Rank inequality done by free probability

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Linear algebra: Let A, B be two matrices in $M_n(\mathbb{C})$. Then

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We can improve it: for any $\lambda \in \mathbb{C}$

$$\begin{aligned}
\text{rank}(A+B) &= \text{rank}(A+\lambda+B-\lambda) \\
&\leq \text{rank}(A+\lambda) + \text{rank}(B-\lambda) \\
&\leq \inf_{\lambda \in \mathbb{C}} (rank(A+\lambda) + \text{rank}(B-\lambda)) \\
&= \min_{\lambda \in \sigma(-A) \cup \sigma(B)} (rank(A+\lambda) + \text{rank}(B-\lambda)) \\
&= \min\{n, \min_{\lambda \in \sigma(-A) \cap \sigma(B)} (rank(A+\lambda) + \text{rank}(B-\lambda))\}
\end{aligned}$$

Alternatively, for two fixed matrices A and B, we have

$$\sup_{U \in U_n(\mathbb{C})} \text{rank}(A + UBU^*) \leq \min\{n, \min_{\lambda \in \sigma(-A) \cap \sigma(B)} (rank(A+\lambda) + \text{rank}(B - \lambda))\},$$

where $U_n(\mathbb{C})$ denotes the group of $n \times n$ unitary matrices.

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Questions

1. Do we actually have

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2. More generally, for a noncommutative polynomial p, can we find a dimensionless optimal rank upper bound for $p(A, UBU^*)$?

Answers by free probability and random matrices

Theorem (Arizmendi-Cébron-Speicher-Y, 24)

Let A, B be two matrices in $M_n(\mathbb{C})$ and $(\hat{A}, \hat{B})$ a free copy of (A, B). Then

$$\operatorname{rank}(p(A, UBU^*)) \leq \operatorname{rank}(p(\hat{A}, \hat{B})).$$

Moreover, this upper bound can be approximated with help of random matrices.

Universality of free random variables: Atoms for non-commutative rational functions, Adv. Math., 2024

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Remarks

1. Same results holds for the general case of finitely many variables.
2. A, B can be normal operators in finite von Neumann algebras.
3. Polynomials can be replaced by noncommutative rational functions.
4. $\dim\ker(\lambda - p(A, UBU^*)) \geq \dim\ker(\lambda - p(\hat{A}, \hat{B})), \forall \lambda \in \mathbb{C}$.

Non-commutative probability space

Definition

A **non-commutative probability space** $(\mathcal{A}, \tau)$ consists of

- a complex unital algebra $\mathcal{A}$
- a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(1_{\mathcal{A}}) = 1$.

An element $a \in \mathcal{A}$ is called **non-commutative random variables**.
Moreover, if $\mathcal{A}$ is a $*$-algebra, then τ is addtionally required to be positive, i.e., $\tau(a^* a) \geq 0$. And we call $(\mathcal{A}, \tau)$ a **$*$-probability space**.

- $(L^\infty(\Omega), \mathbb{E})$, where $(\Omega, \mathbb{P})$ is a probability space.
- $(M_N(\mathbb{C}), \mathrm{tr}_N)$, $\mathrm{tr}_N := \frac{1}{N} \mathrm{Tr}_N$.
- $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a finite von Neumann algebra with its trace τ, called W^***-probability space**.

Analytic distributions of nc random variables

Definition

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space. For a normal random variable $a \in \mathcal{A}$ (i.e., $aa^* = a^*a$), its **analytic distribution** μ_a is the unique probability measure given by

$$\mu_a = \tau \circ E_a$$

where E_a is the projection-valued measure given by the spectral theorem for normal operators. It is also determined by moments, i.e., it is the unique measure satisfying $\tau(p(a, a^*)) = \int_{\mathbb{C}} p(z, \bar{z})\mu_a(z), \quad \forall p \in \mathbb{C}[x, x^*]$,

Atoms and von Neumann rank

For a nc random variable a, we define

$$\mathrm{rank}(a) := \tau(p_{\overline{\mathrm{im}(a)}}) \text{ and } \dim\ker(a) := \tau(p_{\ker a}),$$

where $p_{\overline{\mathrm{im}(a)}}$ is the projection onto the closure of the image of a and $p_{\ker a}$ is the projection onto the kernel of a. Then we have

$$\mathrm{rank}(a) + \dim\ker(a) = 1.$$

Recall that λ is called an **atom** of μ_a if $\mu_a(\{\lambda\}) > 0$ for a normal random variable a. Then

$$\mu_a(\{\lambda\}) = 1 - \mathrm{rank}(a - \lambda).$$

Analytic distributions of nc random variables

Example: eigenvalue distribution

Let $X \in M_N(\mathbb{C})$ be a Hermitian matrix in $(M_N(\mathbb{C}), \mathrm{tr}_N)$. Then the analytic distribution of X is given by

$$\mu_X := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where λ_i, $i = 1, \ldots, N$ are eigenvalues of X.

For $\lambda \in \mathbb{C}$,

$$\mu_X(\{\lambda\}) = \frac{1}{N} \dim \ker(X - \lambda) = 1 - \frac{1}{N} \mathrm{rank}(X - \lambda)$$

is the normalized dimension of the eigenspace corresponding to λ.

Analytic distributions of nc random variables

Definition

A unitary random variable u ($u^*u = uu^* = 1$) in some $*$-probability space $(\mathcal{A}, \tau)$ is **Haar unitary** if $\tau(u^n) = \tau((u^*)^n) = 0$, $\forall n \geq 1$.

Its analytic distribution μ_u is the Haar measure on its spectrum $\sigma(u) = \mathbb{T}$. We call u a **Haar unitary** random variable.

Example: group element

Let G be a discrete group. We consider the nc probability space $(\mathbb{C}G, \tau)$. If $g \in G$ is torsion-free, i.e., $g^n \neq e$, $\forall n \geq 1$, then δ_g is a Haar unitary random variable.

$\mathbb{C}G := \{\sum \alpha_g \delta_g | \alpha_g = 0 \text{ except finitely many } g\}$

$$\tau(\delta_g) := \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

Free independence

Definition

Let $(\mathcal{A}, \tau)$ be a nc probability space and $(\mathcal{A}_i)_{i\in I}$ a family of subalgebras of $\mathcal{A}$ containing 1. We call $(\mathcal{A}_i)_{i\in I}$ **freely independent** if for any $n \geqslant 1$,

$$\tau\left(x_{i_1}\cdots x_{i_n}\right) = 0$$

whenever we have $i_1, \ldots, i_n \in I$ s.t.

- $x_{i_k} \in \mathcal{A}_{i_k}$ with $\tau\left(x_{i_k}\right) = 0$, $k = 1, \cdots, n$,
- $i_k \neq i_{k+1}$.

In particular, we say two random variables $x, y \in \mathcal{A}$ are **freely independent** if the subalgebras generated by $\{1, x\}$ and $\{1, y\}$ are freely independent.

Free products of nc probability space

Definition (Free product)

Let $(\mathcal{A}_i, \tau_i)$ be nc probability spaces. We can construct a linear function τ on the

$$*_{i\in I}\mathcal{A}_i := \mathbb{C}1 \oplus (\bigoplus_{k=1}^{\infty} \bigoplus_{i_1\neq i_2\neq\cdots\neq i_k} \mathcal{A}_{i_1}^{\circ} \otimes \mathcal{A}_{i_2}^{\circ} \otimes \cdots \mathcal{A}_{i_k}^{\circ}),$$

where $\mathcal{A}_i^{\circ} := \ker \tau_i$ such that $\mathcal{A}_i$ are freely independent in $*_{i\in I}\mathcal{A}_i$.

Thus, a free copy of two matrices $A, B \in M_n(\mathbb{C})$ is living in $M_n(\mathbb{C}) * M_n(\mathbb{C})$ and cannot be realized as finitely dimensional matrices. The rank over $M_n(\mathbb{C}) * M_n(\mathbb{C})$ is given through $\mathrm{tr}_n * \mathrm{tr}_n$.

Free convolution

Theorem (Bercovici-Voiculescu, 98)

Let x *and* y *be two freely independen selfadjoint random variables. Then* $\mathrm{rank}(x+y) < 1$ *if and only if there exists* $\lambda \in \mathbb{C}$ *such that* $\mathrm{rank}(x-\lambda) + \mathrm{rank}(y+\lambda) < 1$. *Moreover, in such a case, we have*

$$\mathrm{rank}(x+y) = \mathrm{rank}(x-\lambda) + \mathrm{rank}(y+\lambda).$$

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Combining the argument in the beginning, we see

$$\begin{aligned}\mathrm{rank}(A+UBU^*) &\le \min\{1, \min_{\lambda\in\sigma(A)\cap\sigma(-B)} \mathrm{rank}(A-\lambda) + \mathrm{rank}(B+\lambda)\} \\ &= \mathrm{rank}(\hat{A}+\hat{B}),\end{aligned}$$

where $(\hat{A}, \hat{B})$ is a free copy of Hermitian matrices $A, B \in M_n(\mathbb{C})$.

Free independence by free Haar unitaries

Lemma

Let u be a Haar unitary random variable. Let $\{x, y\}$ be a set of random variables that is free from u. Then x, uyu^* are freely independent.

So (x, uyu^*) is a free copy of (x, y). In particular, taking x, y to be matrices A, B, we obtain a free copy $(A, uBu^*) \in (M_n(\mathbb{C}) * \mathbb{C}\mathbb{Z})^2$ of (A, B). Then our goal is to show that for any unitary matrix $U \in M_n(\mathbb{C})$,

$$\operatorname{rank}(p(A, UBU^*)) \leq \operatorname{rank}(p(A, uBu^*))$$

for any polynomial p.

Free compressions of Haar unitaries

For that purpose, we need to build our Haar unitary random variables also in the matrix form for a given $n \in \mathbb{N}$.

Lemma

Let u be a free Haar unitary random variable. For each integer n, with the help of free compressions by matrix units, we can construct $\hat{U} = (u_{ij})_{i,j=1}^{n}$ such that

1. $\hat{U}$ is a Haar unitary random variable.
2. $\hat{U}$ is free from $M_n(\mathcal{M})$ if $\mathcal{M}$ is free from $\{u_{ij}\}$.
3. u_{ij}, $i, j = 1, \ldots, n$ form an algebra isomorphic to the polynomials of n^2 formal variables with rank preserving.

(1)(2) are well-known results in free probability. (3) follows from a result by Mai-Speicher-Y., 23.

Matrix model of free random variables

Let $A, B \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{M})$. Then Item (1)(2) of our free compression lemma for Haar unitaries says that there is a Haar unitary

$$\hat{U} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix} \in M_n(\mathcal{M}).$$

that is free from A, B. Hence our goal now is show that for any unitary matrix U,

$$\operatorname{rank}(p(A, UBU^*)) \leq \operatorname{rank}(p(A, \hat{U}B\hat{U}^*))$$

for any polynomial p.

Inner rank

Definition

Let A be a $n \times n$ matrix over nc polynomials in formal variables. The **inner rank** of A, denoted by $\rho(A)$, is the minimal integer r such that the factorizations $A = PQ$ holds, where P, Q are $n \times r$ and $r \times n$ matrices respectively.

1. $\rho(A + B) \leq \rho(A) + \rho(B)$.
2. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.
3. It actually can be defined on any ring R. For any evalution map Φ from polynomials to a ring R and we denote it by ρ_R, we have
$$\rho_R(\Phi(A)) \leq \rho(A).$$
4. For a matrix $A \in M_n(\mathbb{C})$, $\rho_{\mathbb{C}}(A) = \text{rank}(A)$.

Rank decreasing along homorphisms

Let $x_{ij}, i, j = 1, \ldots, n$ be formal variables and $U = (U_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ a unitary matrix. Consider the homomorphism Φ defined by $\Phi(x_{ij}) = U_{ij}$, we have

$$\mathrm{rank}(P(U_{ij})) = \rho_{\mathbb{C}}(P(U_{ij})) \leq \rho(P)$$

for any matrix P over $x_{ij}, i, j = 1, \ldots, n$.
Then it remains to show that

$$\rho(P) \leq \mathrm{rank}(P(u_{ij})),$$

where $\hat{U} = (u_{ij})_{i,j=1}^n$ is built out of the free compression of a Haar unitary. Actually Item (3) of our compression lemma claims that

$$\rho(P) = \mathrm{rank}(P(u_{ij})).$$

Summary: universality of free variables

Let $A, B \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{M})$. Then for the Haar unitary

$$\hat{U} = (u_{ij})_{i,j=1}^{n} \in M_n(\mathcal{M}).$$

that is free from A, B, through a matrix $\bar{U}$ of formal variables $x_{ij}, i, j = 1, \ldots, n$

$$\bar{U} = (x_{ij})_{i,j=1}^{n},$$

We have

$$\mathrm{rank}(p(A, UBU^*)) \leq \rho(p(A, \bar{U}B\bar{U}^{-1})) = \mathrm{rank}(p(A, \hat{U}B\hat{U}^*))$$

for any polynomial p.

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for any polynomial p.
Caution: $\bar{U}^{-1}$ actually breaks down the homorphism but a linearization trick can save the rank inequality.

Asymptotic freeness of random matrices

Theorem

Let U_N be a Haar unitary random matrices, i.e., matrix-valued random variable sampled according to the Haar measure on $N \times N$ unitary matrices. Let X_N, Y_N be deterministic matrices such that $\{\mu_{X_N}\}_{N\geq 1}, \{\mu_{Y_N}\}_{N\geq 1}$ converge in distribution to probability measures μ_x, μ_y respectively. Then $Y_N := U_N X_N U_N^*$ is asymptotic free from X_N. That is, for freely independent random variables $x, y \in (\mathcal{M}, \tau)$ with analytic distributions μ_x, μ_y, we have

$$\lim_{N\to\infty} \mathrm{tr}_n(p(X_N, U_N Y_N U_N^*)) = \tau(p(x, y)),$$

for any polynomial p.

In particular, we see that for two fixed matrices A, B in $M_n(\mathbb{C})$, we have $(X_N, Y_N) := (A \otimes I_N, U_{Nn}(B \otimes I_N)U_{Nn}^*)$ converge in distribution towards a free copy $(\hat{A}, \hat{B})$ of (A, B), where U_{Nn} is a sequence of Haar unitary random matrices.
It follows that for any polynomial p and

$$P := \begin{pmatrix} 0 & p \\ p^* & 0 \end{pmatrix}$$

$\mu_{P(X_m, Y_m)}$ converges weakly towards $\mu_{P(\hat{A}, \hat{B})}$. Thanks to Portmanteau theorem, we conclude

$$\limsup_{m \to \infty} \mu_{P(X_m, Y_m)}(\{\lambda\}) \leq \mu_{P(\hat{A}, \hat{B})}(\{\lambda\}), \quad \forall \lambda \in \mathbb{C}.$$

This implies that

$$\lim_{m \to \infty} \mu_{P(X_m, Y_m)}(\{\lambda\}) = \mu_{P(\hat{A}, \hat{B})}(\{\lambda\}), \quad \forall \lambda \in \mathbb{C}$$

and

$$\lim_{m \to \infty} \operatorname{rank}(p(X_m, Y_m)) = \operatorname{rank}(p(\hat{A}, \hat{B}).$$

Dimensionless optimal rank upper bound

Theorem (Arizmendi-Cébron-Speicher-Y, 24)

Let A, B *be two matrices in* $M_n(\mathbb{C})$. *We consider the set of matrices*

$$\chi = \amalg_{N=1}^{\infty} \left\{ (X, Y) \in M_N(\mathbb{C})^2 \;\middle|\; \begin{array}{l} \frac{1}{n}\operatorname{rank}(\lambda - A) = \frac{1}{N}\operatorname{rank}(\lambda - X), \\ \frac{1}{n}\operatorname{rank}(\lambda - B) = \frac{1}{N}\operatorname{rank}(\lambda - Y) \end{array} \right\}.$$

Then for any polynomial $p \in \mathbb{C}\langle x, y\rangle$

$$\sup_{X,Y\in\chi} \frac{1}{N}\operatorname{rank} p(X, Y) = \frac{1}{n}\operatorname{rank}(p(\hat{A}, \hat{B})) = \frac{1}{n}\rho(A, \bar{U}B\bar{U}^{-1}),$$

where $\hat{A}, \hat{B}$ *are freely independent copies of* A, B *and* $\bar{U} = (x_{ij})_{i,j=1}^{n}$ *is a matrix over polynomials in* $x_{ij}, i, j = 1, \ldots, n$.

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where $\hat{A}, \hat{B}$ are freely independent copies of A, B and $\bar{U} = (x_{ij})_{i,j=1}^{n}$ is a matrix over polynomials in x_{ij}, $i, j = 1, \ldots, n$.

The upper bound is not optimal for a fixed dimension. For example, $p = (xy - yx)^2 z - z(xy - yx)^2$ is vanishing on 2×2 matrices but $\sup_{X,Y,Z\in\chi} \frac{1}{N}\operatorname{rank}(p(X, Y, Z)) > 0$ for some matrices A, B, C.

Examples

Theorem (Arizmendi-Cébron-Speicher-Y, 24)

Let A, B be two matrices. Then we have

- $\mathrm{rank}(A+B) \leq \min\{n, \min_{\lambda\in\sigma(A)\cap\sigma(-B)}(\mathrm{rank}(A-\lambda)+\mathrm{rank}(B+\lambda))\}$;
- $\mathrm{rank}(AB-BA) \leq \min\{n, \min_{\lambda\in\mathbb{C}} 2\,\mathrm{rank}(A-\lambda), \min_{\lambda\in\mathbb{C}} 2\,\mathrm{rank}(B-\lambda)\}$;
- $\mathrm{rank}(AB+BA) \leq \min\{n, 2\,\mathrm{rank}(A), 2\,\mathrm{rank}(B), \mathrm{rank}(A)+\min_{\lambda\neq 0}\mathrm{rank}(B-\lambda), \min_{\lambda\neq 0}\mathrm{rank}(A-\lambda)+\mathrm{rank}(B)\}$

where the quantities on right hand side come from $\mathrm{rank}(\hat{A}+\hat{B})$, $\mathrm{rank}(\hat{A}\hat{B}-\hat{B}\hat{A})$, $\mathrm{rank}(\hat{A}\hat{B}+\hat{B}\hat{A})$ *for a free copy* $(\hat{A},\hat{B})$ *of* (A,B).

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Thank you!