Topological Recursion Revised

(on a series of joint papers with A.Alexandrov, B.Bychkov, P.Dunin-Barkowski, and S.Shadrin)

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IASM-BIRS Workshop
Noncommutative Geometry Meets Topological Recursion

Hangzhou, September 22-27, 2024

Topological recursion: an overview

Chekhov-Eynard-Orantin '06
Eynard-Orantin '07

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Goal: to compute a system of quantities (*correlators*)

$$\left\{f^{(g)}_{k_1,\dots,k_n}\right\}, \quad g \geq 0, \quad (k_1,\dots,k_n) \vdash d = \sum k_i$$

Examples:

- Hurwitz numbers (simple, double, monotone, weighted etc.);
- enumeration of maps (hypermaps, fully simple, weighted etc.);
- correlators of matrix models;
- correlators of CohFT's (GW invariants);
- WP volumes, MV volumes, etc.

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Potential (free energy) F; partition function (tau function) $Z = e^F$:

$$F(p_1,p_2,\dots;\hbar) = \sum_{g\geq 0,\ n\geq 1} \frac{\hbar^{2g-2+n}}{n!} \sum_{k_1,\dots,k_n\geq 1} f^{(g)}_{k_1,\dots,k_n} p_{k_1}\dots p_{k_n}$$

n-point function:

$$H^{(g)}_n(w_1,\dots,w_n) = \sum_{k_1,\dots,k_n} f^{(g)}_{k_1,\dots,k_n}\, w_1^{k_1}\dots w_n^{k_n}, \quad n = 1,2,\dots.$$

Topological recursion computes $H^{(g)}_n$ in a closed form inductively in g and n

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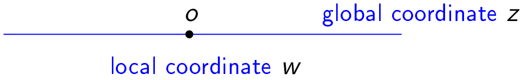
Spectral curve: $\Sigma = \mathbb{C}P^1$



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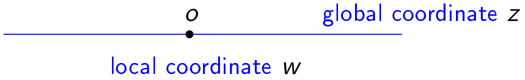

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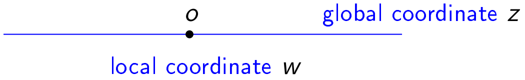

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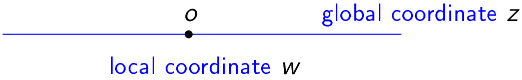

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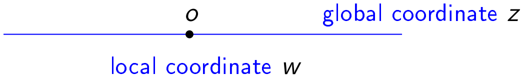

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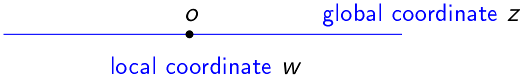

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$$\omega_n^{(g)} = d_1 \dots d_n H_n^{(g)} + \delta_{g,0}\delta_{n,2} \frac{dw_1 dw_2}{(w_1 - w_2)^2}$$

Example

Hurwitz numbers: $d=\sum_{i=1}^{n} k_i,\ m=2g-2+n+d,$

$$f^{(g)}_{k_1,\dots,k_n}=\frac{|\mathrm{Aut}(k_1,\dots,k_n)|}{m!d!}\#\left\{(\tau_1,\dots,\tau_m)\ \middle|\ \begin{array}{l}1)\ \tau_i{\in}S(d)\text{ a transposition}\\ 2)\ \tau_1{\circ}{\cdots}{\circ}\tau_m\text{ has cyclic type }(k_1,\dots,k_n)\\ 3)\ \text{connectness condition}\end{array}\right\}$$

$$\omega^{(g)}_n=\sum_{k_1,\dots,k_n} f^{(g)}_{k_1,\dots,k_n}\prod_{i=1}^{n} k_i w_i^{k_i-1}dw_i+\delta_{g,0}\delta_{n,2}\tfrac{dw_1\,dw_2}{(w_1-w_2)^2}$$

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$$\begin{aligned} w(z)&=z\,e^{-z}\\ &=z-z^2+\frac{z^3}{2}-\dots \end{aligned}\quad\Rightarrow\quad \begin{aligned}\omega^{(0)}_3&=\tfrac{dz_1\,dz_2\,dz_3}{(1-z_1)^2(1-z_2)^2(1-z_3)^2},\\ \omega^{(1)}_1&=\tfrac{(4-z_1)\,z_1}{24\,(1-z_1)^4}\,dz_1,\end{aligned}\qquad \mathcal{P}=\{z=1\},$$

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$$y(x)=z,\qquad x(z)=\log(w(z))=\log z-z,$$

$$\omega^{(0)}_1=y_1\;dx_1=(1-z_1^2)\,dz_1,\qquad \omega^{(0)}_2=\tfrac{dz_1\,dz_2}{(z_1-z_2)^2}.$$

Topological recursion: initial data

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- $\Sigma = \mathbb{C}P^1$ (generalization: a smooth algebraic complex curve);
- $B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$ (generalization: a bidifferential on Σ^2 with similar singularity on the diagonal)
- dx, dy meromorphic differentials on Σ
- $\mathcal{P} = \{q_1, \dots, q_N\}$ a set of *simple* zeroes of dx such that $dy|_{q_j} \neq 0$

Initial differentials:

$$\omega_1^{(0)}(z_1) = y(z_1)\, dx(z_1), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

The higher ω-differentials are computed by a recursive procedure inductively in g and n

Topological recursion: two step induction

2g − 2 + n > 0: $\quad K=\{2,\dots,n\},\ z_K=(z_2,\dots,z_n),$

First Step: $z\approx q_j\in\mathcal{P},\ x(z)=x(\sigma(z))$

$$\tilde\omega^{(g)}_n(z,z_K)=\frac{\omega^{(g-1)}_{n+1}(z,\sigma(z),z_K)+\sum\limits_{\substack{g_1+g_2=g,\ J_1\sqcup J_2=K\\(g_i,|J_i|+1)\neq(0,1)}}\omega^{(g_1)}_{|J_1|+1}(z,z_{J_1})\omega^{(g_2)}_{|J_2|+1}(\sigma(z),z_{J_2})}{(y(z)-y(\sigma(z)))\,dx(z)}.$$

Second Step:

$$\omega^{(g)}_n(z,z_K)=\tilde\omega^{(g)}_n(z,z_K)+(\text{holomorphic in } z),\quad z\to q_j,\quad j=1,\dots,N.$$

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Equivalently,

$$\omega^{(g)}_n(z_1, z_K) = \sum_{j=1}^{N} \operatorname*{res}_{z=q_j} \tilde{\omega}^{(g)}_n(z, z_K) \int^{z} B(\cdot, z_1).$$

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Requirement: compatibility with limits under degenerations of the spectral curve data

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$$x_\epsilon(z) = z^5 + \epsilon\, z, \qquad \begin{array}{l} \text{(a) } y_\epsilon = \frac{1}{z^3} \\ \text{(b) } y_\epsilon = \frac{1}{z^3+\epsilon} \\ \text{(c) } y_\epsilon = \frac{z^2}{x_\epsilon} = \frac{z}{z^4+\epsilon} \end{array}$$

How CEO TR differentials of these families behave as $\epsilon \to 0$?

Two examples

Answer:

- (1), $k \leq 3$; (2b): NO LIMIT!
- (1), $k \geq 4$; (2a):
  - the limit does exist and is govern by GenTR
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- (2c):
  - the limit does exist but it is different from that one of the case (2a)
  - is govern by BE recursion
  - a closed formula for the TR differentials of this family is also available

Closed expression for GetTR differentials

$$(1): \quad x = z^2 + \frac{\epsilon}{z^{k-2}}, \quad y = z^2, \quad k \geq 4$$

$$(2a): \quad x = z^5 + \epsilon z, \quad y = z^{-3}$$

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Input data: two functions $x(z)$, $y(z)$ such that dx and dy are meromorphic

$$\hat{z}(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \quad \hat{z}_i^{\pm} = \hat{z}(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2} - e^{-u/2}}{u},$$

$$\mathbb{W}_n^{\vee}(z_1, v_1, \dots, z_n, v_n) = \sum_{g \geq 0} \hbar^{2g-2+n} \mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^{n} \left(e^{v_i(\mathcal{S}(v_i \hbar \partial_{y_i}) - 1)x_i} \sqrt{\frac{d\hat{z}_i^+}{dz_i} \frac{d\hat{z}_i^-}{dz_i}} \; \frac{dz_i}{dx_i} \right) (-1)^{n-1} \sum_{\sigma \in \mathrm{cycl}(n)} \prod_{i=1}^{n} \frac{1}{\hat{z}_i^+ - \hat{z}_{\sigma(i)}^-}$$

$$\frac{(-1)^n \omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1, \dots, k_n \geq 0} (-\partial_{x_1})^{k_1} \dots (-\partial_{x_n})^{k_n} \, [v_1^{k_1} \dots v_n^{k_n}] \, \mathbb{W}_n^{\vee,(g)}$$

Generalized TR: overview

CEO TR: an explicit formula for $\tilde{\omega}^{(g)}_n(z, z_2, \ldots, z_n)$

Then, $\omega^{(g)}_n(z_1, z_2, \ldots, z_n) = \sum_{q_j \in \mathcal{P}} \operatorname*{res}_{z=q_j} \tilde{\omega}^{(g)}_n(z, z_2, \ldots, z_n) \int^{z} B(\cdot, z_1).$

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CEO TR: $$\tilde{\omega}^{(0)}_3(z, z_2, z_3) = \frac{B(z, z_2)B(\sigma(z), z_3) + B(z, z_3)B(\sigma(z), z_2)}{(y(z) - y(\sigma(z)))\, dx(z)}$$

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Generalized TR: overview

$o \in \Sigma$, z local coordinate $\qquad \begin{array}{l} x = a\,z^r + \text{h.o.t}, \\ y = b\,z^s + \text{h.o.t}, \end{array} \qquad a\,b \neq 0,\ r, s \in \mathbb{Z}$

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Definition

The point $o \in \Sigma$ is called
special, if $r + z > 0$ and $(r, s) \neq (1, 1)$,
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Initial data of GenTR: $(\Sigma, dx, dy, B, \mathcal{P})$;

- Σ, B the same as for CEO TR
- dx, dy arbitrary meromorphic differentials (with no restriction on zeroes and poles)
- $\mathcal{P}$ is an arbitrary subset in the set of special points

Generalized TR: basic properties

- $\omega_1^{(0)} = y\,dx$, $\omega_2^{(0)} = B$
- $2g-2+n>0$: $\omega_n^{(g)}$ is global meromorphic, symmetris, and has poles at $z_i = q_j$, $q_j \in \mathcal{P}$.
- Two-step recursion for $\omega_n^{(g)}$:
    - $\tilde{\omega}_n^{(g)}(z, z_2, \dots, z_n)$ is given by an *explicit formula* (below)
    - It is global meromorphic in z
    - its poles in z are at special points and also at $z_2, \dots, z_n$
    - $\omega_n^{(g)}$ selects those poles of $\tilde{\omega}_n^{(g)}$ which are *key-special*

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Theorem (Compatibility with known versions of TR)

- $(r,s) = (2,1) \Leftrightarrow$ *CEO*
- $(r,s) = (2,-1) \Leftrightarrow$ *Chekhov-Norbury irregular recursion*
- $r>0$, $s=\pm1 \Leftrightarrow$ *BE recursion*
- $(r,s) = (1,0) \Leftrightarrow$ *LogTR of* [ABDKS23]

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Remark. GenTR is not compatible with BE TR if $s \neq \pm 1$

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

GenTR is compatible with limits of the spectral curve data as long as key-special points and key$^\vee$-special points do not collapse together

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See e.g. Example (1), $k \geq 4$, Exampe (2a)

Example

$$x = z^2, \quad y = \frac{1}{z+s}, \quad \mathcal{P} = \{0\}$$

This TR is compatible with the limit as $s \to 0$

$s \neq 0$: CEO TR $\rightsquigarrow$ KW potential (with properly rescaled times)

$s = 0$: CN irregular TR $\rightsquigarrow$ BGW potential

Generalized TR: compatibility with limits

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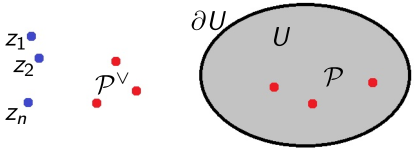
Proof



$$\omega_n^{(g)}(z_1, z_K) = \frac{1}{2\pi} \int_{z\in\partial U} \left(\tilde{\omega}_n^{(g)}(z, z_K) \int^z B(\cdot, z_1) \right), \qquad z_1, \dots, z_n \in \Sigma \setminus U.$$

Generalized TR: xy swap duality

xy duality transformation: an explicit closed formula $\quad \{\omega_n^{(g)}\} \quad \longleftrightarrow \quad \{\omega_n^{\vee,(g)}\}$

Theorem (Compatibility with xy swap)

$$\begin{tikzcd}
(\Sigma, dx, dy, B, \mathcal{P}) \arrow[d, "GenTR"'] & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \arrow[d, "GenTR"] \\
\{\omega_n^{(g)}\} \arrow[r, leftrightarrow, "xy\ \ swap"] & \{\omega_n^{\vee,(g)}\}
\end{tikzcd}$$

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\end{tikzcd}
$$

Remark.

GenTR relation for $\omega_n^{(g)}$ $\quad \Leftrightarrow \quad$ $\omega_n^{\vee,(g)}$ is regular at $\mathcal{P}$

Generalized TR: xy swap duality

Corollary

$\Sigma = \mathbb{C}P^1$, $\mathcal{P}^\vee = \emptyset$ (all special points are key-special). Then, $\omega_n^{\vee,(g)} = 0$ for $2g-2+n>0$ and an explicit formula $(*)$ for $\omega_n^{(g)}$ holds.

Generalized TR: xy swap duality

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$$\hat z(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \qquad \hat z_i^{\pm} = \hat z(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2}-e^{-u/2}}{u},$$

$$\mathbb{W}_n^{\vee}(z_1,v_1,\dots,z_n,v_n) = \sum_{g\geq 0} \hbar^{2g-2+n}\mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^{n}\left(e^{v_i(\mathcal{S}(v_i\hbar\partial_{y_i})-1)x_i}\sqrt{\frac{d\hat z_i^{+}}{dz_i}\frac{d\hat z_i^{-}}{dz_i}}\,\frac{dz_i}{dx_i}\right)(-1)^{n-1}\sum_{\sigma\in \mathrm{cycl}(n)}\prod_{i=1}^{n}\frac{1}{\hat z_i^{+}-\hat z_{\sigma(i)}^{-}}$$

$$\frac{(-1)^n\omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1,\dots,k_n\geq 0}(-\partial_{x_1})^{k_1}\dots(-\partial_{x_n})^{k_n}\,[v_1^{k_1}\dots v_n^{k_n}]\,\mathbb{W}_n^{\vee,(g)}$$

Generalized TR: KP integrability

Theorem (KP integrability)

If $\Sigma = \mathbb{C}P^1$, then GenTR differentials are KP integrable

(see the talk of Sasha Alexandrov for details)

Corollary

GenTR potential for $\begin{cases} dx = z^{r-1}dz \\ dy = z^{s-1}dz \end{cases}$ *is a solution of KP hierarchy for any* (r,s), $r+s>0$

Example: $(r,s)=(1,2)$

Example

$$\begin{cases} x = z, \\ y = z^2. \end{cases}$$

Special points $= \{0\}$

$\mathcal{P}$	$\mathcal{P}^\vee$	GenTR	GenTR$^\vee$
$\emptyset$	$\{0\}$	trivial	KW
$\{0\}$	$\emptyset$	new!	trivial

Expansion point: $z=\infty$, expansion local coordinate: $1/z$

$$F = -\tfrac{1}{48}p_2\hbar + \left(\tfrac{1}{96}p_1^4 - \tfrac{1}{96}p_2^2\right)\hbar^2 + \left(\tfrac{1}{48}p_2p_1^4 + \tfrac{1}{24}p_4p_1^2 - \tfrac{1}{144}p_2^3 - \tfrac{9}{1280}p_6\right)\hbar^3 + \left(\tfrac{9}{640}p_3p_1^5 + \tfrac{1}{32}p_2^2p_1^4 + \tfrac{125}{1152}p_5p_1^3 + \tfrac{9}{256}p_3^2p_1^2 + \tfrac{1}{8}p_2p_4p_1^2 + \tfrac{343}{2880}p_7p_1 + \tfrac{29}{2880}p_4^2 - \tfrac{1}{192}p_2^4 - \tfrac{27}{1280}p_2p_6\right)\hbar^4 + O(\hbar^5)$$

This potential is a solution of KP hierarchy

Example: $(r,s)=(1,2)$

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CEO TR deformation: $\begin{cases} x = z + \frac{\epsilon}{z^2}, \\ y = z^2 \end{cases}$

To be explained

Still missing:

- GenTR recursion formula for $\tilde{\omega}_n^{(g)}$ (L2)
- compatibility with CEO and other versions of TR (L3)
- xy swap formula (L2)
- symplectic duality as a generalization of xy duality (L3)
- closed formulas for BE TR differentials as a special case of symplectic duality (L3)
- definition of KP integrability (the talk of A. Alexandrov)

xy swap transformation at a nice point

$$\begin{matrix} \{\omega_n^{(g)}\} & & \{\omega_n^{\vee,(g)}\} \\ \omega_1^{(0)}(z) = y\,dx & \longleftrightarrow & \omega_1^{\vee,(0)}(z) = x\,dy \end{matrix}$$

xy swap transformation at a nice point

$$\begin{array}{ccc} \{\omega_n^{(g)}\} & \longleftrightarrow & \{\omega_n^{\vee,(g)}\} \\ \omega_1^{(0)}(z){=}y\,dx & & \omega_1^{\vee,(0)}(z){=}x\,dy \end{array}$$

Definition

A point $o \in \Sigma$ is called *nice* if $x = \log z + O(z)$, $y = \log z + O(z)$

$$dx = \frac{dz}{z} + (\text{holomorphic}), \quad dy = \frac{dz}{z} + (\text{holomorphic})$$

Then, $X = e^x$ and $Y = e^y$ can serve as local coordinates

$$dx = \frac{dX}{X}, \quad dy = \frac{dY}{Y}, \quad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y.$$

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \xleftrightarrow{(o,X)} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \xleftrightarrow{(o,Y)} \{(-1)^n \omega_n^{\vee,(g)}\}$$

$$Q = \frac{1}{2}\sum_{i,j}\Big((i+j)p_i p_j \tfrac{\partial}{\partial p_{i+j}} + i\,j\,p_{i+j}\tfrac{\partial^2}{\partial p_i \partial p_j}\Big)$$

$$\omega_n = \sum_{g\geq 0} \hbar^{2g-2+n}\omega_n^{(g)} \qquad \omega_n^\vee = \sum_{g\geq 0} \hbar^{2g-2+n}\omega_n^{(g)}$$

$$\omega_n - \delta_{n,2}\tfrac{dX_1 dX_2}{(X_1-X_2)^2} - \delta_{n,1}\hbar^{-1}x_1 dx_1 = \sum_{k_1,\dots,k_n\geq 1} \frac{\partial^n F}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \textstyle\prod_{i=1}^n d(X_i^{k_i})$$

$$(-1)^n\omega_n^\vee - \delta_{n,2}\tfrac{dY_1 dY_2}{(Y_1-Y_2)^2} + \delta_{n,1}\hbar^{-1}y_1 dy_1 = \sum_{k_1,\dots,k_n\geq 1} \frac{\partial^n F^\vee}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \textstyle\prod_{i=1}^n d(Y_i^{k_i})$$

xy swap transformation at a nice point

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Theorem

The composition $\{\omega_n^{(g)}\} \longmapsto \{\omega_n^{\vee,(g)}\}$ is given by a closed finite expression that extends to a transformation of global meromorphic differentials and does not involve any information on a chosen expansion point o.

The obtained transformation is called the *xy swap duality*

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

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Remark. The potentials F and $F^\vee$ do depend on the expansion point an a choice of local coordinates. The treatment of xy duality as the action of e^{-uQ} on the corresponding tau function is valid for a *nice point* only

xy swap: the formula

$$W_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\,\frac{\omega_n}{\prod_{i=1}^n dx_i}, \qquad \mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \prod_{i=1}^n \frac{dx_i}{u_i\hbar}\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\geq 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\,\mathbb{W}_n^{(g)}$$

Γ_n is the set of *hypergraphs* (graphs with hyperedges) with n marked vertices

xy swap: the formula

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Corrections and details. 1. The dependence of $\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n^{(g)}=[\hbar^{2g-2+n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n$ in u-variables is polynomial.

2. If $|e|=2$ and $e(1)=e(2)$, use the regularized differential $\omega_2(\tilde z_1,\tilde z_2)-\frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}$ instead in the definition of the edge contribution $W_{|e|}$.

xy swap: basic properties

1. $\omega_1^{\vee,(0)} = x\,dy$, $\omega_2^{\vee,(0)} = \omega_2^{(0)}$
2. $2g-2+n>0$: $\omega_n^{\vee,(g)}$ is globally defined and meromorphic
3. Moreover, it is *regular on diagonals*
4. The *inverse transformation* is given by the same formulas with x and y swapped

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3. Moreover, it is *regular on diagonals*
4. The *inverse transformation* is given by the same formulas with x and y swapped

Remark. We aware of no direct combinatorial proof of the last two properties. The arguments we are using involve computation in the space of power expansions at a (nice) point

Example

$$\omega_3^{(0)} + \omega_3^{\vee,(0)} + d_1 \frac{B(z_1,z_2)B(z_1,z_3)}{dx_1\,dy_1} + d_2 \frac{B(z_2,z_3)B(z_2,z_1)}{dx_2\,dy_2} + d_3 \frac{B(z_3,z_1)B(z_3,z_2)}{dx_3\,dy_3} = 0$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \xleftrightarrow{(o,X)} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \xleftrightarrow{(o,Y)} \{(-1)^n \omega_n^{\vee,(g)}\}$$

Notation: $\langle F \rangle = F\,\big|_{p=0}$, 'taking the free term of a series',

$$J^+(X) = \sum_{k=1}^{\infty} k X^k \partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2}\frac{X_1 X_2}{(X_1-X_2)^2} = \sum_{k_1,\dots,k_n} \frac{\partial^n F}{\partial p_{k_1}\dots\partial p_{k_n}}\,\Big|_{p=0} \prod_{i=1}^n k_i X_i^{k_i}$$
$$= \langle J^+(X_1)\dots J^+(X_n) F \rangle$$

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Derivation of xy swap formula, Step 1: inclusion/exclusion

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where the 'connected' correlators are defined through inclusion/exclusion

$$\langle J^+(X) e^F \rangle = \langle J^+(X) e^F \rangle^\circ$$

$$\langle J^+(X_1) J^+(X_2) e^F \rangle = \langle J^+(X_1) J^+(X_2) e^F \rangle^\circ + \langle J^+(X_1) e^F \rangle^\circ \langle J^+(X_2) e^F \rangle^\circ$$

$$\dots$$

$$\langle J^+(X_1) \dots J^+(X_n) e^F \rangle = \sum_{\sqcup I_\alpha = \{1,\dots,n\}} \prod_\alpha \langle \prod_{i \in I_\alpha} J^+(X_i) e^F \rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

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Moreover, define

$$J(X) = \sum_{k=-\infty}^{\infty} X^k J_k, \qquad J_k = \begin{cases} k\,\partial_{p_k}, & k > 0, \\ 0, & k = 0, \\ p_{-k}, & k < 0 \end{cases}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} = \langle J(X_1) \dots J(X_n) e^F \rangle^\circ$$

(with the singular $(0,2)$ correction taken into account automatically)

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \xleftrightarrow{(o,X)} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \xleftrightarrow{(o,Y)} \{(-1)^n \omega_n^{\vee,(g)}\}$$

$$Q = \frac{1}{2} \sum_{i,j} \left((i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + i\, j\, p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right)$$

Similarly,

$$\frac{(-1)^n \omega_n^\vee}{\prod_{i=1}^n dy_i} = \left\langle J(Y_1) \dots J(Y_n)\, e^{F^\vee} \right\rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

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$$Q = \frac{1}{2}\sum_{i,j}\Bigl((i+j)p_ip_j\tfrac{\partial}{\partial p_{i+j}} + i\,j\,p_{i+j}\tfrac{\partial^2}{\partial p_i\partial p_j}\Bigr)$$

Similarly,

$$\begin{aligned}\frac{(-1)^n\omega_n^\vee}{\prod_{i=1}^n dy_i} &= \bigl\langle J(Y_1)\dots J(Y_n)\;e^{F^\vee}\bigr\rangle^\circ\\ &= \bigl\langle J(Y_1)\dots J(Y_n)\;e^{-\hbar Q}e^F\bigr\rangle^\circ\\ &= \bigl\langle \mathbb{J}(Y_1)\dots\mathbb{J}(Y_n)\;e^F\bigr\rangle^\circ, \qquad \mathbb{J}(Y) = e^{\hbar Q}J(Y)e^{-\hbar Q}\end{aligned}$$

The next step: to compute the operator $\mathbb{J}(Y) = e^{\hbar Q}J(Y)e^{-\hbar Q}$ acting on $\mathbb{C}[[p_1,p_2,\dots]]$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q} J(Y) e^{-\hbar Q}$

Main tool: *bosonic representation of $\widehat{\mathfrak{gl}}(\infty)$* on $\mathbb{C}[[p_1, p_2, \dots]]$

$$\sum_{i,j\in\mathbb{Z}} z_1^j z_2^{-i-1} E_{i,j} = \frac{e^{\sum_{i<0} \frac{z_1^i - z_2^i}{i} J_i} e^{\sum_{i>0} \frac{z_1^i - z_2^i}{i} J_i} - 1}{z_1 - z_2}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q} J(Y) e^{-\hbar Q}$

Main tool: *bosonic representation of* $\widehat{\mathfrak{gl}}(\infty)$ on $\mathbb{C}[[p_1, p_2, \dots]]$

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$z_1 = Xe^{u/2}$, $z_2 = Xe^{-u/2}$, $\partial_x = X\partial_X$:

$$\mathcal{E}(X,u) = \sum_{k,m\in\mathbb{Z}} X^m e^{u(k+\frac{1-m}{2})} E_{k-m,m} + \tfrac{1}{u\mathcal{S}(u)} = \frac{e^{\sum_{i<0} u\mathcal{S}(ui)X^i J_i} e^{\sum_{i>0} u\mathcal{S}(ui)X^i J_i}}{u\mathcal{S}(u)}$$

$$= \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i} e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q}J(Y)e^{-\hbar Q}$

Main tool: *bosonic representation of $\widehat{\mathfrak{gl}}(\infty)$* on $\mathbb{C}[[p_1,p_2,\dots]]$

$$\mathcal{E}(X,u)=\sum_{k,m\in\mathbb{Z}}X^m e^{u(k+\frac{1-m}{2})}E_{k-m,m}+\frac{1}{u\mathcal{S}(u)}=\frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0}X^iJ_i}e^{u\mathcal{S}(u\partial_x)\sum_{i>0}X^iJ_i}}{u\mathcal{S}(u)}$$

All operators involved belong to $\widehat{\mathfrak{gl}}(\infty)$:

$$J_m=[X^mu^0]\mathcal{E}(X,u)=\sum_{k\in\mathbb{Z}}E_{k-m,k},\qquad J(X)=[u^0]\mathcal{E}(X,u),$$

$$Q=[X^0u^2]\mathcal{E}(X,u)=\frac{1}{2}\sum_{k\in\mathbb{Z}}\left(k+\tfrac{1}{2}\right)^2E_{k,k}$$

$$\mathbb{J}(Y)=e^{\hbar Q}J(Y)e^{-\hbar Q}=\sum_{k,m\in\mathbb{Z}}\frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}}Y^mE_{k-m,k}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q}J(Y)e^{-\hbar Q}$

Lemma

$$\mathbb{J}(Y)=e^{\hbar Q}J(Y)e^{-\hbar Q}=\sum_{j=0}^{\infty}(-\partial_y)^j[u^j]e^{-u(y-x)}\frac{dx}{dy}\mathcal{E}(X,u\hbar)$$

Or, taking the coefficient of $E_{k-m,k}$,

$$\frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}}Y^m=\sum_{j=0}^{\infty}(-Y\partial_Y)^j[u^j]\left(\tfrac{X}{Y}\right)^u e^{u\hbar\left(k+\frac{1-m}{2}\right)}\frac{dX}{X}\frac{Y}{dY}X^m$$

Substituting, we obtain

$$\frac{(-1)^n\omega_n^\vee}{\prod_{i=1}^n dy_i}=\Big\langle\prod_{i=1}^n\mathbb{J}(Y_i)\,e^F\Big\rangle^\circ=\sum_{k_1,\dots,k_n\ge0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}[u_1^{k_1}\dots u_n^{k_n}]\left(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\right)\mathbb{W}_n$$

where $\quad\mathbb{W}_n=\left(\prod_{i=1}^n e^{u_ix_i}dx_i\right)\Big\langle\prod_{i=1}^n\mathcal{E}(X_i,u_i\hbar)\,e^F\Big\rangle^\circ$

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

$$\mathbb{W}_n = \Big(\prod_{i=1}^{n} e^{u_i x_i} dx_i\Big) \, \big\langle \mathcal{E}(X_n, u_n\hbar) \dots \mathcal{E}(X_n, u_n\hbar) \, e^F \big\rangle^{\circ}$$

- Insert $\mathcal{E}(X, u\hbar) = \frac{e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i<0} X^i J_i} e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i>0} X^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

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- Insert $\mathcal{E}(X, u\hbar) = \frac{e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i<0} X^i J_i}\, e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i>0} X^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

$$W_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\ \frac{\omega_n}{\prod_{i=1}^{n} dx_i},$$

$$\mathbb{W}_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} \frac{dx_i}{u_i\hbar}\Big) \sum_{\gamma\in\Gamma_n} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{e\in E(\gamma)} W_{|e|}(z_{e_1}, u_{e_1}, \dots, z_{e_{|e|}}, u_{e_{|e|}})$$

$$
\begin{tikzcd}
\omega_n \sim \left\langle \prod J(X_i)\, e^F \right\rangle^\circ \arrow[rr, "xy \text{ swap}"] \arrow[dr, "\substack{\text{summation}\\ \text{over graphs}}"'] & & \omega_n^\vee \sim \left\langle \prod J(Y_i)\, e^{F^\vee} \right\rangle^\circ = \left\langle \prod \mathbb{J}(Y_i)\, e^F \right\rangle^\circ \\
& \mathbb{W}_n \sim \left\langle \prod \mathcal{E}(X_i, u_i\hbar)\, e^F \right\rangle^\circ \arrow[ur, "\prod_{i=1}^{n} \sum_{k\ge 0} \partial_{y_i}^k [u_i^k]"'] &
\end{tikzcd}
$$

$$
J(X) = \sum_{k=-\infty}^{\infty} J_k X^k, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q},
$$

$$
\mathcal{E}(X,u) = \frac{e^{u\mathcal{S}(u\partial_x)\sum\limits_{i<0} X^i J_i}\, e^{u\mathcal{S}(u\partial_x)\sum\limits_{i>0} X^i J_i}}{u\mathcal{S}(u)}, \qquad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y
$$

xy swap: the formula (reminding)

$$W_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\,\frac{\omega_n}{\prod_{i=1}^n dx_i}, \qquad \mathcal{S}(u) = \frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n \frac{dx_i}{u_i\hbar}\Big)\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

(If $|e|=2$ and $e(1)=e(2)$, use the regularized differential $\omega_2(\tilde z_1,\tilde z_2)-\frac{d\tilde x_1\,d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}$ instead in the definition of the edge contribution $W_{|e|}$)

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\geq 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\,\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\,\mathbb{W}_n^{(g)}$$

More properties of xy swap

- The xy swap transformation produces no singularities apart from the special points: if $\omega_n^{(g)}$ is regular at some non-special point for all (g,n) with $2g-2+n>0$, then the same holds for $\omega_n^{\vee,(g)}$
- This property motivates the definition of GenTR: it is *defined* by the requirement that all xy dual differentials are holomorphic at the key-special points. Then the compatibility GenTR with xy swap becomes a reformulation of the definition:

$$
\begin{tikzcd}
(\Sigma, dx, dy, B, \mathcal{P}) \arrow[d, "\mathrm{GenTR}"'] & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \arrow[d, "\mathrm{GenTR}"] \\
\{\omega_n^{(g)}\} \arrow[r, leftrightarrow, "xy\ \mathrm{swap}"] & \{\omega_n^{\vee,(g)}\}
\end{tikzcd}
$$

More concretely, this idea is realized below

Partial xy swap duality and definition of GenTR

$$
\begin{tikzcd}
\omega_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ J(X)\ e^F \Big\rangle^{\circ} \arrow[rr] \arrow[dr, "\text{combinatorial expression}"'] & & \omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ \mathbb{J}(Y)\ e^F \Big\rangle^{\circ} \\
& \mathcal{W}_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ \mathcal{E}(X, u\hbar)\ e^F \Big\rangle^{\circ} \arrow[ur, "\sum_{k\geq 0} \partial_y^k [u^k]"'] &
\end{tikzcd}
$$

- $\omega^{(g)}_{n-1,1}$ extends as a global meromorphic n-differential on Σ^n
- $\omega_{n-1,1}$ admits a closed expression in terms of ω_m's with to differentiation in $z_1, \dots, z_{n-1}$
- Moreover, it also admits a closed expression in $\omega^{\vee}_m$'s with to differentiation in $z = z_n$
- **Corollary**. $\omega^{\vee,(g)}_n$ is holomorpfic at $z_i = q$ for $i = 1, \dots, n$ and all (g, n) iff $\omega^{(g)}_{n-1,1}$ *is holomorphic at* $z_n = q$ for all (g, n)

This is yet another reformulation of the definition of GenTR (for $q \in \mathcal{P}$)

Partial xy swap duality and definition of GenTR

$$\begin{tikzcd}[column sep=small]
\omega_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\, J(X)\, e^F \Big\rangle^\circ \arrow[rr] \arrow[dr, "\text{combinatorial expression}"'] & & \omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\, \mathbb{J}(Y)\, e^F \Big\rangle^\circ \\
& \mathcal{W}_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\, \mathcal{E}(X, u\hbar)\, e^F \Big\rangle^\circ \arrow[ur, "\sum_{k\geq 0} \partial_y^k [u^k]"'] &
\end{tikzcd}$$

$$\mathcal{T}_n(z_{[\![n-1]\!]}; z, u) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde z_i \to z}\, u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde x_i}) \tfrac{1}{d\tilde x_i} \Big) \big(\omega_{n-1+k}(z_{[\![n-1]\!]}, \tilde z_{[\![k]\!]}) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]}; z, u) = \frac{dx}{u\hbar}\, e^{\mathcal{T}_1(z,u)} \sum_{\substack{[\![n]\!] = \sqcup_\alpha J_\alpha \\ J_\alpha \neq \emptyset}} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha}; z, u)$$

$$-\frac{\omega^{(g)}_{n-1,1}}{dy} = \sum_{r\geq 0} (-\partial_y)^r [u^r] e^{-u\,y}\, \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}$$

$$\mathcal{T}_n(z_{[\![n-1]\!]}; z, u) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde z_i \to z}\, u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde x_i}) \tfrac{1}{d\tilde x_i} \Big) \big(\omega_{n-1+k}(z_{[\![n-1]\!]}, \tilde z_{[\![k]\!]}) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]}; z, u) = \frac{dx}{u\hbar}\, e^{\mathcal{T}_1(z,u)} \sum_{[\![n]\!] = \sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_{\alpha} \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha}; z, u)$$

$$\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]}, z) = -dy \sum_{r \geq 0} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}$$

$$= -\omega^{(g)}_n(z_{[\![n-1]\!]}, z) - \underbrace{dy \sum_{r \geq 1} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}}_{\text{involves } \omega(g')_{n'}\text{'s with } 2g'-2+n' < 2g-2+n}$$

$$\mathcal{T}_n(z_{[\![n-1]\!]}; z, u) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde z_i \to z} u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde x_i}) \tfrac{1}{d\tilde x_i} \Big) \big(\omega_{n-1+k}(z_{[\![n-1]\!]}, \tilde z_{[\![k]\!]}) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2} \big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]}; z, u) = \frac{dx}{u\hbar} e^{\mathcal{T}_1(z,u)} \sum_{[\![n]\!] = \sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha}; z, u)$$

$$\begin{aligned} \omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]}, z) &= -dy \sum_{r\geq 0} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy} \\ &= -\omega^{(g)}_n(z_{[\![n-1]\!]}, z) + \tilde\omega^{(g)}_n(z_{[\![n-1]\!]}, z) \end{aligned}$$

Definition (Differentials $\tilde\omega^{(g)}_n$ of Generalized Topological Recursion)

$$\tilde\omega^{(g)}_n = -dy \sum_{r\geq 1} (-\partial_y)^r [u^r] e^{-u\,y} \frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]}; z, u)}{dy}$$

This definition implies that $\omega^{(g)}_{n-1,1}$ is holomorphic in z at key-special points.
This implies, in turn, the compatibility of GenTR with xy swap

To be explained

Still missing:

- compatibility with CEO and other versions of TR
- symplectic duality as a generalization of xy duality
- closed formulas for BE TR differentials as a special case of symplectic duality

Loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relation for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

The differentials

$$\omega_m^{(g)}(z, z_K) + \omega_m^{(g)}(\sigma(z), z_K)$$

$$\frac{1}{dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, \sigma(z), z_K) + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(\sigma(z), z_{J_2})\Big)$$

are holomorphic at $z = q$.

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relation for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

Equivalently, the differentials

$$\begin{aligned}\Omega_n^{(g),0} &= \omega_m^{(g)}(z, z_K)\\ \Omega_n^{(g),1} &= \frac{1}{dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, z, z_K) + \sum_{\substack{g_1+g_2=g\\ J_1 \sqcup J_2 = K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(z, z_{J_2})\Big)\end{aligned}$$

have a pole at $z = q$ with odd principal part with respect to σ

Higher loop equations for CEO TR differentials

Define $K = (2, \ldots, n)$,

$$\mathcal{T}_n(z, u; z_K) = \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(\lfloor_{\tilde{z}_i \to z}\, u\hbar \mathcal{S}(u\hbar \tfrac{d}{d\tilde{x}_i}) \tfrac{1}{d\tilde{x}_i} \Big) \big(\omega_{n-1+k}(\tilde{z}_{[\![k]\!]}, z_K) - \delta_{n,1}\delta_{k,2} \tfrac{d\tilde{x}_1 d\tilde{x}_2}{(\tilde{x}_1 - \tilde{x}_2)^2} \big),$$

$$\mathcal{W}_n(z, u; z_K) = \frac{dx}{u\hbar}\, e^{\mathcal{T}_1(z,u)} \sum_{K = \sqcup_\alpha J_\alpha,\ J_\alpha \neq \emptyset} \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z, u; z_{J_\alpha})$$

$$\Omega_n^{(g),k} = k![u^k]\mathcal{W}_n = y(z)^k \omega_n^{(g)}(z, z_K) + \begin{pmatrix} \text{terms containing } \omega_{n'}^{(g')} \\ \text{with } 2g' - 2 + n' < 2g - 2 + n \end{pmatrix}$$

Then, $\Omega_n^{(g),0} = [u^0]\Omega_n^{(g)}$, $\Omega_n^{(g),1} = 2[u^1]\Omega_n^{(g)}$ are the same as above

Theorem (Higher Loop Equations for CEO TR differentials)

The pole of $\Omega_n^{(g),k}$ at $z = q \in \mathcal{P}$ has odd principal part for any $k \geq 0$.

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data

$q \in \mathcal{P}$ a key-special point with exponents (r, s) such that $r \geq 2$ and $s = 1$, that is:

- x has a critical point at q of multiplicity $r - 1$
- dy is holomorphic and nonzero at q

Higher loop equations

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Definition

Ξ_q spanned by differentials $\left(d\frac{1}{dx}\right)^k \alpha$ where $k \geq 0$ and α is holomorphic at q

Theorem (Loop Equation for GenTR differentials)

$\Omega_n^{(g),k} \in \Xi_q$ *for any* $k \geq 0$.

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Definition

Ξ_q spanned by differentials $\left(d\frac{1}{dx}\right)^k \alpha$ where $k \geq 0$ and α is holomorphic at q

Theorem (Loop Equation for GenTR differentials)

$\Omega_n^{(g),k} \in \Xi_q$ *for any* $k \geq 0$.

Remark. 1. For $r = 2$ these loop equations are equivalent to those discussed above

2. The first r loop equations (with $k = 0, 1, \ldots, r - 1$) determine the principal part of the pole of $\omega_n^{(g)}$ at $z = q$ uniquely and can serve as an alternative of GenTR in this case. Then, the higher loop equations (for $k \geq r$) are satisfied automatically.

Higher Loop Equations

$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

$$\mathcal{W}^\vee_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,u) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\begin{tikzcd}
 & \mathcal{W}^\vee_{n-1,1,0} \arrow[dl, "\sum \partial_x^k[v^k]"'] & \\
\omega_{n,0} \arrow[dr, "\mathrm{comb}"'] & & \omega_{n-1,1} \arrow[ul, "\mathrm{comb}"'] \\
 & \mathcal{W}_n \arrow[ur, "\sum \partial_y^k[u^k]"'] &
\end{tikzcd}$$

Higher Loop Equations

$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

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$$\mathcal{W}^\vee_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,u) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\begin{tikzcd}
 & \mathcal{W}^\vee_{n-1,1,0} \arrow[dl, "\sum \partial_x^k {[v^k]}"'] \arrow[dd, shift right, "\sum \partial_x^k {[v^k]} e^{-vx}"'] & \\
\omega_{n,0} \arrow[dr, "\mathrm{comb}"'] & & \omega_{n-1,1} \arrow[ul, "\mathrm{comb}"'] \\
 & \mathcal{W}_n \arrow[uu, shift right, "\sum \partial_y^k {[u^k]} e^{-uy}"'] \arrow[ur, "\sum \partial_y^k {[u^k]}"'] &
\end{tikzcd}$$

Higher Loop Equations

$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

$$\mathcal{W}^\vee_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,u) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\begin{aligned}
\omega_{n-1,1} &= -dx \sum_{k\ge 0} (-\partial_x)^k [v^k] e^{-v\,x} \frac{\mathcal{W}^\vee_{n-1,1,0}}{dx} \\
&= -\sum_{k\ge 0} \Big(-d\,\tfrac{1}{dx}\Big)^k [v^k] e^{-v\,x} \mathcal{W}^\vee_{n-1,1,0}
\end{aligned}$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\ge 0} \Big(-d\,\tfrac{1}{dx}\Big)^k e^{u\,y} [v^k] e^{-v\,x} \mathcal{W}^\vee_{n-1,1,0}$$

Higher Loop Equations

$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^{F} \Big\rangle^{\circ}$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^{F} \Big\rangle^{\circ}$$

$$\mathcal{W}^{\vee}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,u) e^{-\hbar Q}\; e^{F} \Big\rangle^{\circ}$$

$$\omega_{n-1,1} = -dx \sum_{k\ge 0} (-\partial_x)^k [v^k] e^{-v\,x} \frac{\mathcal{W}^{\vee}_{n-1,1,0}}{dx}$$

$$= -\sum_{k\ge 0} \Big(-d\,\frac{1}{dx}\Big)^k [v^k] e^{-v\,x} \mathcal{W}^{\vee}_{n-1,1,0}$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\ge 0} \Big(-d\,\frac{1}{dx}\Big)^k \underbrace{e^{u\,y} [v^k] e^{-v\,x} \mathcal{W}^{\vee}_{n-1,1,0}}_{\text{holomorphic}}$$

Proof of loop equations

The proof is based on a computation at a nice point. Along with the operator identity

$$\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q} = \sum_{j=0}^{\infty} (-\partial_y)^j [u^j] e^{-u(y-x)} \frac{dx}{dy} \mathcal{E}(X, u\hbar)$$

we have also

Lemma

$$\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q} = \sum_{j=0}^{\infty} (-\partial_y)^j [u^j] e^{-u(y-x)} \frac{dx}{dy} \mathcal{E}(X, u\hbar)$$

Thank you!