Classification and counting of multi-orientable tensor models with a mixed $\mathrm{U}(N)$ and $\mathrm{O}(D)$ symmetry

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Noncommutative Geometry Meets Topological Recursion

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introducing random tensor models

Random tensor models and tensor field theories

- Consider a field theory defined by a complex field $\phi : G^d \to \mathbb{C}$, where G is a compact Lie group admitting Peter-Weyl decomposition.
- The Fourier transform of ϕ yields an order-d complex tensor $\phi_{\mathbf{P}}$[1], with $\mathbf{P} = (p_1, p_2, \ldots, p_d)$ a multi-index, where $p_1, p_2, \ldots, p_d \in \mathbb{Z}$.
- $\bar{\phi}_{\mathbf{P}}$ denotes its complex conjugate.

The partition function is $\quad \mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\; e^{-(S^{\text{kinetic}}[\bar{\phi},\phi] + S^{\text{interaction}}[\bar{\phi},\phi])}\,,$

where the action is given by convolutions of tensors

$$\begin{aligned} S^{\text{kinetic}}[\bar{\phi},\phi] &= \mathrm{Tr}_2(\bar{\phi} \cdot K \cdot \phi) \\ &= \sum_{\mathbf{P},\,\mathbf{P}'} \bar{\phi}_{\mathbf{P}}\, K(\mathbf{P};\mathbf{P}')\, \phi_{\mathbf{P}'}\,, \end{aligned}$$

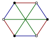
where $\mathrm{Tr}_{n_{\mathcal{B}}}$ are sums over all indices p_s of $\mathbf{P}$ on $n_{\mathcal{B}}$ tensors ϕ and $\bar{\phi}$.

[1] Considering $\phi_{\mathbf{P}}$ as a tensor is a slight abuse because the modes p_s range up to infinity. We cut off at N, then $\phi_{\mathbf{P}}$ is a tensor.

Random tensor models and tensor field theories

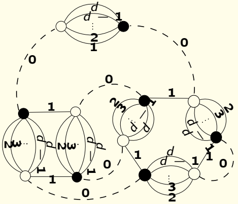
$$S^{\text{kinetic}}[\phi,\bar{\phi}] = \mathrm{Tr}_2(\bar{\phi}\cdot K\cdot \phi)$$

$$S^{\text{interaction}}[\phi,\bar{\phi}] = \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\, \mathrm{Tr}_{2n_{\mathcal{B}}}(\bar{\phi}^{n_{\mathcal{B}}}\cdot \mathcal{V}_{\mathcal{B}}\cdot \phi^{n_{\mathcal{B}}})$$

$\stackrel{d=3}{=}$ $\lambda_2^{(3)}$  $+\,\lambda_4^{(3)}$  $+\,\lambda_{6,1}^{(3)}$  $+\,\lambda_{6,2}^{(3)}$  $+\,\lambda_{6,3}^{(3)}$  $+\cdots$

$\stackrel{d=4}{=}$ $\lambda_2^{(4)}$  $+\,\lambda_{4,1}^{(4)}$  $+\,\lambda_{4,2}^{(4)}$  $+\,\lambda_{6,1}^{(4)}$  $+\,\lambda_{6,2}^{(4)}$  $+\,\lambda_{6,3}^{(4)}$  $+\cdots$

After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs, e.g.,



Remark

- *If $K(\mathbf{P};\mathbf{P}') = \delta_{\mathbf{P};\mathbf{P}'}$ (trivial delta function), then this model is a statistical model i.e., tensor models.*
- *Otherwise, if nontrivial propagator e.g., $K(\mathbf{P};\mathbf{P}') = \delta_{\mathbf{P};\mathbf{P}'}\mathbf{P}^{2b}$, then this is a QFT, i.e., tensor field theories (generalisation of Grosse-Wulkenhaar model).*

Random tensor models and tensor field theories

- After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs.
- $(d+1)$-edge-colored graphs (also, called graph encoding manifolds (GEM)) are dual to simplicial triangulations of piecewise linear (PL) d-dimensional pseudo-manifolds.
  [Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986]
- In other words, tensor models generate discrete (pseudo-)manifolds, and the path integral formulation provides us a way to sum over all of them.

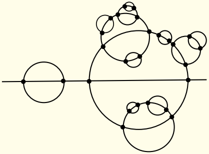
Relevant for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$.

- Encouragingly, the lower dimensional counterpart ($d = 2$), matrix models generate the Brownian sphere at criticality and are rigorously proven to be equivalent to 2-dimensional Liouville quantum gravity.
  [Le Gall, Miermont 2011; Miller, Sheffield 2015]

Promising for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$!

Tensor models

Melons dominate in the large N (size of tensors) limit.
[Gurau Rivasseau 2011]

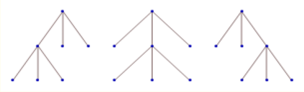


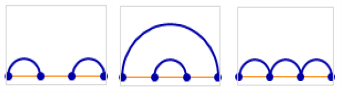
The melonic 2-point function admits the following expansion

$$G_{\mathrm{melonic}}(t) = \sum_{n=0}^{\infty} t^n \, FC_n^{(d+1)} \,, \qquad FC_n^{(d+1)} = \frac{1}{(d+1)n+1} \binom{(d+1)n+1}{n} .$$

Fuss-Catalan numbers $FC_n^{(d+1)}$ ($d = 1$ is Catalan) correspond to

- the numbers of planar $(d+1)$-ary trees with n vertices and with $dn+1$ leaves.
- the numbers of non-crossing partitions of the set $\{1, 2, \cdots, dn\}$ that contain only subsets of size d.
- etc.



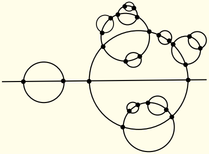


$(d = 2,\ n = 2)$

Tensor models

Melons dominate and they are branched polymers.

[Bonzom, Gurau, Riello, Rivasseau, "*Critical behavior of colored tensor models in the large N limit,*" Nucl. Phys. B 853, 174 (2011)]



[Gurau, Ryan, "*Melons are branched polymers,*" Annales Henri Poincare 15, no. 11, 2085 (2014).]

Corresponding to a branched polymer (tree-like) geometric structure, melons are undesirable in the (random geometric) quantum gravity context.

$\sim$

Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$\mathcal{Z}_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\; e^{-S_{\text{gravity}}} \quad \to \quad \sum_{\text{random triangulations}} e^{-S_{\text{discretised gravity}}}$$

$d = 2$: matrix models

- Large N expansion of partition function is controlled by a topological invariant (genus g),

$$\mathcal{Z} = \sum_g N^{2-2g} Z_g, \quad \text{where } \; Z_g \sim |\lambda - \lambda_c|^{(2-\gamma)(2-2g)/2}\, f_g\,.$$

  (for $g = 0$ sphere, $Z_{g=0} \sim |\lambda - \lambda_c|^{2-\gamma}$, where $\gamma = -1/2$ (“Brownian sphere"))

- A clear relation between topology and the critical behavior in the double scaling limit ($N \to \infty$, $\lambda \to \lambda_c$ while κ being constant),

$$\mathcal{Z} = \sum_g \kappa^{2g-2} f_g, \quad \text{where } \; \kappa^{-1} = N|\lambda - \lambda_c|^{(2-\gamma)/2}\,.$$

Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$\mathcal{Z}_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\; e^{-S_{\text{gravity}}} \quad \to \quad \sum_{\text{random triangulations}} e^{-S_{\text{discretised gravity}}}$$

$d > 2$: tensor models

- Large N limit is controlled by Gurau degree ω, having strong influence from combinatorics and encoding geometrical information via the number of bicolor cycles (faces), $F(\mathcal{G})$,

$$\mathcal{Z} = \sum_{\omega} N^{d-\frac{2}{(d-1)!}\omega} Z_{\omega}, \quad \text{where } \omega(\mathcal{G}) = \frac{(d-1)!}{2}\Big(\frac{d(d-1)}{2} p(\mathcal{G}) + d - F(\mathcal{G})\Big)$$
$$= \sum_{\text{jackets}, \mathcal{J}(\mathcal{G})} g_{\mathcal{J}(\mathcal{G})} \geq 0\,.$$

- Then, $\omega(\mathcal{G}) = 0$ (subclass of sphere, "melons") dominate.
  For melons, $Z_{\omega=0} \sim |\lambda - \lambda_c|^{2-\gamma}$, where $\gamma = 1/2$ (“branched polymer").
- A systematic analysis of subleading contributions ($\omega(\mathcal{G}) > 0$) turns out to be hard. (how to go beyond melons ?)

$\sim$

Outline

- Random tensor models
  - with $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ symmetry
    - critical limits of double and triple scalings
    - classification of the Feynman graphs
  - with $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ symmetry
    - counting of invariants

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor (multi-matrix, multi-orientable) model

multi-scaling limits
and
classification of its Feynman graphs

(with Dario Benedetti, Sylvain Carrozza, Guillaume Valette)
[Ann. Inst. H. Poincare D Comb. Phys. Interact. 9 (2022) 2]

(with Remi Avohou, Matthias Vancraeynest)
[arXiv:2310.13789]

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model

$$\begin{aligned} \mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D) \text{ tensor} \quad &= \quad N \times N \times D \text{ tensor of order-3} \\ &= \quad \text{a collection of } D \text{ matrices of size } N \times N \\ (\sim \quad & \text{affinity to matrix models}) \end{aligned}$$

Therefore, denote this tensor as a vector of complex matrices of size $N \times N$,

$$X_\mu \in M_N(\mathbb{C}), \qquad 1 \le \mu \le D,$$

and in index notation

$$(X_\mu)_{ab}, \quad 1 \le a, b \le N\,,$$

with mixed symmetry $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$: $\quad X_\mu \to X'_\mu = O_{\mu\mu'} U_{(L)} X_{\mu'} U^\dagger_{(R)}$, where $O \in \mathrm{O}(D)$, $U_{(L)}, U_{(R)} \in \mathrm{U}(N)$ (independent unitary matrices in <u>distinct</u> $\mathrm{U}(N)$).

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model admits an expansion in **two parameters** owing to the presence of N and D (g and ℓ) and yields a more **refined classification** of Feynman graphs generated by the model.

$\to$ possibility of finding a new universality class.

Some background

- A tensor model with $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ symmetry shares similar properties with the holographic SYK model with complex fermions at large N, which can be potentially useful in understanding black holes.

  [Klebanov, Tarnopolsky, Phys. Rev. D 95 (2017) no.4, 046004]

- D-number of matrices of size $N \times N$ can be interpreted as transverse excitations of strings, $(X_\mu)_{ab}$, where a, b is associated with $\mathrm{U}(N)$ and the transverse directions μ to the branes is with $\mathrm{O}(D)$.

  String theory interprets the large D limit of the $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model as the limit of large spacetime dimension d in general relativity, where important features of classical black holes may be kept.

  [Emparan, Suzuki, Tanabe, JHEP, 06, (2013), 009.]

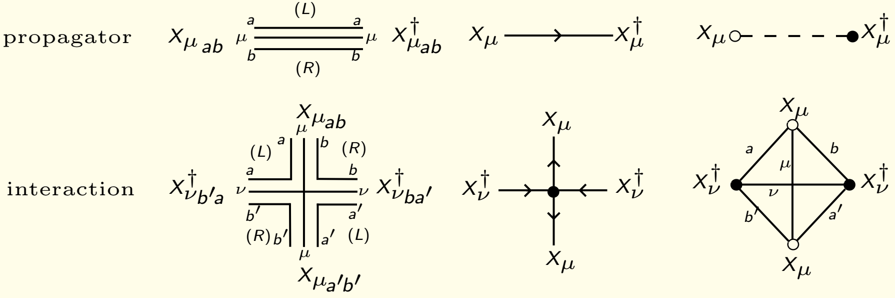
Our model

Free energy

$$\mathcal{F}(\lambda) = \log \int [dX] e^{-S[X,X^\dagger]}\,, \quad \text{where} \quad [dX] = \prod_{\mu,a,b} d\mathrm{Re}(X_\mu)_{ab}\, d\mathrm{Im}(X_\mu)_{ab}\,,$$

$$\text{and} \quad S[X,X^\dagger] = N\,D\Big(\sum_{\mu=1}^{D} \mathrm{Tr}[X_\mu X_\mu^\dagger] - \frac{\lambda}{2}\sqrt{D}\sum_{\mu,\nu=1}^{D} \mathrm{Tr}[X_\mu X_\nu^\dagger X_\mu X_\nu^\dagger]\Big)\,.$$

Perturbative expansion in λ admits a graphical representation in terms of Feynman graphs.



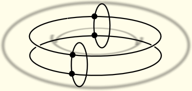
stranded rep. Feynman graph rep. colored rep.

Free energy

$$\to \mathcal{F}(\lambda) = \sum_{g\in\mathbb{N}} N^{2-2g} \sum_{\ell\in\mathbb{N}} D^{1+g-\ell/2} \mathcal{F}_{g,\ell}(\lambda) \qquad \text{[Ferrari 2017]}$$

- genus $g \equiv g_{LR}$ ($\mathrm{U}(N)^{\otimes 2}$ ribbon graph)
- grade ℓ: $\frac{\ell}{2} = g_{OR} + g_{OL} = 1 + g + \frac{v}{2} - \varphi$,

  where φ is the number of $\mathrm{O}(D)$ faces/loops,
  v is the number of vertices.

e.g., $\ell = 0$, $g = 1$ graph 

$\to$ If $D = N$, $\mathcal{F}(\lambda) = \sum_{\omega\in\frac{\mathbb{N}}{2}} N^{3-\omega} \mathcal{F}_\omega(\lambda)$, with $\omega = g + \frac{\ell}{2}$ (Gurau degree).

$\to$ reorganise further, and $\mathcal{F}(\lambda) = D^2 \sum_{g\in\mathbb{N}} \left(\frac{N}{\sqrt{D}}\right)^{2-2g} \sum_{\ell\in\mathbb{N}} D^{-\frac{\ell}{2}} \mathcal{F}_{g,\ell}$,

where $\frac{N}{\sqrt{D}} =: M$ (double scaling parameter).

Free energy

$\rightarrow$ reorganise further, and $\mathcal{F}(\lambda) = D^2 \sum_{g\in\mathbb{N}} \left(\frac{N}{\sqrt{D}}\right)^{2-2g} \sum_{\ell\in\mathbb{N}} D^{-\frac{\ell}{2}} \mathcal{F}_{g,\ell}$,

where $\frac{N}{\sqrt{D}} =: M$ (double scaling parameter).

Take a double scaling limit to collect only $\ell = 0$;

$$\lim_{\substack{N\to\infty \\ D\to\infty \\ M=\frac{N}{\sqrt{D}}\ \textbf{finite}}} \frac{1}{D^2}\mathcal{F}(\lambda) = \sum_{g\in\mathbb{N}} M^{2-2g}\mathcal{F}_{g,\ell=0} \equiv \mathcal{F}^{(0)}(M,\lambda)$$

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2]
  - characterisation of $\ell = 0$ graphs, any g.
  - critical behavior of $\mathcal{F}_{g,\ell=0}$
  - triple scaling limit

Advertisement of the results

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433]
  - characterisation of $\ell = 0$ graphs, any g.
  - critical behavior of $\mathcal{F}_{g,\ell=0}$
  - triple scaling limit

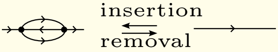
$$\left.\begin{array}{l} \lambda \to \lambda_c \\ M = \frac{N}{\sqrt{D}} \to \infty \\ \text{keep } \kappa^{-1} := M(\lambda - \lambda_c)^{2/b} \text{ fixed} \end{array}\right\}$$

$$\lim_{\substack{M \to \infty \\ \lambda \to \lambda_c \\ \kappa^{-1} := M(\lambda - \lambda_c)^{2/b} \text{ fixed}}} \frac{1}{M^2(\lambda - \lambda_c)^{2-a}} \mathcal{F}^{(0)}(M, \lambda) = \sum_{g \in \mathbb{N}} \kappa^{2g} f_g \quad \text{can be resummed.}$$

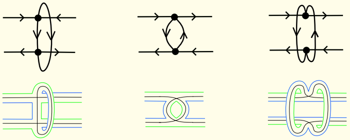
- [Avohou, Toriumi, Vancraeynest, [arXiv: 2310.13789[math-ph]]]
  (motivated by topological recursion)
  - characterisation of $\ell = 1, 2, (3)$ graphs, for any g.
  - characterisation of any ℓ, $g = 0$ graphs with one $\mathrm{O}(D)$ face ($\varphi = 1$).

key graph theoretical objects

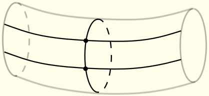
-Melon:



-Dipoles:

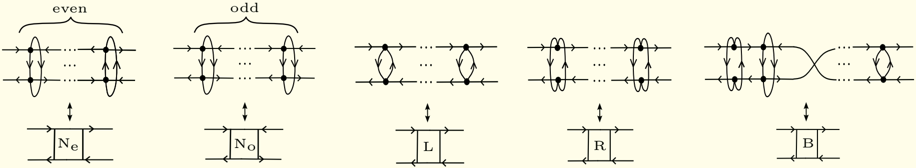


N—dipole L—dipole R—dipole



N—dipole

-Ladders:



N_e—ladder N_o—ladder L—ladder R—ladder B—ladder

-Scheme: equivalent classes of Feynman graphs, up to insertion/deletion of infinite families of subgraphs, leaving ℓ and g invariant. [Gurau Shaeffer 2013; Fusy Tanasa 2015]

central tools and ideas [Avohou, Toriumi, Vancraeynest, [arXiv:2310.13789]]

Lemma

- *Each Feynman graph with $\ell = 1$ and $g = 0$ has at least one $\mathrm{O}(D)$-loop with length 2.*
- *For each Feynman graph with $\ell = 0, 1, 2, 3$, with any $g \geq 1$, it always contains at least one $\mathrm{O}(D)$-loop of length 2.*

Corollary

For any Feynman graph (and its corresponding scheme) of $\ell = 0, 1, 2, 3$ with $g \geq 1$, there exists an N-dipole.

$\rightarrow$ allows **recursive construction** of $\ell = 0, 1, 2, 3$ order by order in g.

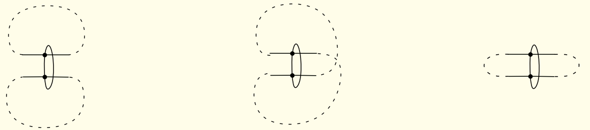
i.e., can construct higher genus graphs from lower genus graphs.

Theorem

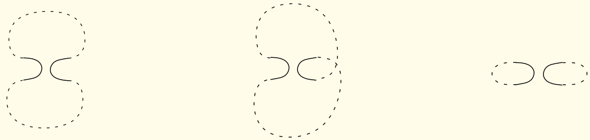
Any $\ell = 0$ resp. $1, 2, 3$ scheme of genus g can be reconstructed from $\ell = 0$, resp. $1, 2, 3$ schemes of genus $g' < g$.

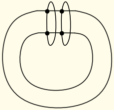
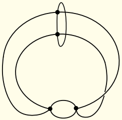
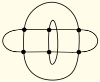
key tools for recursive construction of graphs

- insertion and removal of non-separating N-dipoles



$\Delta\ell = 0$, $\Delta g = 1$ $\uparrow\downarrow$ $\Delta\ell = 0$, $\Delta g = -1$ | $\Delta\ell = 2$, $\Delta g = 1$ $\uparrow\downarrow$ $\Delta\ell = -2$, $\Delta g = -1$ | $\Delta\ell = 4$, $\Delta g = 1$ $\uparrow\downarrow$ $\Delta\ell = -4$, $\Delta g = -1$

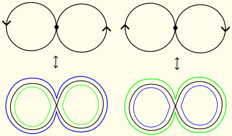


e.g.,  e.g.,  e.g., 

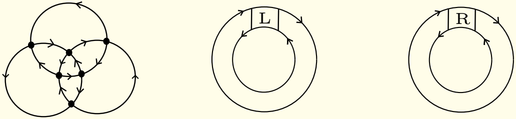
2PI $g = 0$ (planar) graphs [Avohou, Toriumi, Vancraeynest, [arXiv:2310.13789]]

- $\ell = 0,\ g = 0$

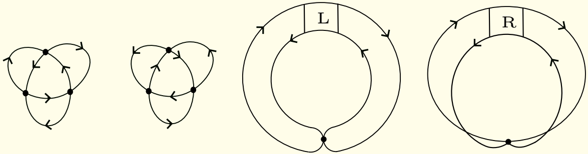
- $\ell = 1,\ g = 0$

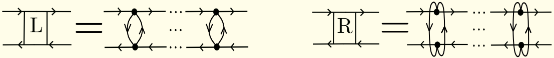


- $\ell = 2,\ g = 0$



- $\ell = 3,\ g = 0$





higher ℓ graphs (planar $g = 0$)

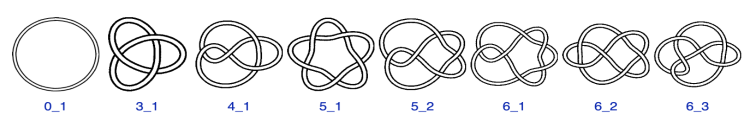
Q. What about higher ℓ graphs?

A. Use 1-1 correspondence between 4-regular planar diagrams and alternating knot diagrams.

For $\ell \geq 4$ but still $g = 0$ (planar),

Theorem ([Avohou, Toriumi, Vancraeynest, [arXiv: 2310.13789[math-ph]]])

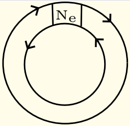
Each 2PI (except the infinity graph) 4-regular planar graph with $\varphi = 1$ **(one $\mathrm{O}(D)$-face/loop)** *and any ℓ, ignoring the orientation assignment on the edges, is in one-to-one correspondance with reduced alternating knot diagrams with ℓ crossings which are*
1) projections of the prime knots as listed in the Rolfsen knot table, or
2) obtained after performing the Tait flyping moves.
Furthermore, we can correspond each 2PR 4-regular planar graphs with $\varphi = 1$ and any ℓ to an alternating knot diagram obtained by performing a connected sum or a Reidemeister move I on the reduced alternating knot diagrams referred above.

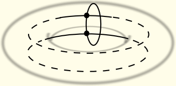


etc.

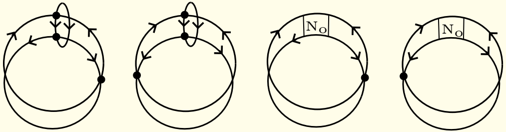
2PI g=1 graphs

- $\ell = 0$, $g = 1$

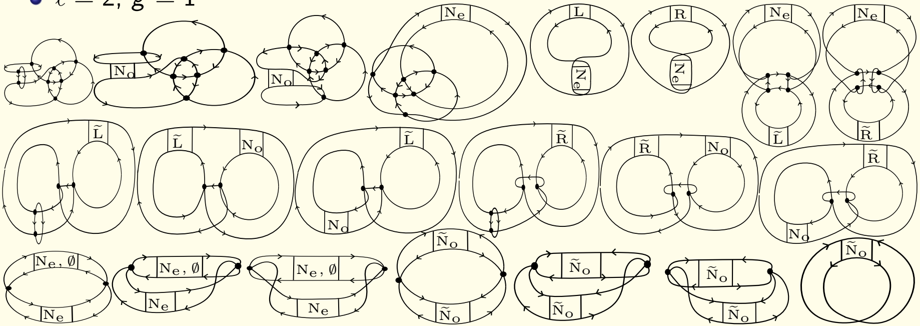




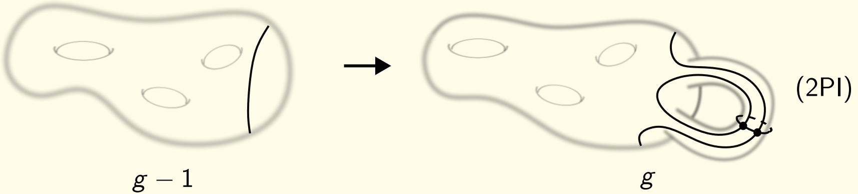
- $\ell = 1$, $g = 1$

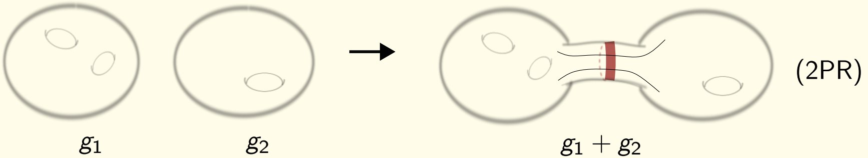


- $\ell = 2$, $g = 1$



topological picture



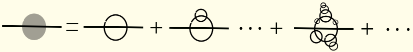


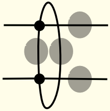
critical behavior of double scaling limit

To resum $\displaystyle\sum_{\substack{\ell=0 \text{ schemes}, S \\ \text{(finite sum)}}} \mathcal{F}_S$ is still too hard (double scaling limit).

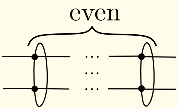
Focus on the subclass of schemes ("**dominant schemes**") that contribute to the dominant singularity, which arises from resumming of melons and ladders.

$$G_{\mathrm{melon}}(\lambda) \sim \frac{1}{3}\left(4 - \sqrt{\frac{8}{3}}\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}\right), \quad \lambda_c = (3^3/4^4)^{1/2}$$



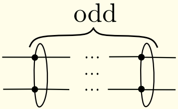


$\longleftrightarrow$ $\lambda^2 G^4_{\mathrm{melon}}(\lambda) =: u$



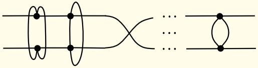
N$_\mathrm{e}$–ladder

$\frac{u^2}{1-u^2}$



N$_\mathrm{o}$–ladder

$\frac{u^3}{1-u^2}$



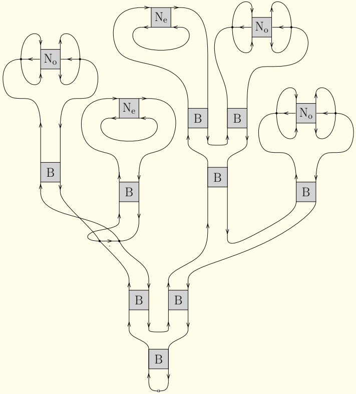
B–ladder

$\frac{6u^2}{(1-3u)(1-u)}$

critical behavior of double scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2])

Dominant schemes *of $\ell = 0$, genus g have $2g-1$ B-ladder vertices, and in one-to-one correspondence with (decorated)* **plane binary trees**.



A dominant scheme of genus $g = 5$. It has the structure of a rooted binary tree with: $g = 5$ leaves, $g - 1 = 4$ inner vertices, and $2g - 1 = 9$ edges.

triple scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433])

Dominant schemes *of* $\ell = 0$, *genus* g *have* $2g-1$ *B-ladder vertices, and in one-to-one correspondence with (decorated)* **plane binary trees**.

- can count with $(g-1)^{\text{th}}$ Catalan numbers. $C_g = \frac{1}{2g-1}\binom{2g-1}{g-1}$.

$$\mathcal{F}_{g,\ell=0} \sim C_g \left(\frac{5}{48}\right)^g \left(\sqrt{1-\frac{\lambda^2}{\lambda_c^2}}\right)^{1-2g} .$$

- can resum

$$\to \sum_g \mathcal{F}_{g,\ell=0} \quad \sim \quad \frac{1}{2}\left(1-\sqrt{1-4\frac{5}{48}\kappa^2}\right), \quad \text{where} \quad \kappa = \left(M\sqrt{1-\frac{\lambda^2}{\lambda_c^2}}\right)^{-1} .$$

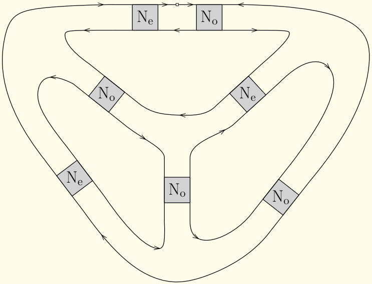
$$\langle g \rangle = \frac{1}{2}\kappa\partial_\kappa \ln\left(\sum_g \mathcal{F}_{g,\ell=0}\right) = \frac{1}{2\sqrt{1-\frac{\kappa^2}{\kappa_c^2}}} \;\to\; \text{diverges}, \quad \kappa_c = \sqrt{\frac{12}{5}} .$$

Large random trees (2PR) dominate, representing surfaces with large g.

critical behavior of double scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2])

2PI dominant schemes *of $\ell = 0$, genus g have $3g - 2$ N-ladder vertices, and are in one-to-one correspondence with (decorated) cubic, and bridgeless* **planar maps**.



A 2PI-dominant scheme of $\ell = 0$, genus $g = 3$. It has $3g - 2 = 7$ N-vertices.

triple scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433])

2PI dominant schemes *of $\ell = 0$, genus g have $3g-2$ N-ladder vertices, and are in one-to-one correspondence with (decorated) cubic, and bridgeless* **planar maps**.

$$\sum_g \mathcal{F}_g \sim \left(1 - \frac{\kappa^2}{\kappa_c^2}\right)^{3/2}, \quad \kappa^{-1} = M(1-\lambda)^{3/2}, \quad \kappa_c = \sqrt{\frac{2^5}{3^3}}$$

$$\langle g \rangle \sim \left(1 - \frac{\kappa^2}{\kappa_c^2}\right)^{1/2} \kappa \quad = \quad \text{finite.}$$

Conclusions so far, for $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model

We studied

- "double" scaling limit ($\rightarrow$ organisation in ℓ)
  - $N \rightarrow \infty$ (size of matrices),
  - $D \rightarrow \infty$ (number of matrices),
  - keeping $\frac{N}{\sqrt{D}} =: M$ fixed and finite.
- "triple" scaling limit
  - $M \rightarrow \infty$ (i.e., $N \gg D$),
  - $\lambda \rightarrow \lambda_c$,
  - keeping $M(\lambda_c - \lambda)^{\alpha}$ fixed and finite.

and obtained

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2]
  - characterisation of $\ell = 0$ graphs, any g.
  - in the triple scaling limit (continuum limit),
    - trees (2PR).
    - 2D quantum gravity (2PI).
- [Avohou, Toriumi, Vancraeynest, [arXiv: 2310.13789[math-ph]]]
  - characterisation of $\ell = 1, 2, (3)$ graphs, for any g.
  - characterisation of any ℓ, $g = 0$ graphs with one $\mathrm{O}(D)$ face ($\varphi = 1$).

$\rightarrow$ topological recursion?

$\sim$

Enumeration of $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

(with Rémi Cocou Avohou, Joseph Ben Geloun)

Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404 [hep-th]]

Summary of the main results

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

We enumerated $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants using group theoretic formulas.

- Our enumerations unveiled a wide array of novel integer sequences that have not been previously known.
- For a general order (r, q), the counting can be interpreted as the partition function of a topological quantum field theory (TQFT) with the symmetric group as the gauge group. We identified the 2-complex pertaining to the enumeration of the invariants, which in turn defines the TQFT, and establish a correspondence with countings associated with covers of diverse topologies, in general with branched points.
- At order $(r, q) = (1, 1)$, the numbers of invariants corresponds to the numbers of certain cicular words with pattern avoidance, offering insights into enumerative combinatorics and potentially to linguistics.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

Consider

- A tensor T transforms under the action of the fundamental representation of the Lie group $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$.

$$T_{a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_q} \to U^{(1)}_{a_1 c_1} U^{(2)}_{a_2 c_2} \ldots U^{(r)}_{a_r c_r} O^{(1)}_{b_1 d_1} O^{(2)}_{b_2 d_2} \ldots O^{(q)}_{b_q d_q} T_{c_1,c_2,\ldots,c_r,d_1,d_2,\ldots,d_q}\,.$$

- A $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$ invariant (UO-invariant) is constructed by contractions of complex tensors of order $r+q$ (of a given number, n, of tensors T and the same number of complex conjugate $\bar{T}$.)

  $\rightarrow$ Therefore, UO invariants are tensor model observables.

- An UO-invariant is algebraically denoted

$$\mathrm{Tr}_{K_n}(T,\bar{T}) = \sum_{a^i_k, b^i_k, a'^i_k, b'^i_k} K_n(\{a^i_k, b^i_k\};\{a'^i_k, b'^i_k\}) \prod_{i=1}^{n} T_{a^i_1,a^i_2,\ldots,a^i_r,b^i_1,b^i_2,\ldots,b^i_q} \bar{T}_{a'^i_1,a'^i_2,\ldots,a'^i_r,b'^i_1,b'^i_2,\ldots,b'^i_q}\,.$$

K_n is a kernel composed of a product of Kronecker delta functions that match the indices of n copies of T's and those of n copies of $\bar{T}$'s. A given tensor contraction dictates the pattern of an edge-colored graph, which can, in turn, be used to label the invariant.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

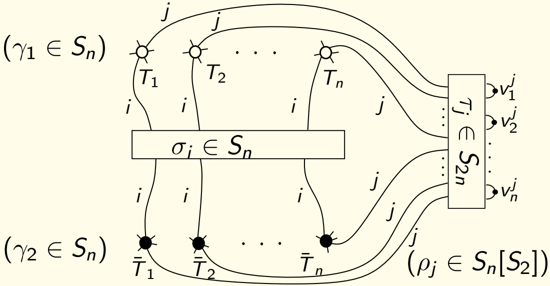


Diagram of contraction of n tensors T and n tensors $\bar{T}$. For a given color $i = 1, 2, \ldots, r$, σ_i represents the contraction in the unitary sector and, for any color $j = 1, 2, \ldots, q$, τ_j represents the contraction in the orthogonal sector.

Consider $(r, q) = (3, 3)$. An UO-invariant is defined by a $(3+3)$-tuple of permutations $(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ from the product space $(S_n)^{\times 3} \times (S_{2n})^{\times 3}$.

We will remove the vertex labeling (two configurations are equivalent if their resulting unlabeled graphs coincide), which introduces more permutations $\gamma_1, \gamma_2 \in S_n$, and $\varrho_1, \varrho_2, \varrho_3 \in S_n[S_2]$ the so-called wreath product subgroup of S_{2n}.

The equivalence relation is

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \sim (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3)$$

Counting UO tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

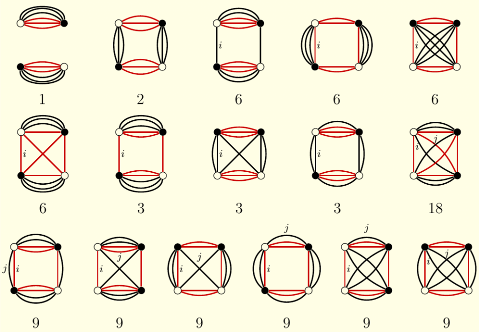
Therefore the counting of UO invariants of order (r, q) is

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q}$$
$$\sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\varrho_1,\ldots,\varrho_q \in S_n[S_2]} \sum_{\substack{\sigma_1,\ldots,\sigma_r \in S_n \\ \tau_1,\ldots,\tau_q \in S_{2n}}} \Big[\prod_{i=1}^{r} \delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\Big]\Big[\prod_{i=1}^{q} \delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\Big] .$$

example: $\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

$\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants are enumerated in the increasing number of tensors: $1, \color{magenta}{108}, 20385, 27911497, 101270263373, 808737763302769, \ldots$



UO-invariant graphs at order $(r, q) = (3, 3)$ with 4 tensors $(n = 2)$. The integer below each graph enumerates various possibilities based on index colors, summing to 108 for all configurations. Black edges are in the U-sector, and red are in the O-sector.

TQFT (lattice gauge theories)

- On a **cellular complex** X, we can define a partition function for a finite **group** G by assigning a group element g_e to each edge and to each **plaquette** P a weight $w_P\left(\prod_{e\in P} g_e\right)$. The partition function of this **lattice gauge theory** is

$$Z[X;G] = \frac{1}{|G|^V} \sum_{g_e} \prod_P w_P\left(\prod_{e\in P} g_e\right),$$

  with V the number of vertices in the cell decomposition.

- The theory is **topological** because it is invariant under refinement of the cellular decomposition.
- When $G = S_n$ (symmetric group or permutation group), it has applications to QFT combinatorics. [Ben Geloun, Ramgoolam, Ann. Inst. H. Poincare Comb. Phys. Interact. 1 (2014) 1]
- The partition function for a topological space X counts equivalence classes of homomorphisms from $\pi_1(X)$ to S_n, i.e., counts equivalence classes of covering spaces of X of degree n counted with a certain weight.

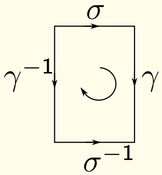
an example of permutation TQFT

e.g., Consider the torus realised as a rectangle.

- The partition function of this lattice gauge theory is given by

$$Z(T^2; S_n) = \frac{1}{n!} \sum_{\sigma,\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1})\,.$$

- $\delta(\gamma\sigma\gamma^{-1}\sigma^{-1})$ or $\gamma\sigma\gamma^{-1}\sigma^{-1} = \mathrm{id}$ is represented by the torus and γ and σ are the generators of the fundamental group of the torus.



- $Z(T^2; S_n)$ counts n-fold covers of the torus.

permutation TQFT for UO tensor invariants

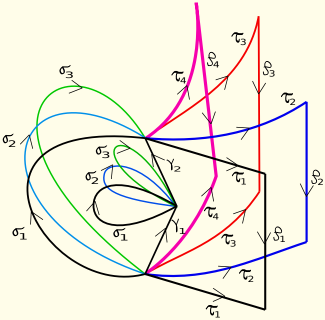
[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

Recall the counting of UO invariants of order (r, q)

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q}$$

$$\sum_{\gamma_1,\gamma_2\in S_n} \sum_{\varrho_1,\dots,\varrho_q\in S_n[S_2]} \sum_{\substack{\sigma_1,\dots,\sigma_r\in S_n \\ \tau_1,\dots,\tau_q\in S_{2n}}} \left[\prod_{i=1}^{r} \delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\right]\left[\prod_{i=1}^{q} \delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\right].$$

TQFT reformulates our enumeration.



2-cellular complex associated with the TQFT_2 of $Z_{(3,4)}$ made of 3+4 cylinders sharing boundaries.

permutation TQFT for UO tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

The counting of UO invariants of order (r, q) can be massaged:

$$Z_{(r,q)}(n) = \frac{1}{n!} \sum_{\gamma \in S_n} Z^q_{n;\gamma} \sum_{\sigma_0, \sigma_2, \sigma_3 \dots, \sigma_r \in S_n} \Big[\prod_{i=2}^{r} \delta(\gamma^{-1} \sigma_i \gamma \sigma_i^{-1}) \Big] \delta(\gamma^{-1} \sigma_0 \gamma \sigma_0^{-1}) \delta(\sigma_0 \prod_{i=2}^{r} \sigma_i) \,,$$

$$\text{with} \quad Z^q_{n;\gamma} = \frac{1}{(n!)[n!(2!)^n]^q} \sum_{\substack{\sigma_1 \in S_n \\ \tau_1, \dots, \tau_q \in S_{2n} \\ \tau_i^{-1} \sigma_1 \gamma^{-1} \sigma_1^{-1} \gamma \tau_i \in S_n[S_2]}} 1 \,.$$

We are counting equivalence classes of r permutations σ_i under the conjugation $\sigma_i \sim \gamma \sigma_i \gamma^{-1}$, and the group generated by r generators subject to one relation by the last constraint $\sigma_0 \prod_{i=2}^{r} \sigma_i = \mathrm{id}$, i.e., the fundamental group of the 2-sphere with r-punctures.

Therefore, $Z_{(r,q)}(n)$ counts the covers of the r-punctured sphere with each cover weighted by $Z^q_{n;\gamma}$, i.e., enumerates $Z^q_{n;\gamma}$-weighted equivalence classes of branched covers (with r branched points of degree n) of the sphere.

Consequences and Outlook

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

The counting of tensor invariants, in addition to their essential role in the pertubative analysis of tensor models in theoretical physics, uncovers unexpected connections between combinatorics, algebra, and topology.

- We added more correspondence between the enumeration of tensor invariants and 2-dimensional permutation TQFT.
- The sequences of numbers corresponding to our enumerations are all new and unknown before in OEIS (Online Encyclopedia of Integer Sequences).
- So far, regardless of whether the invariants are unitary, orthogonal, or UO symmetric, we consistently find a correspondence with (branched) covers of either the sphere or the torus (possibly with punctures). We ask what about non-orientable manifolds, e.g., the Klein bottle (as a closed manifold)? Which types of tensors, transformations, and tensor contractions may lead to the enumeration of covers of nonorientable manifolds?

the end