb{W}_n \sim \Big\langle \prod \mathcal{E}(X_i, u_i\hbar)\, e^F \Big\rangle^\circ$$

$\nearrow$ $\prod_{i=1}^{n} \sum_{k\geq 0} \partial_{y_i}^k [u_i^k]$

$$J(X) = \sum_{k=-\infty}^{\infty} J_k X^k, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q},$$

$$\mathcal{E}(X,u) = \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i}\, e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}, \qquad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y$$

xy swap transformation

The action on *global meromorphic differentials*:

$$W_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\,\frac{\omega_n}{\prod_{i=1}^n dx_i}, \qquad \mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n \frac{dx_i}{u_i\hbar}\Big)\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

(with a regularization of certain singular $(0,2)$ contributions)

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\geq 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\,\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\,\mathbb{W}_n^{(g)}$$

Example: intersection numbers with r-spin Chiodo classes

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases} , \qquad dx = (1 - r\, z^r)\frac{dz}{z}$$

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases}, \quad dx = (1 - r\,z^r)\frac{dz}{z}, \qquad \begin{cases} x^\vee = z^s \\ y^\vee = \log z + O(z) \end{cases}$$

n-point functions of the dual (generalized) TR for $r = 3$, $s = 2$:

$$H_3^{\vee,(0)} = 0, \quad H_1^{\vee,(1)} = -\frac{1}{48} z_1^{-2}, \quad H_2^{\vee,(1)} = -\frac{1}{48}(z_1^{-1} z_2^{-3} + z_1^{-3} z_2^{-1})$$

Example: intersection numbers with r-spin Chiodo classes

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n-point functions of the original (CEO) TR, by xy duality:

$$H_3^{(0)} = \tfrac{3}{2} \frac{(z_1 + z_2 + z_3)(3z_1 z_2 z_3 + 1) + 3(z_1^2 z_2^2 + z_3^2 z_2^2 + z_1^2 z_3^2)}{(1 - 3z_1^3)(1 - 3z_2^3)(1 - 3z_3^3)}$$

$$H_1^{(1)} = \tfrac{1}{16}(3\,\partial_{x_1} - 1)\frac{z_1}{1 - 3z_1^3}$$

$$H_2^{(1)} = \tfrac{3}{32} \sum_{k_1,k_2,k_3} (-\partial_{y_1})^{k_1} (-\partial_{y_2})^{k_2} (-\partial_{y_3})^{k_3} [u_1^{k_1} u_2^{k_2} u_3^{k_3}]$$

$$\frac{(u_1 + u_2) z_1 z_2 + 3(u_1^2 + u_2 u_1 + u_2^2)(z_1^2 + z_2 z_1 + z_2^2) - 2(z_1^2 + z_2^2)}{(1 - 3z_1^3)(1 - 3z_2^3)}$$

Partial xy swap duality and definition of GenTR

In the power expansions at a nice point:

$$\omega_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; J(X)\; e^F \Big\rangle^\circ \longrightarrow \omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathbb{J}(Y)\; e^F \Big\rangle^\circ$$

combinatorial expression

$$\sum_{k\geq 0} \partial_y^k [u^k]$$

$$\mathcal{W}_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X, u\hbar)\; e^F \Big\rangle^\circ$$

- ω_n, $\mathcal{W}_n$, $\omega^{(g)}_{n-1,1}$ extend globally on Σ^n
- $\omega^{(g)}_n$ is holomorpfic in $z = z_n$ at $q \in \mathcal{P}$ for $i = 1, \dots, n$ and all (g, n) iff $\omega^{\vee,(g)}_n$ is holomorphic at q for all (g, n) (in all z_i's)

Definition of GenTR

$$\mathcal{T}_n(z_{[\![n-1]\!]}; z, u) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde z_i \to z}\, u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde x_i}) \tfrac{1}{d\tilde x_i} \Big) \big(\omega_{n-1+k}(z_{[\![n-1]\!]}, \tilde z_{[\![k]\!]}) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]}; z, u) = \frac{dx}{u\hbar}\, e^{\mathcal{T}_1(z,u)} \sum_{[\![n]\!]=\sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha}; z, u)$$

$$\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]}, z) = -dy \sum\nolimits_{r\geq 0} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}$$

$$= -\omega^{(g)}_n(z_{[\![n-1]\!]}, z) - \underbrace{dy \sum_{r\geq 1} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}}_{\text{involves } \omega(g')_{n'}\text{'s with } 2g'-2+n' < 2g-2+n}$$

Definition of GenTR

$$\mathcal{T}_n(z_{[\![n-1]\!]}; z, u) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde z_i \to z}\, u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde x_i}) \tfrac{1}{d\tilde x_i} \Big) \big(\omega_{n-1+k}(z_{[\![n-1]\!]}, \tilde z_{[\![k]\!]}) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]}; z, u) = \frac{dx}{u\hbar}\, e^{\mathcal{T}_1(z,u)} \sum_{[\![n]\!] = \sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha}; z, u)$$

$$\begin{aligned} \omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]}, z) &= -dy \sum\nolimits_{r\geq 0} (-\partial_y)^r [u^r] e^{-u\,y} \tfrac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy} \\ &= -\omega^{(g)}_n(z_{[\![n-1]\!]}, z) + \tilde\omega^{(g)}_n(z_{[\![n-1]\!]}, z) \end{aligned}$$

Definition (Differentials $\tilde\omega^{(g)}_n$ of Generalized Topological Recursion)

$$\tilde\omega^{(g)}_n = -dy \sum_{r\geq 1} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}$$

Compatibility of GenTR with xy duality

$$\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]}, z) = -\omega^{(g)}_{n}(z_{[\![n-1]\!]}, z) + \tilde{\omega}^{(g)}_{n}(z_{[\![n-1]\!]}, z)$$

$\{\omega^{(g)}_n\}$ satisfy GenTR

$\Longrightarrow$ $\omega^{(g)}_{n-1,1}$ is holomorphic in z at key-special points

$\Longrightarrow$ $\omega^{(g)}_n$ is holomorphic in z_i's at key-special points

By the same reason,

$\{\omega^{(g)}_n\}$ are holomorphic at $q \in \mathcal{P}^\vee$ $\Longrightarrow$ $\{\omega^{\vee,(g)}_n\}$ satisfy GenTR$^\vee$

Loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relation for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

The differentials

$$\omega_n^{(g)}(z, z_K) + \omega_n^{(g)}(\sigma(z), z_K)$$
$$\frac{1}{dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, \sigma(z), z_K) + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(\sigma(z), z_{J_2})\Big)$$

are holomorphic at $z = q$.

Loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relation for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

Equivalently, the differentials

$$\mathcal{W}_n^{(g),0} = \omega_n^{(g)}(z, z_K)$$
$$\mathcal{W}_n^{(g),1} = \frac{1}{dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, z, z_K) + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(z, z_{J_2})\Big)$$

have a pole at $z = q$ with odd principal part with respect to σ

Higher loop equations for CEO TR differentials

Define $K = (2, \dots, n)$,

$$\mathcal{T}_n(z,u;z_K) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde{z}_i \to z} u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde{x}_i}) \tfrac{1}{d\tilde{x}_i} \Big) \Big(\omega_{n-1+k}(\tilde{z}_{[\![k]\!]}, z_K) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde{x}_1 d\tilde{x}_2}{(\tilde{x}_1 - \tilde{x}_2)^2} \Big),$$

$$\mathcal{W}_n(z,u;z_K) = \frac{dx}{u\hbar} e^{\mathcal{T}_1(z,u)} \sum_{K = \sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z,u;z_{J_\alpha})$$

$$\mathcal{W}_n^{(g),k} = k![u^k]\mathcal{W}_n^{(g)} = y(z)^k \omega_n^{(g)}(z, z_K) + \begin{pmatrix} \text{terms containing } \omega_{n'}^{(g')} \\ \text{with } 2g' - 2 + n' < 2g - 2 + n \end{pmatrix}$$

Then, $\quad \mathcal{W}_n^{(g),0} = [u^0]\mathcal{W}_n^{(g)}, \qquad \mathcal{W}_n^{(g),1} = 2[u^1]\mathcal{W}_n^{(g)}$ are the same as above

Theorem (Higher Loop Equations for CEO TR differentials)

The pole of $\mathcal{W}_n^{(g),k}$ at $z = q \in \mathcal{P}$ has odd principal part for any $k \geq 0$.

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data

$q \in \mathcal{P}$ a key-special point with exponents (r, s) such that $r \geq 2$ and $s = 1$, that is:

- x has a critical point at q of multiplicity $r - 1$
- dy is holomorphic and nonzero at q

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data
$q \in \mathcal{P}$ a key-special point with exponents (r, s) such that $r \geq 2$ and $s = 1$

Definition

Ξ_q spanned by differentials $\left(d\frac{1}{dx}\right)^k \alpha$ where $k \geq 0$ and α is holomorphic at q

Theorem (Loop Equation for GenTR differentials)

$\mathcal{W}_n^{(g),k} \in \Xi_q$ *for any* $k \geq 0$.

Higher loop equations

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Theorem (Loop Equation for GenTR differentials)

$\mathcal{W}_n^{(g),k} \in \Xi_q$ for any $k \geq 0$.

Remark. 1. For $r = 2$ these loop equations are equivalent to those discussed above

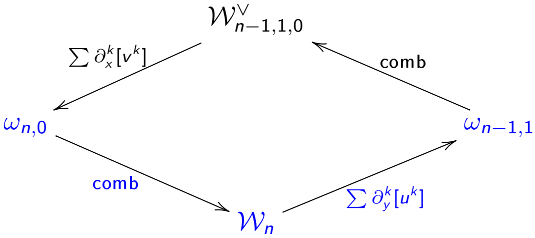
2. $$\mathcal{W}_n^{(g),k} = y(z)^k \omega_n^{(g)}(z, z_K) + \left(\begin{array}{c}\text{terms containing } \omega_{n'}^{(g')} \\ \text{with } 2g' - 2 + n' < 2g - 2 + n\end{array}\right)$$

The first r loop equations (with $k = 0, 1, \ldots, r-1$) $\rightsquigarrow$ unique polar part of $\omega_n^{(g)}$ at $z = q$.

This identification of polar part of $\omega_n^{(g)}$ is equivalent to GenTR relations

Then, the higher loop equations (for $k \geq r$) are satisfied automatically

Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

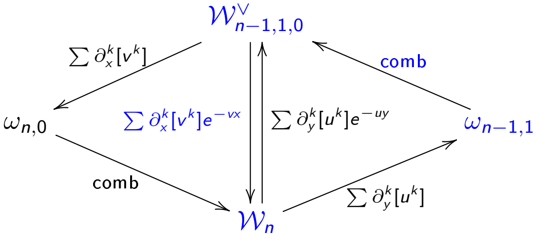
$$\mathcal{W}^\vee_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,v) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\omega_{n-1,1} = -dy \sum_{k\ge 0} (-\partial_y)^k [u^k] e^{-u\,y} \frac{\mathcal{W}_{n-1,1,0}}{dy}$$

$$= -\sum_{k\ge 0} \Big(-d\,\tfrac{1}{dy}\Big)^k [u^k] e^{-u\,y} \mathcal{W}_{n-1,1,0}$$

$$\omega_n = -\sum_{k\ge 0} \Big(-d\,\tfrac{1}{dx}\Big)^k [v^k] e^{-v\,x} \mathcal{W}^\vee_{n-1,1,0}$$

Proof of Loop Equations for GenTR



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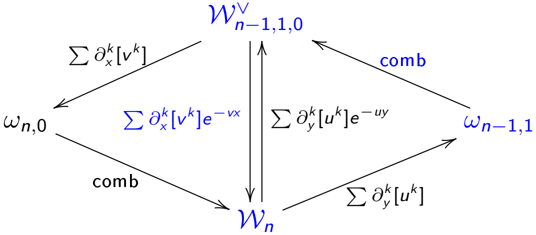
$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\geq 0} \Big(-d\,\tfrac{1}{dx}\Big)^{k} e^{u\,y} [v^k] e^{-v\,x} \mathcal{W}^{\vee}_{n-1,1,0}$$

Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\, e^{\hbar Q} J(Y) e^{-\hbar Q}\, e^F \Big\rangle^\circ$$

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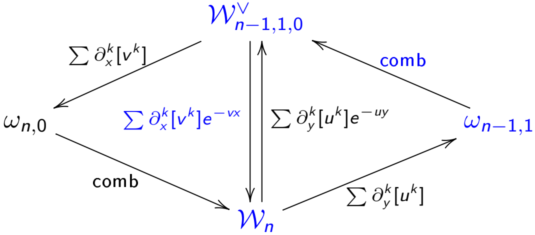
Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ e^{\hbar Q} J(Y) e^{-\hbar Q}\ e^F \Big\rangle^\circ$$

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The identity holds globally, but the only proof available involves computations in the expansion at a nice point

Example