Weighted Alexandrov-Fenchel type inequalities for hypersurfaces in $\mathbb{R}^n$

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Recent Advances in Comparison Geometry

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Outline

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Isoperimetric inequality

- Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Then the classical isoperimetric inequality states that

$$|\partial\Omega| \geq \sqrt{4\pi}|\Omega|^{\frac{1}{2}},$$

where $|\partial\Omega|$ is the length of the boundary $\partial\Omega$, $|\Omega|$ denotes the area of Ω.
Equality holds if and only if Ω is a disk.

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- For a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, there holds

$$\frac{|\Sigma|}{\omega_{n-1}} \geq \left(\frac{Vol(\Omega)}{\frac{\omega_{n-1}}{n}}\right)^{\frac{n-1}{n}},$$

where $|\Sigma|$ denotes the area of Σ and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$.
Equality holds iff Σ is a round sphere.

The classical Minkowski inequality

- For a closed convex surface Σ in $\mathbb{R}^3$, there holds

$$\int_\Sigma H d\mu \geq \sqrt{16\pi|\Sigma|}.$$

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- For a closed convex surface Σ in $\mathbb{R}^n$, we have

$$\int_\Sigma H d\mu \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}}.$$

Equality holds if and only if Σ is a round sphere.

Alexandrov-Fenchel inequalities

Theorem (Alexandrov-Fenchel inequalities (1937))

For convex hypersurface Σ in $\mathbb{R}^n$, we have

$$\frac{1}{\omega_{n-1}}\int_\Sigma H_k d\mu \geq \left(\frac{1}{\omega_{n-1}}\int_\Sigma H_j d\mu\right)^{\frac{n-1-k}{n-1-j}}, \quad 0 \leq j < k \leq n-1 \tag{1}$$

where $H_k = \frac{\sigma_k}{C_{n-1}^k}$ is the k-th normalized mean curvature.

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Extention to non-convex hypersurfaces: Guan-Li, Huisken, Chang-Wang, Qiu, Agostiniani-Fogagnolo-Mazzieri

Guan-Li's theorem

Theorem (Guan-Li, 2009)

Alexandrov-Fenchel inequalities (1) hold for star-shaped and k-convex hypersurfaces.

- $\Sigma^{n-1} \subseteq \mathbb{R}^n$ is called star-shaped, if its support function $u = \langle X, \nu \rangle > 0$ on Σ, where X is the position vector of Σ, ν is the unit outer-normal.
- Σ is called k-convex if $H_j > 0$ for all $1 \leq j \leq k$. In particular, 1-convex is called mean convex.

The flow approach

Method: Inverse curvature flow

$$\frac{\partial}{\partial t}X = \frac{H_{k-1}}{H_k}\nu \quad \text{in} \quad \mathbb{R}^n \tag{2}$$

- $\mathcal{I}_{k,j}(\Sigma_t) = \left(\int_{\Sigma_t} H_j d\mu_t\right)^{-\frac{n-1-k}{n-1-j}} \int_{\Sigma_t} H_k d\mu_t$ is $\searrow$ along flow (2)

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$$\frac{d}{dt}\int_{\Sigma_t} H_k = (n-1-k)\int_{\Sigma_t} H_{k+1}\frac{H_{k-1}}{H_k} \leq (n-1-k)\int_{\Sigma_t} H_k$$

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Theorem (C. Gerhardt/ J. Urbas(1990))

If Σ_0 is star-shaped, k-convex, then Σ_t expands to infinity as $t \to \infty$ and $e^{-t}\Sigma_t \to \mathbb{S}^n(r_0)$ smoothly.

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If Σ_0 is star-shaped, k-convex, then Σ_t expands to infinity as $t \to \infty$ and $e^{-t}\Sigma_t \to \mathbb{S}^n(r_0)$ smoothly.

- Gerhardt/Urbas's theorem implies that

$$\mathcal{I}_{k,j}(\Sigma_0) \geq \lim_{t\to\infty} \mathcal{I}_{k,j}(\Sigma_t) = \mathcal{I}_{k,j}(S_{r_0}) = \omega_{n-1}^{\frac{k-j}{n-1-j}}.$$

Weighted AF inequalities in $\mathbb{H}^n$

- Brendle-Hung-Wang (2015)

$$\int_\Sigma \left(VH_1 - \langle \bar{\nabla} V, \nu \rangle \right) d\mu \geq \omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}},$$

provided Σ is star-shaped and mean convex hypersurface in $\mathbb{H}^n$, where $V = coshr$.

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- Ge-Wang-W. (2015)

$$\int_\Sigma VH_{2k+1} d\mu \geq \omega_{n-1} \left(\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{(k+1)(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k-2}{(k+1)(n-1)}} \right)^{k+1},$$

provided that Σ be a horospherical convex hypersurface in $\mathbb{H}^n$.

Inequality involve r^2-weighted mean curvature

Theorem (K. Kwong-P. Miao, 2014)

Let Σ be a smooth, star-shaped hypersurface with positive mean curvature in $\mathbb{R}^n$. Then

$$\int_{\Sigma} r^2 H_1 d\mu \geq nVol(\Omega), \quad (3)$$

where $Vol(\Omega)$ is the volume of Ω enclosed by Σ, r is the distance to a fixed point O. Equality holds iff Σ is a geodesic sphere.

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- $Q(t) = e^{-\frac{n-2}{n-1}t}\left[\int_{\Sigma_t} r^2 H_1 d\mu - nVol(\Omega_t)\right]$ is $\searrow$ along the IMCF: $\frac{\partial}{\partial t}X = \frac{1}{H_1}\nu$.

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- Using Reilly's formula, (3) holds for convex hypersurface.

$$\int_\Omega (\triangle u)^2 - |\nabla^2 u|^2 - Ric(\nabla u, \nabla u)dvol = \int_\Sigma 2u_\nu \triangle_\Sigma u + H(u_\nu)^2 + II(\nabla^\Sigma u, \nabla^\Sigma u)d\mu.$$

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- A star-shaped and mean convex hypersurface $\xrightarrow{IMCF}$ convex .

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Theorem (Girão-Rodrigues, 2020)

Let Σ be a smooth, star-shaped hypersurface with positive mean curvature in $\mathbb{R}^n$. Then

$$\int_\Sigma r^2 H_1 \geq \omega_{n-1}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}. \tag{4}$$

Equality holds iff Σ is a geodesic sphere.

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- Consider $E(t) = |\Sigma_t|^{-\frac{n}{n-1}} \int_{\Sigma_t} r^2 H_1 d\mu_t$ that

$$E'(t) \leq \frac{2}{|\Sigma_t|^{\frac{n}{n-1}}}\left(nVol(\Omega_t) - \int_{\Sigma_t} r^2 H_1 d\mu_t\right)$$
$$\leq 0 \qquad \text{(by Kwong-Miao's result)}$$

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- $E(0) \geq lim_{t\to\infty} E(t) = \omega_{n-1}^{-\frac{1}{n-1}}$.

Theorem (K. Kwong-P. Miao, 2015)

Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $\mathbb{R}^n (n \geq 3)$. Then for all $k = 2, \cdots, n-1$, there holds

$$\int_\Sigma r^2 H_k d\mu \geq \int_\Sigma H_{k-2} d\mu.$$

Equality holds if and only if Σ is a round sphere.

Approach: use of a generalized Hsiung-Minkowski formula.

Inequality involve weighted k-th mean curvature

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Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$. Then for all $k = 2, \cdots, n-1$, there holds

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- Consider $Q(t) = \left(\int_{\Sigma_t} H_{k-1} \right)^{-\frac{n-k+1}{n-k}} \int_{\Sigma_t} r^2 H_k d\mu_t,$

$$Q'(t) \leq 2 \left(\int_{\Sigma_t} H_{k-1} \right)^{-\frac{n-k+1}{n-k}} \left(\int_{\Sigma_t} H_{k-2} d\mu_t - \int_{\Sigma_t} r^2 H_k d\mu_t \right) \overset{Kwong-Miao}{\leq} 0.$$

A natural question

- Is there a direct proof of (4) and (5) without using Kwong-Miao's results?

Outline

Sharp weighted inequality involving three distinct quermassintegrals

Theorem (W.)

Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $\mathbb{R}^n (n \geq 3)$. Then for each $k = 1, \cdots, n-1$, there holds

$$\int_\Sigma r^2 H_k d\mu + \frac{2(k-1)}{n-k+1} \int_\Sigma H_{k-2} d\mu \geq \frac{n+k-1}{n-k+1} \omega_{n-1} \left(\frac{\int_\Sigma H_{k-1} d\mu}{\omega_{n-1}} \right)^{\frac{n-k+1}{n-k}}.$$

Equality holds iff Σ is a round sphere.

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Equality holds iff Σ is a round sphere.

- $k = 1$: reduces to Girão and Rodrigues's result (make convention that $\int_\Sigma H_{-1} d\mu = n Vol(\Omega)$).
- Kwong-Wei, 2023.

Idea of Proof

- Key point: Use the normalized(rescaled) inverse curvature flow

$$\frac{\partial}{\partial t}X = \left(\frac{H_{k-1}}{H_k} - u\right)\nu, \qquad u = \langle X, \nu\rangle.$$

- The above flow is equivalent to the un-normalized one by rescaling.

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- The above flow is equivalent to the un-normalized one by rescaling.
- Choose functional

$$Q_k(t) = \left(\frac{\int_{\Sigma_t} H_{k-1}}{\omega_{n-1}}\right)^{-\frac{n-k+1}{n-k}} \left(\int_{\Sigma_t} r^2 H_k + \frac{2(k-1)}{n+1-k}\int_{\Sigma_t} H_{k-2}\right)$$

- Monotonicity: $Q_k(t)$ is $\searrow$ along above flow.

Monotonicity of $Q_k(t)$

- Apply the variation formula along $\frac{\partial X}{\partial t} = F\nu$ that

$$\frac{d}{dt}\int_{\Sigma_t} H_{k-1} d\mu_t = (n-k)\int_{\Sigma_t} H_k F d\mu_t,$$

$$\frac{d}{dt}\int_{\Sigma_t} r^2 H_k d\mu_t = \int_{\Sigma_t} \Big((n-1-k)\, r^2 H_{k+1} + 2(k+1) u H_k - 2k H_{k-1} \Big) F d\mu_t.$$

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- We have

$$\frac{d}{dt}\left(\int_{\Sigma_t} r^2 H_k d\mu_t + \frac{2(k-1)}{n+1-k}\int_{\Sigma_t} H_{k-2} d\mu_t\right)$$
$$= \int_{\Sigma_t}\Big((n-1-k) r^2 H_{k+1} + 2(k+1)uH_k - 2H_{k-1}\Big)\left(\frac{H_{k-1}}{H_k} - u\right)$$
$$= (n-1-k)\int_{\Sigma_t} r^2\left(\frac{H_{k+1}H_{k-1}}{H_k} - uH_{k+1}\right) d\mu_t$$
$$+ 2k\int_{\Sigma_t}(uH_{k-1} - u^2 H_k) d\mu_t - 2\int_{\Sigma_t} H_k\left(\frac{H_{k-1}}{H_k} - u\right)^2 d\mu_t.$$

Lemma

Let (Σ^{n-1}, g) be a closed hypersurface in the Euclidean space $\mathbb{R}^n$. For $k = 1, \cdots, n-1$, we have

$$\int_\Sigma r^2 u H_k d\mu = \int_\Sigma r^2 H_{k-1} d\mu + \frac{1}{2kC_{n-1}^k} \int_\Sigma (T_{k-1})^{ij} \nabla_i (r^2) \nabla_j (r^2) d\mu,$$

$$\int_\Sigma u^2 H_k d\mu = \int_\Sigma u H_{k-1} d\mu + \frac{1}{4kC_{n-1}^k} \int_\Sigma (T_{k-1})^{ij} h_i^s \nabla_j (r^2) \nabla_s (r^2) d\mu.$$

Here, $(T_k)_j^i := \frac{\partial \sigma_{k+1}}{\partial h_i^j}$ *and* $T_k^{ij} = (T_k)_s^i g^{js}$.

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Here, $(T_k)_j^i := \frac{\partial \sigma_{k+1}}{\partial h_i^j}$ *and* $T_k^{ij} = (T_k)_s^i g^{js}$.

$$\nabla_i(T_{ij}^{k-1} \nabla_j r^2) = 2((n-k)\sigma_{k-1} - k\sigma_k u) = 2kC_{n-1}^k (H_{k-1} - uH_k).$$

- Multiply by the function r^2 and integration by parts
- Multiply by u and notice $\nabla_i u = h_i^s \nabla_s \left(\frac{1}{2} r^2\right)$.

Monotonicity of $Q_k(t)$

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\Sigma_t} r^2 H_k d\mu_t + \frac{2(k-1)}{n+1-k}\int_{\Sigma_t} H_{k-2} d\mu_t\right)\\
=&(n-1-k)\int_{\Sigma_t} r^2\left(\frac{H_{k+1}H_{k-1}}{H_k} - uH_{k+1}\right)d\mu_t\\
&+2k\int_{\Sigma_t}(uH_{k-1}-u^2H_k)d\mu_t - 2\int_{\Sigma_t} H_k\left(\frac{H_{k-1}}{H_k}-u\right)^2 d\mu_t\\
\leq&(n-1-k)\int_{\Sigma_t} r^2(H_k - uH_{k+1})d\mu_t - \frac{1}{2C_{n-1}^k}\int_{\Sigma_t}(T_{k-1})^{ij}h_i^s\nabla_j(r^2)\nabla_s(r^2)d\mu_t\\
=&(n-1-k)\int_{\Sigma_t} r^2(H_k - uH_{k+1})d\mu_t - \frac{1}{2C_{n-1}^k}\int_{\Sigma_t}\left(-(T_k)^{js}+\sigma_k g^{js}\right)\nabla_j(r^2)\nabla_s(r^2)\\
=&(n-1-k)\int_{\Sigma_t} r^2(H_k - uH_{k+1}) + \frac{2(k+1)C_{n-1}^{k+1}}{2C_{n-1}^k}\int_{\Sigma_t} r^2(uH_{k+1}-H_k)\\
&-\frac{1}{2C_{n-1}^k}\int_{\Sigma_t}\sigma_k g^{jl}\nabla_j(r^2)\nabla_l(r^2)d\mu_t\\
=&-\frac{1}{2}\int_{\Sigma_t} H_k|\nabla(r^2)|^2 d\mu_t \leq 0.
\end{aligned}
$$

Monotonicity of $Q_k(t)$ and Rigidity

- Monotonicity: The kth quermassintegral is preserved by the Minkowski formula

$$\frac{d}{dt}\int_{\Sigma_t} H_{k-1}d\mu_t = (n-k)\int_{\Sigma_t} H_k\left(\frac{H_{k-1}}{H_k} - u\right)d\mu_t = 0.$$

$$\Longrightarrow Q_k(t) = \left(\frac{\int_{\Sigma_t} H_{k-1}}{\omega_{n-1}}\right)^{-\frac{n-k+1}{n-k}}\left(\int_{\Sigma_t} r^2 H_k + \frac{2(k-1)}{n+1-k}\int_{\Sigma_t} H_{k-2}\right)$$

is $\searrow$ along the flow.

- Limit: Note that $Q_k(t)$ is a scaling invariant, we have

$$Q_k(0) \geq \lim_{t\to\infty} Q_k(t) = \frac{n+k-1}{n-k+1}\omega_{n-1}.$$

- Rigidity: If equality holds, then $\frac{d}{dt}Q_k(t) = 0$ holds on the solution of Σ_t of the flow for all time $t \Longrightarrow$ the initial hypersurface Σ is a coordinate sphere.

Outline

1. Background and motivations
2. Sharp weighted inequality involving three distinct quermassintegrals
3. Weighted Alexandrov-Fenchel inequalities in $\mathbb{R}^n$

Weighted Alexandrov-Fenchel inequalities

Theorem (W.)

Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$. Then for all $k = 1, \cdots, n-1$; $j = 0, \cdots, k-1$ and $1 \leq \alpha \in \mathbb{R}$, there holds

$$\int_\Sigma r^{2\alpha} H_k d\mu \geq \omega_{n-1} \left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}} \right)^{\frac{n-1-k+2\alpha}{n-1-j}}. \qquad (6)$$

Equality holds if and only if Σ is a round sphere.

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$$\int_\Sigma r^{2\alpha} H_k d\mu \geq \omega_{n-1} \left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}} \right)^{\frac{n-1-k+2\alpha}{n-1-j}}. \tag{6}$$

Equality holds if and only if Σ is a round sphere.

- Isoperimetric type result: In class of star-shaped and k-convex hypersurfaces with fixed $\int_\Sigma H_j d\mu$ $(0 \leq j \leq k-1)$, minimum of $\int_\Sigma r^{2\alpha} H_k d\mu$ $(\alpha \geq 1)$ is achieved by and only by round spheres.

A reformulation

- Observe above can be reformulated as

$$\left(\frac{\int_\Sigma r^{2\alpha} H_k d\mu}{\omega_{n-1}}\right)^{\frac{1}{n-1-k+2\alpha}} \geq \left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}}\right)^{\frac{1}{n-js-l}}. \qquad (7)$$

- Let $\rho_+(\Sigma)$ be the outer radius of a bounded domain Ω with boundary Σ that $\rho_+(\Sigma) = \inf\{\rho : \Omega \subset B_\rho(O)\}$, then one has

$$\lim_{\alpha\to+\infty} \left(\frac{\int_\Sigma r^{2\alpha} H_k d\mu}{\omega_{n-1}}\right)^{\frac{1}{n-1-k+2\alpha}} = \rho_+(\Sigma).$$

Theorem (W.)

Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$. Then

$$\rho_+(\Sigma) \geq \left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-j}}, \quad j = 0, \cdots, k-1. \tag{8}$$

Application

Theorem (W.)

Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$. Then

$$\rho_+(\Sigma) \geq \left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-j}}, \quad j = 0, \cdots, k-1. \tag{8}$$

- Inequality (8) can be also interpreted as an upper bound of the curvature integrals.
- For $j = 0$, it was proved by Scheuer-Xia (2019) in the ambient space of non-positive radial curvature.

A conjecture

- Inequality (8) is known for convex hypersurfaces.
- For the case of $j = k - 1$, (8) is equivalent to

$$\rho_+(\Sigma) \geq \left(\frac{\int_\Sigma H_k d\mu}{\omega_{n-1}} \right)^{\frac{1}{n-1-k}}, \tag{9}$$

  provided that Σ is a star-shaped and $(k+1)$-convex hypersurface.
- Comparing with the Alexandrov-Fenchel inequality, can one generalize this result to k-convex hypersurfaces?

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- Inequality (8) is known for convex hypersurfaces.
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- Comparing with the Alexandrov-Fenchel inequality, can one generalize this result to k-convex hypersurfaces?

Conjecture

The inequality (9) holds for a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$.

Sketch of Proof

- Approach: Use the normalized(rescaled) inverse curvature flow

$$\frac{\partial}{\partial t}X = \left(\frac{H_{k-1}}{H_k} - u\right)\nu, \qquad u = \langle X, \nu\rangle.$$

- Monotonicity:

$$Q_\alpha(t) = \left(\frac{\int_\Sigma H_s d\mu}{\omega_{n-1}}\right)^{-\frac{n-1-k+2\alpha}{n-1-s}} \int_{\Sigma_t} r^{2\alpha} H_k d\mu_t$$

is $\searrow$ along the above flow.

- Limit: Note that $Q_\alpha(t)$ is a scaling invariant, we have

$$Q_\alpha(0) \geq \lim_{t\to\infty} Q_\alpha(t) = \omega_{n-1}.$$

General evolution equation

Proposition

Let Σ_t be a smooth family of closed hypersurfaces in the Euclidean space $\mathbb{R}^n$ satisfying a general flow

$$\frac{\partial}{\partial t}X = F\nu, \tag{10}$$

where ν is the outward unit normal of Σ_t and F is a smooth function on Σ_t. Then for all $k = 1, \cdots, n-1$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma \Phi^\alpha H_k d\mu_t \\ &= \int_\Sigma \Big((n-1-k)\,\Phi^\alpha H_{k+1} + \alpha(k+1)\Phi^{\alpha-1} u H_k - \alpha k \Phi^{\alpha-1} H_{k-1} \\ &\qquad - \frac{\alpha(\alpha-1)}{C_{n-1}^k}\Phi^{\alpha-2} T_{k-1}^{ij}\nabla_i\Phi\nabla_j\Phi \Big) F d\mu_t, \end{aligned} \tag{11}$$

where $\Phi = \Phi(r) = \frac{1}{2}r^2$.

Monotonicity of $Q_\alpha(t)$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t}\Phi^\alpha H_k d\mu_t\\
=&\int_{\Sigma_t}\left((n-1-k)\Phi^\alpha H_{k+1}+\alpha(k+1)\Phi^{\alpha-1}uH_k-\alpha k\Phi^{\alpha-1}H_{k-1}\right)\left(\frac{H_{k-1}}{H_k}-u\right)\\
&-\int_{\Sigma_t}\frac{\alpha(\alpha-1)}{C_{n-1}^k}\Phi^{\alpha-2}(T_{k-1})^{ij}\nabla_i\Phi\nabla_j\Phi\left(\frac{H_{k-1}}{H_k}-u\right)d\mu_t\\
=&\int_{\Sigma_t}(n-1-k)\Phi^\alpha\left(\frac{H_{k+1}H_{k-1}}{H_k}-uH_{k+1}\right)d\mu_t+\int_{\Sigma_t}\alpha k\Phi^{\alpha-1}(uH_{k-1}-u^2H_k)\\
&-\int_{\Sigma_t}\alpha\Phi^{\alpha-1}H_k\left(\frac{H_{k-1}}{H_k}-u\right)^2d\mu_t-\int_{\Sigma_t}\alpha(k-1)\Phi^{\alpha-1}\left(\frac{H_{k-1}^2}{H_k}-uH_{k-1}\right)\\
&-\int_{\Sigma_t}\frac{\alpha(\alpha-1)}{C_{n-1}^k}\Phi^{\alpha-2}T_{k-1}^{ij}\nabla_i\Phi\nabla_j\Phi\left(\frac{H_{k-1}}{H_k}-u\right)\\
=&I+II+III+IV+V.
\end{aligned}
$$

Monotinicity of $Q_\alpha(t)$

- Using Newton-Maclaurin inequality,

$$I \le \int_{\Sigma_t} (n-1-k)\Phi^\alpha \Big(H_k - uH_{k+1} \Big) d\mu_t$$

- Multiply $\Phi^{\alpha-1}u$ to both sides of

$$II = \int_{\Sigma_t} \alpha k \Phi^{\alpha-1}(uH_{k-1} - u^2 H_k) d\mu_t = \frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha \Phi^{\alpha-1} u \nabla_i (T_{k-1}^{ij} \nabla_j \Phi)$$

$$= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1)\Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} T_{k-1}^{ij} \nabla_i u \nabla_j \Phi$$

$$= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1)\Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} T_{k-1}^{ij} h_i^s \nabla_s \Phi \nabla_j \Phi$$

$$= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \alpha(\alpha-1)\Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi + \alpha \Phi^{\alpha-1} (\sigma_k g^{js} - T_k^{js}) \nabla_s \Phi \nabla_j \Phi$$

$$= -\frac{1}{C_{n-1}^k} \int_{\Sigma_t} \Big(\alpha(\alpha-1)\Phi^{\alpha-2} u T_{k-1}^{ij} \nabla_i \Phi \nabla_j \Phi d\mu_t + \alpha \Phi^{\alpha-1} H_k |\nabla \Phi|^2 - \alpha \Phi^{\alpha-1} T_k^{ij} \nabla_i \Phi \nabla_j \Phi \Big) = II_1 + II_2 + II_3$$

Monotinicity of $Q_\alpha(t)$

- Using again the divergence-free property of T_k,

$$II_3 = \frac{1}{C_{n-1}^k}\int_{\Sigma_t} \alpha\Phi^{\alpha-1}T_k^{ij}\nabla_i\Phi\nabla_j\Phi d\mu_t = -\frac{1}{C_{n-1}^k}\int_{\Sigma_t}\Phi^\alpha\nabla_i(T_{ij}^k\nabla_j\Phi)d\mu_t$$

$$= -\frac{1}{C_{n-1}^k}\int_{\Sigma_t}\Phi^\alpha(k+1)C_{n-1}^{k+1}(H_k - uH_{k+1})d\mu_t$$

$$= -(n-1-k)\int_{\Sigma_t}\Phi^\alpha(H_k - uH_{k+1})d\mu_t$$

- $\Longrightarrow I + II_3 \leq 0$.
- $II_1 + V = -\int_{\Sigma_t}\frac{\alpha(\alpha-1)}{C_{n-1}^k}\Phi^{\alpha-2}T_{k-1}^{ij}\frac{H_{k-1}}{H_k}\nabla_i\Phi\nabla_j\Phi d\mu_t$.

$$IV \leq -\int_{\Sigma_t}\alpha(k-1)\Phi^{\alpha-1}(H_{k-2} - uH_{k-1})$$

$$= -\alpha\int_{\Sigma_t}\Phi^{\alpha-1}\frac{1}{C_{n-1}^{k-1}}\nabla_i(T_{k-2}^{ij}\nabla_j\Phi) = \frac{\alpha(\alpha-1)}{C_{n-1}^{k-1}}\int_{\Sigma_t}\Phi^{\alpha-2}T_{k-2}^{ij}\nabla_i\Phi\nabla_j\Phi$$

Monotinicity of $Q_\alpha(t)$

We finally arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t}\Phi^\alpha H_k d\mu_t \\
\leq\ & -\int_{\Sigma_t}\alpha\Phi^{\alpha-1}H_k|\nabla\Phi|^2 d\mu_t-\int_{\Sigma_t}\alpha\Phi^{\alpha-1}H_k\left(\frac{H_{k-1}}{H_k}-u\right)^2 \\
&-\alpha(\alpha-1)\int_{\Sigma_t}\Phi^{\alpha-2}\left(\frac{1}{C_{n-1}^k}T_{k-1}^{ij}\frac{H_{k-1}}{H_k}-\frac{1}{C_{n-1}^{k-1}}T_{k-2}^{ij}\right)\nabla_i\Phi\nabla_j\Phi \\
\leq & -\alpha(\alpha-1)\int_{\Sigma_t}\Phi^{\alpha-2}\left(\frac{1}{C_{n-1}^k}T_{k-1}^{ij}\frac{H_{k-1}}{H_k}-\frac{1}{C_{n-1}^{k-1}}T_{k-2}^{ij}\right)\nabla_i\Phi\nabla_j\Phi \\
\leq\ & 0
\end{aligned}
$$

Monotinicity of $Q_\alpha(t)$

We finally arrive at

$$
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&\frac{d}{dt}\int_{\Sigma_t}\Phi^\alpha H_k d\mu_t \\
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& -\alpha(\alpha-1)\int_{\Sigma_t}\Phi^{\alpha-2}\left(\frac{1}{C_{n-1}^k}T_{k-1}^{ij}\frac{H_{k-1}}{H_k}-\frac{1}{C_{n-1}^{k-1}}T_{k-2}^{ij}\right)\nabla_i\Phi\nabla_j\Phi \\
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\leq\ & 0
\end{aligned}
$$

by the observation

$$
\frac{1}{C_{n-1}^k}T_{k-1}^{ij}\frac{H_{k-1}}{H_k}-\frac{1}{C_{n-1}^{k-1}}T_{k-2}^{ij}=\frac{H_{k-1}^2}{H_k}g^{js}\frac{\partial}{\partial h_i^s}\left(\frac{H_k}{H_{k-1}}\right).
$$

Monotinicity of $Q_\alpha(t)$

- On the other hand, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t} H_j d\mu_t =&(n-1-j)\int_{\Sigma_t} H_{j+1}\left(\frac{H_{k-1}}{H_k}-u\right)d\mu_t\\ \geq&(n-1-j)\int_{\Sigma_t}(H_j-uH_{j+1})d\mu_t=0,\end{aligned}$$

  by the Newton-Maclaurin inequality and Minkowski formula .

- Combining the above together and recalling $\Phi=\Phi(r)=\frac{1}{2}r^2$,

$$Q_\alpha(t)=\left(\frac{\int_\Sigma H_j d\mu}{\omega_{n-1}}\right)^{-\frac{n-j-k+2\alpha}{n-j-l}}\int_{\Sigma_t} r^{2\alpha}H_k d\mu_t$$

  is $\searrow$ along the flow.

A natural question

Theorem (W.)

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Question 1: Does the inequality hold for all $\alpha \in [0,1]$?

- $\alpha = 0$: by the Alexandrov-Fenchel inequalities.
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Let Σ be a smooth, closed, star-shaped and k-convex hypersurface in $R^n (n \geq 3)$. Then for all $k = 1, \cdots, n-1$; $j = 0, \cdots, k-1$ and $1 \leq \alpha \in \mathbb{R}$, there holds

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Question 2: Does the weighted inequality for general α hold for the hypersurface in $\mathbb{H}^n$?

Thank you for your attention!