

# Weighted Alexandrov-Fenchel type inequalities for hypersurfaces in $\mathbb{R}^n$

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Recent Advances in Comparison Geometry

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- 1 Background and motivations
- 2 Sharp weighted inequality involving three distinct quermassintegrals
- 3 Weighted Alexandrov-Fenchel inequalities in  $\mathbb{R}^n$

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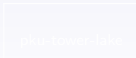
# Isoperimetric inequality

- Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . Then the classical isoperimetric inequality states that

$$|\partial\Omega| \geq \sqrt{4\pi}|\Omega|^{\frac{1}{2}},$$

where  $|\partial\Omega|$  is the length of the boundary  $\partial\Omega$ ,  $|\Omega|$  denotes the area of  $\Omega$ .

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Equality holds if and only if  $\Omega$  is a disk.

- For a smooth bounded domain  $\Omega \subseteq \mathbb{R}^n$  with boundary  $\partial\Omega = \Sigma$ , there holds

$$\frac{|\Sigma|}{\omega_{n-1}} \geq \left( \frac{\text{Vol}(\Omega)}{\frac{\omega_{n-1}}{n}} \right)^{\frac{n-1}{n}},$$

where  $|\Sigma|$  denotes the area of  $\Sigma$  and  $\omega_{n-1}$  is the area of the unit sphere  $\mathbb{S}^{n-1}$ .

Equality holds iff  $\Sigma$  is a round sphere.

# The classical Minkowski inequality

- For a closed **convex** surface  $\Sigma$  in  $\mathbb{R}^3$ , there holds

$$\int_{\Sigma} H d\mu \geq \sqrt{16\pi|\Sigma|}.$$

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- For a closed **convex** surface  $\Sigma$  in  $\mathbb{R}^n$ , we have

$$\int_{\Sigma} H d\mu \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}}.$$

Equality holds if and only if  $\Sigma$  is a round sphere.

# Alexandrov-Fenchel inequalities

## Theorem (Alexandrov-Fenchel inequalities (1937))

For *convex* hypersurface  $\Sigma$  in  $\mathbb{R}^n$ , we have

$$\frac{1}{\omega_{n-1}} \int_{\Sigma} H_k d\mu \geq \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} H_j d\mu \right)^{\frac{n-1-k}{n-1-j}}, \quad 0 \leq j < k \leq n-1 \quad (1)$$

where  $H_k = \frac{\sigma_k}{C_{n-1}^k}$  is the  $k$ -th normalized mean curvature.

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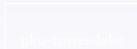
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Extention to **non-convex** hypersurfaces: Guan-Li, Huisken, Chang-Wang, Qiu, Agostiniani-Fogagnolo-Mazzieri



## Theorem (Guan-Li, 2009)

Alexandrov-Fenchel inequalities (1) hold for *star-shaped* and *k-convex* hypersurfaces.

- $\Sigma^{n-1} \subseteq \mathbb{R}^n$  is called *star-shaped*, if its support function  $u = \langle X, \nu \rangle > 0$  on  $\Sigma$ , where  $X$  is the position vector of  $\Sigma$ ,  $\nu$  is the unit outer-normal.
- $\Sigma$  is called *k-convex* if  $H_j > 0$  for all  $1 \leq j \leq k$ .  
In particular, 1-convex is called mean convex.

# The flow approach

Method: Inverse curvature flow

$$\frac{\partial}{\partial t} X = \frac{H_{k-1}}{H_k} \nu \quad \text{in } \mathbb{R}^n \quad (2)$$

- $\mathcal{I}_{k,j}(\Sigma_t) = \left( \int_{\Sigma_t} H_j d\mu_t \right)^{-\frac{n-1-k}{n-1-j}} \int_{\Sigma_t} H_k d\mu_t$  is  $\searrow$  along flow (2)

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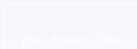
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Theorem (C. Gerhardt/ J. Urbas(1990))

If  $\Sigma_0$  is star-shaped,  $k$ -convex, then  $\Sigma_t$  expands to infinity as  $t \rightarrow \infty$  and  $e^{-t}\Sigma_t \rightarrow \mathbb{S}^n(r_0)$  smoothly.



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- Gerhardt/Urbas's theorem implies that

$$\mathcal{I}_{k,j}(\Sigma_0) \geq \lim_{t \rightarrow \infty} \mathcal{I}_{k,j}(\Sigma_t) = \mathcal{I}_{k,j}(S_{r_0}) = \omega \frac{k-j}{n-1-j}.$$

- Brendle-Hung-Wang (2015)

$$\int_{\Sigma} \left( VH_1 - \langle \bar{\nabla} V, \nu \rangle \right) d\mu \geq \omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}},$$

provided  $\Sigma$  is **star-shaped and mean convex** hypersurface in  $\mathbb{H}^n$ , where  $V = \cosh r$ .

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- Ge-Wang-W. (2015)

$$\int_{\Sigma} VH_{2k+1} d\mu \geq \omega_{n-1} \left( \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{(k+1)(n-1)}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k-2}{(k+1)(n-1)}} \right)^{k+1},$$

provided that  $\Sigma$  be a **horospherical convex** hypersurface in  $\mathbb{H}^n$ .



# Inequality involve $r^2$ -weighted mean curvature

Theorem (K. Kwong-P. Miao, 2014)

Let  $\Sigma$  be a smooth, *star-shaped* hypersurface with *positive* mean curvature in  $\mathbb{R}^n$ . Then

$$\int_{\Sigma} r^2 H_1 d\mu \geq n \text{Vol}(\Omega), \quad (3)$$

where  $\text{Vol}(\Omega)$  is the volume of  $\Omega$  enclosed by  $\Sigma$ ,  $r$  is the distance to a fixed point  $O$ . Equality holds iff  $\Sigma$  is a geodesic sphere.

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- $Q(t) = e^{-\frac{n-2}{n-1}t} \left[ \int_{\Sigma_t} r^2 H_1 d\mu - n \text{Vol}(\Omega_t) \right]$  is  $\searrow$  along the IMCF:  
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- Using Reilly's formula, (3) holds for *convex* hypersurface.

$$\int_{\Omega} (\Delta u)^2 - |\nabla^2 u|^2 - \text{Ric}(\nabla u, \nabla u) d\text{vol} = \int_{\Sigma} 2u_{\nu} \Delta_{\Sigma} u + H(u_{\nu})^2 + II(\nabla^{\Sigma} u, \nabla^{\Sigma} u) d\mu.$$

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- A star-shaped and mean convex hypersurface  $\xrightarrow{\text{IMCF}}$  convex .

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## Theorem (Girão-Rodrigues, 2020)

Let  $\Sigma$  be a smooth, *star-shaped* hypersurface with *positive* mean curvature in  $\mathbb{R}^n$ . Then

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- Use IMCF:  $\frac{\partial}{\partial t} X = \frac{1}{H_1} \nu$
- Consider  $E(t) = |\Sigma_t|^{-\frac{n}{n-1}} \int_{\Sigma_t} r^2 H_1 d\mu_t$  that

$$\begin{aligned} E'(t) &\leq \frac{2}{|\Sigma_t|^{\frac{n}{n-1}}} \left( n \text{Vol}(\Omega_t) - \int_{\Sigma_t} r^2 H_1 d\mu_t \right) \\ &\leq 0 \quad (\text{by Kwong-Miao's result}) \end{aligned}$$

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- $E(0) \geq \lim_{t \rightarrow \infty} E(t) = \omega_{n-1}^{-\frac{1}{n-1}}$ .

## Theorem (K. Kwong-P. Miao, 2015)

Let  $\Sigma$  be a smooth, closed, *star-shaped* and *k-convex* hypersurface in  $\mathbb{R}^n$  ( $n \geq 3$ ). Then for all  $k = 2, \dots, n - 1$ , there holds

$$\int_{\Sigma} r^2 H_k d\mu \geq \int_{\Sigma} H_{k-2} d\mu.$$

Equality holds if and only if  $\Sigma$  is a round sphere.

**Approach:** use of a generalized Hsiung-Minkowski formula.





# Inequality involve weighted k-th mean curvature

Theorem (Y. Wei-T. Zhou, 2023)

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$$Q'(t) \leq 2 \left( \int_{\Sigma_t} H_{k-1} \right)^{-\frac{n-k+1}{n-k}} \left( \int_{\Sigma_t} H_{k-2} d\mu_t - \int_{\Sigma_t} r^2 H_k d\mu_t \right) \stackrel{\text{Kwong-Miao}}{\leq} 0.$$

- Is there a direct proof of (4) and (5) without using Kwong-Miao's results?

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# Sharp weighted inequality involving three distinct quermassintegrals

## Theorem (W.)

Let  $\Sigma$  be a smooth, closed, *star-shaped* and *k-convex* hypersurface in  $\mathbb{R}^n$  ( $n \geq 3$ ). Then for each  $k = 1, \dots, n-1$ , there holds

$$\begin{aligned} & \int_{\Sigma} r^2 H_k d\mu + \frac{2(k-1)}{n-k+1} \int_{\Sigma} H_{k-2} d\mu \\ & \geq \frac{n+k-1}{n-k+1} \omega_{n-1} \left( \frac{\int_{\Sigma} H_{k-1} d\mu}{\omega_{n-1}} \right)^{\frac{n-k+1}{n-k}}. \end{aligned}$$

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- $k = 1$ : reduces to Girão and Rodrigues's result (make convention that  $\int_{\Sigma} H_{-1} d\mu = n \text{Vol}(\Omega)$ ).
- Kwong-Wei, 2023.

- **Key point:** Use the **normalized(rescaled)** inverse curvature flow

$$\frac{\partial}{\partial t} X = \left( \frac{H_{k-1}}{H_k} - u \right) \nu, \quad u = \langle X, \nu \rangle.$$

- The above flow is equivalent to the un-normalized one by rescaling.

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- The above flow is equivalent to the un-normalized one by rescaling.
- Choose functional

$$Q_k(t) = \left( \frac{\int_{\Sigma_t} H_{k-1}}{\omega_{n-1}} \right)^{-\frac{n-k+1}{n-k}} \left( \int_{\Sigma_t} r^2 H_k + \frac{2(k-1)}{n+1-k} \int_{\Sigma_t} H_{k-2} \right)$$

- **Monotonicity:**  $Q_k(t)$  is  $\searrow$  along above flow.



# Monotonicity of $Q_k(t)$

- Apply the variation formula along  $\frac{\partial X}{\partial t} = F\nu$  that

$$\frac{d}{dt} \int_{\Sigma_t} H_{k-1} d\mu_t = (n-k) \int_{\Sigma_t} H_k F d\mu_t,$$

$$\begin{aligned} \frac{d}{dt} \int_{\Sigma_t} r^2 H_k d\mu_t = \int_{\Sigma_t} \left( (n-1-k) r^2 H_{k+1} + 2(k+1) u H_k \right. \\ \left. - 2k H_{k-1} \right) F d\mu_t. \end{aligned}$$

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- We have

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Sigma_t} r^2 H_k d\mu_t + \frac{2(k-1)}{n+1-k} \int_{\Sigma_t} H_{k-2} d\mu_t \right) \\ &= \int_{\Sigma_t} \left( (n-1-k) r^2 H_{k+1} + 2(k+1) u H_k - 2H_{k-1} \right) \left( \frac{H_{k-1}}{H_k} - u \right) \\ &= (n-1-k) \int_{\Sigma_t} r^2 \left( \frac{H_{k+1} H_{k-1}}{H_k} - u H_{k+1} \right) d\mu_t \\ & \quad + 2k \int_{\Sigma_t} (u H_{k-1} - u^2 H_k) d\mu_t - 2 \int_{\Sigma_t} H_k \left( \frac{H_{k-1}}{H_k} - u \right)^2 d\mu_t. \end{aligned}$$

## Lemma

Let  $(\Sigma^{n-1}, g)$  be a closed hypersurface in the Euclidean space  $\mathbb{R}^n$ . For  $k = 1, \dots, n-1$ , we have

$$\int_{\Sigma} r^2 u H_k d\mu = \int_{\Sigma} r^2 H_{k-1} d\mu + \frac{1}{2kC_{n-1}^k} \int_{\Sigma} (T_{k-1})^{ij} \nabla_i(r^2) \nabla_j(r^2) d\mu,$$
$$\int_{\Sigma} u^2 H_k d\mu = \int_{\Sigma} u H_{k-1} d\mu + \frac{1}{4kC_{n-1}^k} \int_{\Sigma} (T_{k-1})^{ij} h_i^s \nabla_j(r^2) \nabla_s(r^2) d\mu.$$

Here,  $(T_k)_j^i := \frac{\partial \sigma_{k+1}}{\partial h_j^i}$  and  $T_k^{ij} = (T_k)_s^i g^{js}$ .

## Lemma

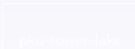
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$$\int_{\Sigma} u^2 H_k d\mu = \int_{\Sigma} u H_{k-1} d\mu + \frac{1}{4kC_{n-1}^k} \int_{\Sigma} (T_{k-1})^{ij} h_i^s \nabla_j(r^2) \nabla_s(r^2) d\mu.$$

Here,  $(T_k)_j^i := \frac{\partial \sigma_{k+1}}{\partial h_j^i}$  and  $T_k^{ij} = (T_k)_s^i g^{js}$ .

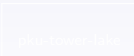
$$\nabla_i(T_{ij}^{k-1} \nabla_j r^2) = 2((n-k)\sigma_{k-1} - k\sigma_k u) = 2kC_{n-1}^k(H_{k-1} - uH_k).$$

- Multiply by the function  $r^2$  and integration by parts
- Multiply by  $u$  and notice  $\nabla_i u = h_i^s \nabla_s(\frac{1}{2}r^2)$ .



# Monotonicity of $Q_k(t)$

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Sigma_t} r^2 H_k d\mu_t + \frac{2(k-1)}{n+1-k} \int_{\Sigma_t} H_{k-2} d\mu_t \right) \\ &= (n-1-k) \int_{\Sigma_t} r^2 \left( \frac{H_{k+1} H_{k-1}}{H_k} - u H_{k+1} \right) d\mu_t \\ & \quad + 2k \int_{\Sigma_t} (u H_{k-1} - u^2 H_k) d\mu_t - 2 \int_{\Sigma_t} H_k \left( \frac{H_{k-1}}{H_k} - u \right)^2 d\mu_t \\ &\leq (n-1-k) \int_{\Sigma_t} r^2 (H_k - u H_{k+1}) d\mu_t - \frac{1}{2C_{n-1}^k} \int_{\Sigma_t} (T_{k-1})^{ij} h_i^s \nabla_j(r^2) \nabla_s(r^2) d\mu_t \\ &= (n-1-k) \int_{\Sigma_t} r^2 (H_k - u H_{k+1}) d\mu_t - \frac{1}{2C_{n-1}^k} \int_{\Sigma_t} \left( -(T_k)^{js} + \sigma_k g^{js} \right) \nabla_j(r^2) \nabla_s(r^2) \\ &= (n-1-k) \int_{\Sigma_t} r^2 (H_k - u H_{k+1}) + \frac{2(k+1)C_{n-1}^{k+1}}{2C_{n-1}^k} \int_{\Sigma_t} r^2 (u H_{k+1} - H_k) \\ & \quad - \frac{1}{2C_{n-1}^k} \int_{\Sigma_t} \sigma_k g^{jl} \nabla_j(r^2) \nabla_l(r^2) d\mu_t \\ &= -\frac{1}{2} \int_{\Sigma_t} H_k |\nabla(r^2)|^2 d\mu_t \leq 0. \end{aligned}$$



# Monotonicity of $Q_k(t)$ and Rigidity

- **Monotonicity:** The  $k$ th quermassintegral is preserved by the Minkowski formula

$$\frac{d}{dt} \int_{\Sigma_t} H_{k-1} d\mu_t = (n-k) \int_{\Sigma_t} H_k \left( \frac{H_{k-1}}{H_k} - u \right) d\mu_t = 0.$$

$$\implies Q_k(t) = \left( \frac{\int_{\Sigma_t} H_{k-1}}{\omega_{n-1}} \right)^{-\frac{n-k+1}{n-k}} \left( \int_{\Sigma_t} r^2 H_k + \frac{2(k-1)}{n+1-k} \int_{\Sigma_t} H_{k-2} \right)$$

is  $\searrow$  along the flow.

- **Limit:** Note that  $Q_k(t)$  is a scaling invariant, we have

$$Q_k(0) \geq \lim_{t \rightarrow \infty} Q_k(t) = \frac{n+k-1}{n-k+1} \omega_{n-1}.$$

- **Rigidity:** If equality holds, then  $\frac{d}{dt} Q_k(t) = 0$  holds on the solution of  $\Sigma_t$  of the flow for all time  $t \implies$  the initial hypersurface  $\Sigma$  is a coordinate sphere.

- 1 Background and motivations
- 2 Sharp weighted inequality involving three distinct quermassintegrals
- 3 Weighted Alexandrov-Fenchel inequalities in  $\mathbb{R}^n$**

# Weighted Alexandrov-Fenchel inequalities

## Theorem (W.)

Let  $\Sigma$  be a smooth, closed, *star-shaped* and *k-convex* hypersurface in  $R^n (n \geq 3)$ . Then for all  $k = 1, \dots, n-1$ ;  $j = 0, \dots, k-1$  and  $1 \leq \alpha \in \mathbb{R}$ , there holds

$$\int_{\Sigma} r^{2\alpha} H_k d\mu \geq \omega_{n-1} \left( \frac{\int_{\Sigma} H_j d\mu}{\omega_{n-1}} \right)^{\frac{n-1-k+2\alpha}{n-1-j}}. \quad (6)$$

Equality holds if and only if  $\Sigma$  is a round sphere.



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Equality holds if and only if  $\Sigma$  is a round sphere.

- **Isoperimetric type result:** In class of star-shaped and  $k$ -convex hypersurfaces with fixed  $\int_{\Sigma} H_j d\mu$  ( $0 \leq j \leq k-1$ ), minimum of  $\int_{\Sigma} r^{2\alpha} H_k d\mu$  ( $\alpha \geq 1$ ) is achieved by and only by round spheres.

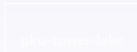
# A reformulation

- Observe above can be reformulated as

$$\left( \frac{\int_{\Sigma} r^{2\alpha} H_k d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-k+2\alpha}} \geq \left( \frac{\int_{\Sigma} H_j d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-j_s-l}}. \quad (7)$$

- Let  $\rho_+(\Sigma)$  be the outer radius of a bounded domain  $\Omega$  with boundary  $\Sigma$  that  $\rho_+(\Sigma) = \inf\{\rho : \Omega \subset B_\rho(O)\}$ , then one has

$$\lim_{\alpha \rightarrow +\infty} \left( \frac{\int_{\Sigma} r^{2\alpha} H_k d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-k+2\alpha}} = \rho_+(\Sigma).$$



## Theorem (W.)

Let  $\Sigma$  be a smooth, closed, *star-shaped* and *k-convex* hypersurface in  $R^n$  ( $n \geq 3$ ). Then

$$\rho_+(\Sigma) \geq \left( \frac{\int_{\Sigma} H_j d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-j}}, \quad j = 0, \dots, k-1. \quad (8)$$

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- Inequality (8) can be also interpreted as an **upper bound of the curvature integrals**.
- For  $j = 0$ , it was proved by **Scheuer-Xia** (2019) in the ambient space of non-positive radial curvature.

# A conjecture

- Inequality (8) is known for **convex** hypersurfaces.
- For the case of  $j = k - 1$ , (8) is equivalent to

$$\rho_+(\Sigma) \geq \left( \frac{\int_{\Sigma} H_k d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-k}}, \quad (9)$$

provided that  $\Sigma$  is a **star-shaped** and  **$(k + 1)$ -convex** hypersurface.

- Comparing with the Alexandrov-Fenchel inequality, can one generalize this result to  **$k$ -convex** hypersurfaces?

# A conjecture

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- Comparing with the Alexandrov-Fenchel inequality, can one generalize this result to  **$k$ -convex** hypersurfaces?

## Conjecture

*The inequality (9) holds for a smooth, closed, star-shaped and  **$k$ -convex** hypersurface in  $R^n (n \geq 3)$ .*

# Sketch of Proof

- **Approach:** Use the **normalized(rescaled)** inverse curvature flow

$$\frac{\partial}{\partial t} X = \left( \frac{H_{k-1}}{H_k} - u \right) \nu, \quad u = \langle X, \nu \rangle.$$

- **Monotonicity:**

$$Q_\alpha(t) = \left( \frac{\int_{\Sigma} H_s d\mu}{\omega_{n-1}} \right)^{-\frac{n-1-k+2\alpha}{n-1-s}} \int_{\Sigma_t} r^{2\alpha} H_k d\mu_t$$

is  $\searrow$  along the above flow.

- **Limit:** Note that  $Q_\alpha(t)$  is a scaling invariant, we have

$$Q_\alpha(0) \geq \lim_{t \rightarrow \infty} Q_\alpha(t) = \omega_{n-1}.$$

## Proposition

Let  $\Sigma_t$  be a smooth family of closed hypersurfaces in the Euclidean space  $\mathbb{R}^n$  satisfying a general flow

$$\frac{\partial}{\partial t} X = F\nu, \quad (10)$$

where  $\nu$  is the outward unit normal of  $\Sigma_t$  and  $F$  is a smooth function on  $\Sigma_t$ . Then for all  $k = 1, \dots, n-1$  and  $\alpha \in \mathbb{R}$ , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} \Phi^\alpha H_k d\mu_t \\ &= \int_{\Sigma} \left( (n-1-k) \Phi^\alpha H_{k+1} + \alpha(k+1) \Phi^{\alpha-1} u H_k - \alpha k \Phi^{\alpha-1} H_{k-1} \right. \\ & \quad \left. - \frac{\alpha(\alpha-1)}{C_{n-1}^k} \Phi^{\alpha-2} T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi \right) F d\mu_t, \end{aligned} \quad (11)$$

where  $\Phi = \Phi(r) = \frac{1}{2}r^2$ .



# Monotonicity of $Q_\alpha(t)$

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma_t} \Phi^\alpha H_k d\mu_t \\ &= \int_{\Sigma_t} \left( (n-1-k) \Phi^\alpha H_{k+1} + \alpha(k+1) \Phi^{\alpha-1} u H_k - \alpha k \Phi^{\alpha-1} H_{k-1} \right) \left( \frac{H_{k-1}}{H_k} - u \right) \\ & \quad - \int_{\Sigma_t} \frac{\alpha(\alpha-1)}{C_{n-1}^k} \Phi^{\alpha-2} (T_{k-1})^{ij} \nabla_i \Phi \nabla_j \Phi \left( \frac{H_{k-1}}{H_k} - u \right) d\mu_t \\ &= \int_{\Sigma_t} (n-1-k) \Phi^\alpha \left( \frac{H_{k+1} H_{k-1}}{H_k} - u H_{k+1} \right) d\mu_t + \int_{\Sigma_t} \alpha k \Phi^{\alpha-1} (u H_{k-1} - u^2 H_k) \\ & \quad - \int_{\Sigma_t} \alpha \Phi^{\alpha-1} H_k \left( \frac{H_{k-1}}{H_k} - u \right)^2 d\mu_t - \int_{\Sigma_t} \alpha(k-1) \Phi^{\alpha-1} \left( \frac{H_{k-1}^2}{H_k} - u H_{k-1} \right) \\ & \quad - \int_{\Sigma_t} \frac{\alpha(\alpha-1)}{C_{n-1}^k} \Phi^{\alpha-2} T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi \left( \frac{H_{k-1}}{H_k} - u \right) \\ &= I + II + III + IV + V. \end{aligned}$$

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# Monotonicity of $Q_\alpha(t)$

- Using Newton-Maclaurin inequality,

$$I \leq \int_{\Sigma_t} (n-1-k)\Phi^\alpha \left( H_k - uH_{k+1} \right) d\mu_t$$

- Multiply  $\Phi^{\alpha-1}u$  to both sides of

$$\begin{aligned} II &= \int_{\Sigma_t} \alpha k \Phi^{\alpha-1} (uH_{k-1} - u^2 H_k) d\mu_t = \frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha \Phi^{\alpha-1} u \nabla_i (T_{k-1}^{ij} \nabla_j \Phi) \\ &= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1) \Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} T_{k-1}^{ij} \nabla_i u \nabla_j \Phi \\ &= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1) \Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} T_{k-1}^{ij} h_i^s \nabla_s \Phi \nabla_j \Phi \\ &= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1) \Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} (\sigma_k g^{js} - T_k^{js}) \nabla_s \Phi \nabla_j \Phi \\ &= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \left( \alpha(\alpha-1) \Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi d\mu_t + \alpha \Phi^{\alpha-1} H_k |\nabla \Phi|^2 \right. \\ &\quad \left. - \alpha \Phi^{\alpha-1} T_k^{ij} \nabla_i \Phi \nabla_j \Phi \right) = II_1 + II_2 + II_3 \end{aligned}$$

# Monotonicity of $Q_\alpha(t)$

- Using again the divergence-free property of  $T_k$ ,

$$\begin{aligned} II_3 &= \frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha \Phi^{\alpha-1} T_k^{ij} \nabla_i \Phi \nabla_j \Phi d\mu_t = -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \Phi^\alpha \nabla_i (T_{ij}^k \nabla_j \Phi) d\mu_t \\ &= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \Phi^\alpha (k+1) C_{n-1}^{k+1} (H_k - uH_{k+1}) d\mu_t \\ &= -(n-1-k) \int_{\Sigma_t} \Phi^\alpha (H_k - uH_{k+1}) d\mu_t \end{aligned}$$

- $\implies I + II_3 \leq 0$ .

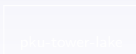
- $II_1 + V = - \int_{\Sigma_t} \frac{\alpha(\alpha-1)}{C_{n-1}^k} \Phi^{\alpha-2} T_{k-1}^{ij} \frac{H_{k-1}}{H_k} \nabla_i \Phi \nabla_j \Phi d\mu_t$ .

$$\begin{aligned} IV &\leq - \int_{\Sigma_t} \alpha(k-1) \Phi^{\alpha-1} (H_{k-2} - uH_{k-1}) \\ &= -\alpha \int_{\Sigma_t} \Phi^{\alpha-1} \frac{1}{C_{n-1}^{k-1}} \nabla_i (T_{k-2}^{ij} \nabla_j \Phi) = \frac{\alpha(\alpha-1)}{C_{n-1}^{k-1}} \int_{\Sigma_t} \Phi^{\alpha-2} T_{k-2}^{ij} \nabla_i \Phi \nabla_j \Phi \end{aligned}$$

# Monotonicity of $Q_\alpha(t)$

We finally arrive at

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma_t} \Phi^\alpha H_k d\mu_t \\ & \leq - \int_{\Sigma_t} \alpha \Phi^{\alpha-1} H_k |\nabla \Phi|^2 d\mu_t - \int_{\Sigma_t} \alpha \Phi^{\alpha-1} H_k \left( \frac{H_{k-1}}{H_k} - u \right)^2 \\ & \quad - \alpha(\alpha-1) \int_{\Sigma_t} \Phi^{\alpha-2} \left( \frac{1}{C_{n-1}^k} T_{k-1}^{ij} \frac{H_{k-1}}{H_k} - \frac{1}{C_{n-1}^{k-1}} T_{k-2}^{ij} \right) \nabla_i \Phi \nabla_j \Phi \\ & \leq - \alpha(\alpha-1) \int_{\Sigma_t} \Phi^{\alpha-2} \left( \frac{1}{C_{n-1}^k} T_{k-1}^{ij} \frac{H_{k-1}}{H_k} - \frac{1}{C_{n-1}^{k-1}} T_{k-2}^{ij} \right) \nabla_i \Phi \nabla_j \Phi \\ & \leq 0 \end{aligned}$$



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by the observation

$$\frac{1}{C_{n-1}^k} T_{k-1}^{ij} \frac{H_{k-1}}{H_k} - \frac{1}{C_{n-1}^{k-1}} T_{k-2}^{ij} = \frac{H_{k-1}^2}{H_k} g^{js} \frac{\partial}{\partial h_i^s} \left( \frac{H_k}{H_{k-1}} \right).$$

# Monotonicity of $Q_\alpha(t)$

- On the other hand, we have

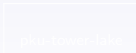
$$\begin{aligned}\frac{d}{dt} \int_{\Sigma_t} H_j d\mu_t &= (n-1-j) \int_{\Sigma_t} H_{j+1} \left( \frac{H_{k-1}}{H_k} - u \right) d\mu_t \\ &\geq (n-1-j) \int_{\Sigma_t} (H_j - u H_{j+1}) d\mu_t = 0,\end{aligned}$$

by the Newton-Maclaurin inequality and Minkowski formula .

- Combining the above together and recalling  $\Phi = \Phi(r) = \frac{1}{2}r^2$ ,

$$Q_\alpha(t) = \left( \frac{\int_{\Sigma} H_j d\mu}{\omega_{n-1}} \right)^{-\frac{n-j-k+2\alpha}{n-j-l}} \int_{\Sigma_t} r^{2\alpha} H_k d\mu_t$$

is  $\searrow$  along the flow.



## Theorem (W.)

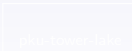
Let  $\Sigma$  be a smooth, closed, *star-shaped* and *k-convex* hypersurface in  $R^n (n \geq 3)$ . Then for all  $k = 1, \dots, n-1$ ;  $j = 0, \dots, k-1$  and  $1 \leq \alpha \in \mathbb{R}$ , there holds

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Equality holds if and only if  $\Sigma$  is a round sphere.

**Question 1:** Does the inequality hold for all  $\alpha \in [0, 1]$ ?

- $\alpha = 0$ : by the Alexandrov-Fenchel inequalities.
- $\alpha = \frac{1}{2}$ : by the Minkowski formula.



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**Question 2:** Does the weighted inequality for general  $\alpha$  hold for the hypersurface in  $\mathbb{H}^n$ ?



*Thank you for your attention!*

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