Two-sided Lorentzian area comparison, integral curvature bounds and singularity theorems

(j/w Kontou, Ohanyan, Schinnerl)

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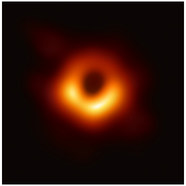
IASM Hangzhou, February 27, 2024

Outline

Singularity Theorems – Popular Science Intro

"Black holes and big bangs/big crunches are fun and these theorems from Lorentzian Geometry predict their existence"



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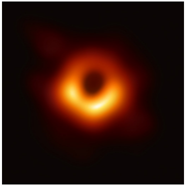

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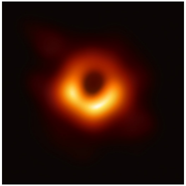

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Lorentzian analogues of well-known Riemannian results like Bonnet-Myers

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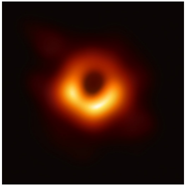

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Importance has been recognized: 2020 Physics Nobel Prize for Roger Penrose (Penrose singularity theorem (1965))

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Riemannian context

Theorem (Sprouse '00)

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} > (n-1)k$ $(k < 0)$. Then for any $R, \delta > 0$ there exists $\varepsilon = \varepsilon(n, k, R, \delta)$ such that if

$$\sup_x \frac{1}{vol(B(x,R))} \int_{B(x,R)} ((n-1) - \mathrm{Ric}_-)_+ dV < \varepsilon$$

then (M, g) is compact, with $diam(M) < \pi + \delta$.

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Proof of [S] based on going from volume integrals to integrals along geodesics via:

"Segment inequality" (Cheeger-Colding '96)

Let (M,g) be Riemannian with $\mathrm{Ric} \geq -(n-1)$. For $A_1, A_2 \subseteq B(p,r)$ with $r \leq R$ and ϕ non-negative, continuous

$$\int_{A_1 \times A_2} \mathcal{F}_\phi(x,y)\, dvol(x) dvol(y) \leq r\, C(n,R)(|A_1| + |A_2|) \int_{B(p,2R)} \phi\, dvol$$

where $\mathcal{F}_\phi(x,y) := \sup_{\gamma \in \Gamma(x,y)} \int_0^{|xy|} \phi(\gamma(s)) ds$

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The proof of [CG] uses a two-sided Bishop-Gromov estimate for the area element.

Goal: Apply similar strategy to Hawking's Singularity theorem

(A variant of) The Hawking singularity theorem (Hawking ('67))

A (smooth) spacetime is future timelike geodesically incomplete if

1. $\mathrm{Ric}(X, X) \geq 0$ for every timelike vector X
2. There exists a smooth spacelike Cauchy hypersurface Σ in M
3. The mean curvature vector of Σ is past pointing timelike, i.e. $\exists \beta > 0$ with $H := -g(\vec{H}, \vec{n}) < \beta < 0$ (where $\vec{n}$=future unit normal to Σ and H=mean curvature w.r.t. $\vec{n}$).

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Remark: This is an inherently asymmetric situation, so

Hawking's Singularity Theorem $\leftrightsquigarrow$ Myers Theorem with boundary

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4. existence of causal geodesics γ from p to q with $L(\gamma) = \max$. Morally: "Lorentzian analogue to assuming completeness in the Riemannian setting"
5. If Σ is smooth and spacelike, then we also get existence of length maximizing past directed timelike geodesics from p to Σ for any $p \in I^+(\Sigma)$

[a] 3. is equivalent to (GH). Hounnonkpe-Minguzzi '19: For $n \geq 3$ and M non-compact it is sufficient to assume $J^+(p) \cap J^-(q)$ $\forall p, q \in M$.

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where $h_{ij}(t,\mathbf{x})dx^i dx^j$ is a family of Riemannian metrics on Σ, $d\sigma(t,\mathbf{x})$ the volume element and $H(t,\mathbf{x})$ the mean curvature (w.r.t. the future unit normal VF $U(t,\mathbf{x})$) of $\{t\} \times \Sigma \hookrightarrow M$

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- Define

$$\mathrm{Reg}_\eta^+(T) := \{\mathbf{x} \in \Sigma : \gamma_{\vec{n}_\mathbf{x}} \text{ does not have a cut point before } T + \eta\} \subseteq \Sigma$$

Two-sided comparison estimates

Same setup as before + $\mathrm{Ric}(U, U) \geq n\kappa$ for some $\kappa < 0$.

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For all $T, \eta > 0$ *there are positive constants* $C^{\square -} = C^{\square -}(n, \kappa, \eta) = (n-1)\sqrt{|\kappa|}\coth(\eta\sqrt{|\kappa|}) > 0$ *and* $C^{\square +} = C^{\square +}(n, \kappa, T)$ *and such that for all* $t \in [0, T]$ *and all* $\mathbf{x} \in \mathrm{Reg}^{+}_{\eta}(T)$,

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Lemma (Area comparison)

For all $T, \eta > 0$ *there are positive constants* $C^{A+} = C^{A+}(n,\kappa,T)$ *and*

$$C^{A-} = C^{A-}(n,\kappa,T,\eta) = \sinh(\eta\sqrt{|\kappa|})^{n-1}\sinh(\sqrt{|\kappa|}(T+\eta))^{-(n-1)}$$

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$$C^{A-} \leq \mathcal{A}(t,\mathbf{x}) \leq C^{A+}.$$

Lorentzian segment type inequality

Same assumptions as before. Define

$$\mathcal{F}_f^T : \Sigma \to [0,\infty], \quad \mathcal{F}_f^T(\mathbf{x}) := \int_0^{\min(T, s^+(\mathbf{x}))} f(\exp_\Sigma^+(t,\mathbf{x}))dt$$

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Then for any $\eta > 0$ and any measurable subset $B \subset \mathrm{Reg}_\eta^+(T)$ such that $0 < \sigma B < \infty$ we have that

$$\inf_{\mathbf{x}\in B} \mathcal{F}_f^T \leq \frac{1}{C^{A-}(n,\kappa,T,\eta)} \frac{1}{\sigma B} \int_{\Omega_T^+(B)} f \, dvol_g,$$

where $C^{A-} = C^{A-}(n,\kappa,T,\eta)$ is the backward area comparison constant and σ is the (Riemannian) volume measure on Σ.

Note: Again asymmetric $\rightsquigarrow$ not a real segment inequality

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Assume that $\mathrm{Ric}(v, v) \geq n\kappa$ for all unit timelike $v \in TM$ and $H \leq \beta$ on Σ for some constants $0 > \kappa, \beta \in \mathbb{R}$ with $\beta \geq -(n-1)\sqrt{|\kappa|}$.[a]

Let $B \subseteq \Sigma$ with $0 < \sigma(B) < \infty$. If for any $0 < T, \eta \in \mathbb{R}$

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In other words:

$$B \subseteq \mathrm{Reg}^+_\eta(T) \implies \sigma(B) \leq \frac{\int_{\Omega^+_T(B)} |\,\mathrm{Ric}(U_p,U_p)_-\,|\, dvol_g(p)}{C^{A-}(n,\kappa,\eta,T)(|\beta| - \frac{n-1}{T})} \qquad \forall T, \eta > 0$$

Consequence 2: An actual singularity theorem

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Let $C^{\square}_{\max}(n, \kappa, \eta) = (n-1)\sqrt{|\kappa|}\coth(\eta\sqrt{|\kappa|})$. *If:*

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(ii) *There exist* $Q_1, Q_2 > 0$ *such that for all* $F \in C_c^\infty(\mathcal{M}^+)$*, we have the following integral bound on the Ricci tensor:*

$$\int_M \mathrm{Ric}(U_p, U_p) F(p)^2 \, dvol_g(p) \geq -Q_1 \|F\|^2_{L^2(M)} - Q_2 \|U(F)\|^2_{L^2(M)}.$$

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(iii) *the mean curvature of* Σ *satisfies*

$$-H \geq \min\left\{(n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}\tau), \nu_*(n,\kappa,\tau)\right\}$$

everywhere on Σ, *where*

$$\nu_*(n,\kappa,\tau) := \min_{\{(T,\eta):T+\eta=\tau\}} \min_{\tau_0\in(0,T)} \left(Q_1 + Q_2\frac{(C^{\square}_{\max}(n,\kappa,\eta))^2}{4} + Q_2\frac{C^{\square}_{\max}(n,\kappa,\eta)}{2T}\right)\frac{T}{3} + Q_2\left(1+\frac{TC^{\square}_{\max}(n,\kappa,\eta)}{2}\right)\left(\frac{1}{\tau_0}+\frac{1}{T-\tau_0}\right) + n|\kappa|\tau_0\frac{2}{3} + \frac{n-1}{T-\tau_0}.$$

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(i) *There is* $\kappa < 0$ *such that* $\mathrm{Ric}(v,v) \geq n\kappa$ *for all unit timelike* $v \in TM$.

(ii) *There exist* $Q_1, Q_2 > 0$ *such that for all* $F \in C_c^\infty(\mathcal{M}^+)$, *we have the following integral bound on the Ricci tensor:*

$$\int_M \mathrm{Ric}(U_p,U_p)F(p)^2\, dvol_g(p) \geq -Q_1\|F\|^2_{L^2(M)} - Q_2\|U(F)\|^2_{L^2(M)}.$$

(iii) *the mean curvature of* Σ *satisfies*

$$-H \geq \min\left\{(n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}\tau), \nu_*(n,\kappa,\tau)\right\}$$

everywhere on Σ, *where*

$$\nu_*(n,\kappa,\tau) := \min_{\{(T,\eta):T+\eta=\tau\}} \min_{\tau_0\in(0,T)} \left(Q_1 + Q_2\frac{(C^{\square}_{\max}(n,\kappa,\eta))^2}{4} + Q_2\frac{C^{\square}_{\max}(n,\kappa,\eta)}{2T}\right)\frac{T}{3}$$
$$+ Q_2\left(1+\frac{TC^{\square}_{\max}(n,\kappa,\eta)}{2}\right)\left(\frac{1}{\tau_0}+\frac{1}{T-\tau_0}\right) + n|\kappa|\tau_0\frac{2}{3} + \frac{n-1}{T-\tau_0}.$$

Then no future-directed timelike curve emanating from Σ *has length greater than* τ *and hence* (M,g) *is future timelike geodesically incomplete.*

Proof idea

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$$\int_0^\tau \mathrm{Ric}(\dot\gamma(t), \dot\gamma(t)) f(t)^2 dt = \int_M \mathrm{Ric}(U_p, U_p) F(p)^2 \, dvol_g(p)$$

$$\geq -Q_1 \|F\|^2_{L^2(M)} - Q_2 \|\dot F\|^2_{L^2(M)} = -Q_1 \|f\|^2_{L^2([0,\tau])} - Q_2 \left|\left| \dot f + f \frac{(\partial_t \mathcal{A})(., \mathbf{x}_0)}{2\mathcal{A}(., \mathbf{x}_0)} \right|\right|^2_{L^2([0,\tau])}$$

$$\geq -\left(Q_1 + \frac{Q_2 (C^\Box_{\max})^2}{4} + \frac{Q_2 C^\Box_{\max}}{2T} \right) \|f\|^2_{L^2(\mathbb{R})} - \left(Q_2 + \frac{Q_2 T C^\Box_{\max}}{2} \right) \|\dot f\|^2_{L^2(\mathbb{R})}.$$

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- Applications to low-regularity singularity theorems:

Work in progress j/w Calisti, Hafemann, Kunzinger, Steinbauer: Hawking's singularity theorem for (locally) Lipschitz metrics

Thank you for your attention!