How to cook involutions on moduli space of sheaves of K3 surfaces

from a derived category side

Y. Prieto–Montanez
ICTP
(work in progress with D. Faenzi and G. Menet)

Workshop *"Geometry of HyperKähler Varieties"*
Hangzhou, China
September 6, 2023

Notation

Notation

- X denotes a projective K3 surface with a $H \in Pic(X)$ an ample class with $H^2 = 2(g-1) > 0$.

Notation

- X denotes a projective K3 surface with a $H \in Pic(X)$ an ample class with $H^2 = 2(g-1) > 0$.
- Moduli spaces of sheaves of K3 surfaces are deformation equivalent to the Hilbert scheme of K3 surfaces.

Notation

- X denotes a projective K3 surface with a $H \in Pic(X)$ an ample class with $H^2 = 2(g-1) > 0$.
- Moduli spaces of sheaves of K3 surfaces are deformation equivalent to the Hilbert scheme of K3 surfaces.
- $D^b(X)$ denotes the derived category of bounded complex of coherent sheaves.

Main ingredients from derived category:

Main ingredients from derived category:

The spherical twist

Main ingredients from derived category:

The spherical twist

Let $\mathcal{S}$ be a spherical object in $\boldsymbol{D}^b(X)$. The **spherical twist** associated to $\mathcal{S}$ is the auto-equivalence $T_{\mathcal{S}}$ on $\boldsymbol{D}^b(X)$ given by the FM-transform with kernel $\mathcal{P}_{\mathcal{S}}$ defined as the cone

$$\mathcal{P}_{\mathcal{S}} := C(q^*\mathcal{S}^\vee \otimes p^*\mathcal{S} \to \mathcal{O}_\Delta),$$

where p, q are the natural projections $X \times X \to X$.

Main ingredients from derived category:

The spherical twist

Let $\mathcal{S}$ be a spherical object in $\mathbf{D}^b(X)$. The **spherical twist** associated to $\mathcal{S}$ is the auto-equivalence $T_{\mathcal{S}}$ on $\mathbf{D}^b(X)$ given by the FM-transform with kernel $\mathcal{P}_{\mathcal{S}}$ defined as the cone

$$\mathcal{P}_{\mathcal{S}} := C(q^*\mathcal{S}^\vee \otimes p^*\mathcal{S} \to \mathcal{O}_\Delta),$$

where p, q are the natural projections $X \times X \to X$.

Note that $\mathcal{P}_{\mathcal{S}}$, is an object that completes

$$q^*\mathcal{S}^\vee \otimes p^*\mathcal{S} \longrightarrow \mathcal{O}_\Delta \longrightarrow \mathcal{P}_{\mathcal{S}} \longrightarrow q^*\mathcal{S}^\vee \otimes p^*\mathcal{S}[1]$$

So,

$$T_{\mathcal{S}}(\mathcal{E}) := C(\bigoplus_i \mathrm{Hom}(\mathcal{S}, \mathcal{E}[i]) \otimes \mathcal{S}[-i] \xrightarrow{ev} \mathcal{E}), \ \forall \mathcal{E} \in \mathbf{D}^b(X).$$

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$,

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$, then $\mathcal{E}^\vee$ is obtained as

$$\cdots \to \mathcal{H}om(\mathcal{E}^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i-1}, \mathcal{O}_X) \to \cdots$$

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$, then $\mathcal{E}^\vee$ is obtained as

$$\cdots \to \mathcal{H}om(\mathcal{E}^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i-1}, \mathcal{O}_X) \to \cdots$$

Note that if X is regular, then the **derived dual** $\mathcal{E}^\vee := \boldsymbol{R}\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \in \boldsymbol{D}^b(X)$ for any $\mathcal{E} \in \boldsymbol{D}^b(X)$ where $\mathcal{E}^\vee$ is not the usual dual sheaf $\mathbf{Hom}(\mathcal{E}, \mathcal{O}_X)$.

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$, then $\mathcal{E}^\vee$ is obtained as

$$\cdots \to \mathcal{H}om(\mathcal{E}^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i-1}, \mathcal{O}_X) \to \cdots$$

Note that if X is regular, then the **derived dual** $\mathcal{E}^\vee := \boldsymbol{R}\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \in \boldsymbol{D}^b(X)$ for any $\mathcal{E} \in \boldsymbol{D}^b(X)$ where $\mathcal{E}^\vee$ is not the usual dual sheaf $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$. Moreover, under the assumption of projectivity,

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$, then $\mathcal{E}^\vee$ is obtained as

$$\cdots \to \mathcal{H}om(\mathcal{E}^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i-1}, \mathcal{O}_X) \to \cdots$$

Note that if X is regular, then the **derived dual** $\mathcal{E}^\vee := \boldsymbol{R}\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \in \boldsymbol{D}^b(X)$ for any $\mathcal{E} \in \boldsymbol{D}^b(X)$ where $\mathcal{E}^\vee$ is not the usual dual sheaf $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$. Moreover, under the assumption of projectivity, due to the compatibilities of derived tensor product and derived local Hom,

more ingredients: the derived dual

Let $\mathcal{E}$ be a complex:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^{i} \to \mathcal{E}^{i+1} \to \cdots$$

with locally free sheaves $\mathcal{E}^i$, then $\mathcal{E}^\vee$ is obtained as

$$\cdots \to \mathcal{H}om(\mathcal{E}^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}^{i-1}, \mathcal{O}_X) \to \cdots$$

Note that if X is regular, then the **derived dual** $\mathcal{E}^\vee := \mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \in \mathbf{D}^b(X)$ for any $\mathcal{E} \in \mathbf{D}^b(X)$ where $\mathcal{E}^\vee$ is not the usual dual sheaf $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$. Moreover, under the assumption of projectivity, due to the compatibilities of derived tensor product and derived local Hom, we obtain

$$\mathcal{F}^\vee \otimes \mathcal{E} \simeq \mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{E} \simeq \mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{E}).$$

more ingredients: tensoring by line bundles,...

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X.

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X. Defines $-\otimes L$ as the map on $\boldsymbol{D}^b(X)$ acting by

$$-\otimes L : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X. Defines $-\otimes L$ as the map on $\boldsymbol{D}^b(X)$ acting by

$$-\otimes L : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

The map $-\otimes L$ can be seen as a FM-transform with Kernel $\iota_* L$.

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X. Defines $-\otimes L$ as the map on $\mathbf{D}^b(X)$ acting by

$$-\otimes L : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

The map $-\otimes L$ can be seen as a FM-transform with Kernel $\iota_* L$.

(Optional ingredients:) (?)

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X. Defines $- \otimes L$ as the map on $\mathbf{D}^b(X)$ acting by

$$- \otimes L : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

The map $- \otimes L$ can be seen as a FM-transform with Kernel $\iota_* L$.

(Optional ingredients:)

The shift functor, $\mathcal{E} \to \mathcal{E}[1]$, is the FM-transform with kernel $\mathcal{O}_\Delta[1]$.

more ingredients: tensoring by line bundles,...

Let L be a line bundle of X. Defines $-\otimes L$ as the map on $\mathbf{D}^b(X)$ acting by

$$-\otimes L : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

The map $-\otimes L$ can be seen as a FM-transform with Kernel $\iota_* L$.

(Optional ingredients:)

The shift functor, $\mathcal{E} \to \mathcal{E}[1]$, is the FM-transform with kernel $\mathcal{O}_\Delta[1]$.
The Serre functor for a K3 surface, is the FM-transform with kernel $\mathcal{O}_\Delta[2]$.

Recipe of an involution

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object and $L \in \mathrm{Pic}(X)$.

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object and $L \in \mathrm{Pic}(X)$. Define $\Phi_{\mathcal{S},L}$ as the following functor on $\boldsymbol{D}^b(X)$:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},L}(\mathcal{E}) = \boldsymbol{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object and $L \in \mathrm{Pic}(X)$. Define $\Phi_{\mathcal{S},L}$ as the following functor on $\mathbf{D}^b(X)$:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},L}(\mathcal{E}) = \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

The functor $\Phi_{\mathcal{S},L}$ on an element $\mathcal{E} \in \mathbf{D}^b(X)$ is explicitly given by

$$\Phi_{\mathcal{S},L} : \mathcal{E} \overset{T_{\mathcal{S}}}{\mapsto} T_{\mathcal{S}}(\mathcal{E}) \overset{()^\vee}{\mapsto} \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X) \overset{-\otimes L}{\mapsto} \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object and $L \in \mathrm{Pic}(X)$. Define $\Phi_{\mathcal{S},L}$ as the following functor on $\mathbf{D}^b(X)$:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},L}(\mathcal{E}) = \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

The functor $\Phi_{\mathcal{S},L}$ on an element $\mathcal{E} \in \mathbf{D}^b(X)$ is explicitly given by

$$\Phi_{\mathcal{S},L} : \mathcal{E} \overset{T_{\mathcal{S}}}{\mapsto} T_{\mathcal{S}}(\mathcal{E}) \overset{()^\vee}{\mapsto} \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X) \overset{-\otimes L}{\mapsto} \mathbf{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

Is $\Phi_{\mathcal{S},L}$ well-defined on $\mathbf{D}^b(X)$?

Recipe of an involution

Definition

Let $\mathcal{S}$ be a spherical object and $L \in \mathrm{Pic}(X)$. Define $\Phi_{\mathcal{S},L}$ as the following functor on $\boldsymbol{D}^b(X)$:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},L}(\mathcal{E}) = \boldsymbol{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

The functor $\Phi_{\mathcal{S},L}$ on an element $\mathcal{E} \in \boldsymbol{D}^b(X)$ is explicitly given by

$$\Phi_{\mathcal{S},L} : \mathcal{E} \overset{T_{\mathcal{S}}}{\mapsto} T_{\mathcal{S}}(\mathcal{E}) \overset{()^{\vee}}{\mapsto} \boldsymbol{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X) \overset{-\otimes L}{\mapsto} \boldsymbol{R}\mathrm{Hom}(T_{\mathcal{S}}(\mathcal{E}), L).$$

Is $\Phi_{\mathcal{S},L}$ well-defined on $\boldsymbol{D}^b(X)$? Is it already an involution?

Ideal prototype.

$$
\begin{array}{ccc}
\mathbf{D}^b(X) & \xrightarrow{\quad\Phi\quad} & \mathbf{D}^b(X) \\
\downarrow & & \downarrow \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\quad\Phi^H\quad} & \widetilde{H}(X,\mathbb{Z}) \\
\downarrow & & \downarrow \\
v^\perp & \xrightarrow{\quad\Phi^H_v\quad} & v^\perp \\
\downarrow & & \downarrow \\
H^2(M_H(X,v),\mathbb{Z}) & \xrightarrow{\Phi^*} & H^2(M_H(X,v),\mathbb{Z})
\end{array}
$$

Ideal prototype.

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\quad \Phi \quad} & D^b(X) \\
\downarrow & & \downarrow \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\quad \Phi^H \quad} & \widetilde{H}(X,\mathbb{Z}) \\
\downarrow & & \downarrow \\
v^{\perp} & \xrightarrow{\quad \Phi^H_v \quad} & v^{\perp} \\
\downarrow & & \downarrow \\
H^2(M_H(X,v),\mathbb{Z}) & \xrightarrow{\ \Phi^* \ } & H^2(M_H(X,v),\mathbb{Z})
\end{array}
$$

Orlov: $\exists\, P \in D^b(X \times X)$ st $\Phi = FM_P$

$\Downarrow$

$\alpha \mapsto p_* q^* \alpha . v(P)$

Ideal prototype.

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\quad\Phi\quad} & D^b(X) \\
\downarrow & & \downarrow \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\quad\Phi^H\quad} & \widetilde{H}(X,\mathbb{Z}) \\
\downarrow & & \downarrow \\
v^\perp & \xrightarrow{\quad\Phi^H_v\quad} & v^\perp \\
\downarrow & & \downarrow \\
H^2(M_H(X,v),\mathbb{Z}) & \xrightarrow{\Phi^*} & H^2(M_H(X,v),\mathbb{Z})
\end{array}
$$

v? st $\Phi^H(v) = v$?

Ideal prototype.

$$
\begin{array}{ccc}
\mathbf{D}^b(X) & \xrightarrow{\quad\Phi\quad} & \mathbf{D}^b(X) \\
\downarrow & & \downarrow \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\quad\Phi^H\quad} & \widetilde{H}(X,\mathbb{Z}) \\
\updownarrow & & \updownarrow \\
v^\perp & \xrightarrow{\quad\Phi^H_v\quad} & v^\perp \\
\downarrow & & \downarrow \\
H^2(M_H(X,v),\mathbb{Z}) & \xrightarrow{\Phi^*} & H^2(M_H(X,v),\mathbb{Z})
\end{array}
$$

Ideal prototype.

$$\begin{array}{ccc}
D^b(X) & \xrightarrow{\quad \Phi \quad} & D^b(X) \\
\downarrow & & \downarrow \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\quad \Phi^H \quad} & \widetilde{H}(X,\mathbb{Z}) \\
\updownarrow & & \updownarrow \\
v^\perp & \xrightarrow{\quad \Phi^H_v \quad} & v^\perp \\
\cong \downarrow & & \downarrow \cong \\
H^2(M_H(X,v),\mathbb{Z}) & \xrightarrow{\ \Phi^* \ } & H^2(M_H(X,v),\mathbb{Z})
\end{array}$$

$+\ v^2 > 0$

Mukai's isometry (Mu, OG, Y)

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector.

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

+ H is v-generic.

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$.

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

$\Rightarrow$ [M, G-H, O6, Ψ] HK mfld of $K3^{[\frac{v^2+2}{2}]}$-type.

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$? $\phi_{S,L}$ on $\widetilde{H}(X,\mathbb{Z})$ is an involution BUT $L \in v(S)^{\perp}$, $\widetilde{NS}(X)$

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a H- semi-stable sheaf for all $\mathcal{E} \in M(v)$?

μ-stable $\Rightarrow$ stable $\Rightarrow$ semistable $\not\Rightarrow$ μ-semistable.

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it? $\mathcal{E}$? st $H^1(X, \mathcal{E}) = 0$? (Weak Brill-Noether condition)

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it?
4. What is the Mukai vector $v(\Phi_{\mathcal{S},L}(\mathcal{E}))$?

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it?
4. What is the Mukai vector $v(\Phi_{\mathcal{S},L}(\mathcal{E}))$?
5. What is the induced map $\Phi^*_{\mathcal{S},L}$ on $H^2(M(v), \mathbb{Z})$?

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it?
4. What is the Mukai vector $v(\Phi_{\mathcal{S},L}(\mathcal{E}))$?
5. What is the induced map $\Phi^*_{\mathcal{S},L}$ on $H^2(M(v), \mathbb{Z})$? Is it symplectic

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X, \mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it?
4. What is the Mukai vector $v(\Phi_{\mathcal{S},L}(\mathcal{E}))$?
5. What is the induced map $\Phi^*_{\mathcal{S},L}$ on $H^2(M(v), \mathbb{Z})$? Is it symplectic or anti-symplectic? $\phi^*_{S,L}\, w_{M(v)} = \pm\, w_{M(v)}$

Is $\Phi_{\mathcal{S},L}$ well-defined on the moduli space of sheaves?

Let $v = (v_o, v_1 H, v_2) \in \widetilde{H}(X,\mathbb{Z})$ be a Mukai vector. Assume that v is primitive and $v^2 > 0$. The Mukai vector associated to a sheaf $\mathcal{E}$ is given by $v(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - rk(\mathcal{E}))$.
Set by

$$M(v) := \{\mathcal{E} \text{ is } H\text{-ss sheaf with } v(\mathcal{E}) = v\} \subset \mathrm{Coh}(X)$$

1. Is $\Phi_{\mathcal{S},L}(\mathcal{E}) \in \mathrm{Coh}(X)$ for all $\mathcal{E} \in \mathrm{Coh}(X)$?
2. What is the order of $\Phi_{\mathcal{S},L}$?
3. Is $\Phi_{\mathcal{S},L}(\mathcal{E})$ a semi-stable sheaf for all $\mathcal{E} \in M(v)$? If not, where is it?
4. What is the Mukai vector $v(\Phi_{\mathcal{S},L}(\mathcal{E}))$?
5. What is the induced map $\Phi^*_{\mathcal{S},L}$ on $H^2(M(v),\mathbb{Z})$? Is it symplectic or anti-symplectic? $\phi^*_{S,L}\, w_{M(v)} = \pm\, w_{M(v)}$

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

From $\Phi \in \mathbf{D}^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$,

From $\Phi \in \mathbf{D}^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T^H_{\mathcal{S}}$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T_{\mathcal{S}}^H$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s\rangle \cdot v(\mathcal{S}).$$

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T^H_{\mathcal{S}}$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s\rangle \cdot v(\mathcal{S}).$$

- If $\Phi = ()^\vee$,

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T^H_{\mathcal{S}}$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s\rangle \cdot v(\mathcal{S}).$$

- If $\Phi = ()^\vee$, then $\Phi^H = \mathbb{D} := (x, yD, z) \mapsto (x, -yD, z)$.

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X, \mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T_{\mathcal{S}}^H$ on $\widetilde{H}(X, \mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s \rangle \cdot v(\mathcal{S}).$$

- If $\Phi = ()^{\vee}$, then $\Phi^H = \mathbb{D} := (x, yD, z) \mapsto (x, -yD, z)$.
- If $\Phi = - \otimes L$,

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T^H_{\mathcal{S}}$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s \rangle \cdot v(\mathcal{S}).$$

- If $\Phi = ()^\vee$, then $\Phi^H = \mathbb{D} := (x, yD, z) \mapsto (x, -yD, z)$.
- If $\Phi = - \otimes L$, then Φ^H is given by

$$(x, yD, z) \overset{-\otimes L^H}{\mapsto} (x, xL + yD, \frac{L^2}{2}x + (D \cdot L)y + xz),$$

From $\Phi \in D^b(X)$ to $\Phi^H \in \mathcal{O}(\widetilde{H}(X,\mathbb{Z}))$

- If $\Phi = T_{\mathcal{S}}$, then its induced action on cohomology $T_{\mathcal{S}}^H$ on $\widetilde{H}(X,\mathbb{Z})$ is given by the reflection R_s in the hyperplane orthogonal to the (-2)-class $s \in NS(X)$,

$$R_s : w \mapsto w + \langle w, s\rangle \cdot v(\mathcal{S}).$$

- If $\Phi = ()^\vee$, then $\Phi^H = \mathbb{D} := (x, yD, z) \mapsto (x, -yD, z)$.
- If $\Phi = - \otimes L$, then Φ^H is given by

$$(x, yD, z) \overset{-\otimes L^H}{\mapsto} (x, xL + yD, \frac{L^2}{2}x + (D \cdot L)y + xz),$$

the multiplication (cup product of cohomology classes) with the Chern character $\mathrm{ch}(L) = \exp(c_1(L)) = (1, L, L^2/2)$.

A sheaf is sending in a sheaf

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$.

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf,

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$,

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf.

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},d}(\mathcal{E}) = R\mathcal{H}om_X(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X(dH)).$$

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},d}(\mathcal{E}) = R\mathcal{H}om_X(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v,

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{S,d}(\mathcal{E}) = R\mathcal{H}om_X(T_S(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v, we obtain

$$R\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(dH)) \to R\mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \Phi_{S,d}(\mathcal{E})$$

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S},d}(\mathcal{E}) = R\mathcal{H}om_X(T_\mathcal{S}(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v, we obtain

$$R\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(dH)) \to R\mathrm{Hom}_X(\mathcal{S},\mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{O}_X(dH) \to \Phi_{\mathcal{S},d}(\mathcal{E})$$

Hence,

$$\mathcal{H}^{-1}(\Phi_{\mathcal{S},d}(\mathcal{E})) \to \mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(dH)) \to \mathrm{Hom}_X(\mathcal{S},\mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{O}_X(dH) \to \mathcal{H}^0(\Phi_{\mathcal{S},d}(\mathcal{E})) \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH)) \to 0$$

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \stackrel{e_v}{\to} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{S,d}(\mathcal{E}) = R\mathcal{H}om_X(T_S(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v, we obtain

$$R\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(dH)) \to R\mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \Phi_{S,d}(\mathcal{E})$$

Hence,

$$\mathcal{H}^{-1}(\Phi_{S,d}(\mathcal{E})) \to \mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(dH)) \to \mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \mathcal{H}^0(\Phi_{S,d}(\mathcal{E})) \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH)) \to 0$$

Under the assumption that $\mathrm{coker}(e_v)$ is torsion sheaf, we have $\mathcal{H}^{-1}(\Phi_{\mathcal{S},d}(\mathcal{E})) = 0$.

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{S,d}(\mathcal{E}) = R\mathcal{H}om_X(T_S(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v, we obtain

$$R\mathcal{H}om_X(\mathcal{E},\mathcal{O}_X(dH)) \to R\mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \Phi_{S,d}(\mathcal{E})$$

Hence,

$$\mathcal{H}^{-1}(\Phi_{S,d}(\mathcal{E})) \to \mathcal{H}om_X(\mathcal{E},\mathcal{O}_X(dH)) \to \mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \mathcal{H}^0(\Phi_{S,d}(\mathcal{E})) \to \mathcal{E}xt^1(\mathcal{E},\mathcal{O}_X(dH)) \to 0$$

Under the assumption that $\mathrm{coker}(e_v)$ is torsion sheaf, we have $\mathcal{H}^{-1}(\Phi_{\mathcal{S},d}(\mathcal{E})) = 0$. Since $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^p(\mathcal{E},\mathcal{O}_X) = 0$ for all $p \geq 2$.

A sheaf is sending in a sheaf

Set by $L = \mathcal{O}_X(dH)$ and by $\Phi_{\mathcal{S},d} := \Phi_{\mathcal{S},\mathcal{O}_X(dH)}$. Assume that $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, and $\mathrm{coker}(\mathrm{Hom}_X(\mathcal{S},\mathcal{E}) \otimes \mathcal{S} \xrightarrow{e_v} \mathcal{E})$ is a torsion sheaf. We can compute $\Phi_{\mathcal{S},d}$ as:

$$\mathcal{E} \mapsto \Phi_{S,d}(\mathcal{E}) = R\mathcal{H}om_X(T_S(\mathcal{E}), \mathcal{O}_X(dH)).$$

Take the dual by $\mathcal{O}_X(dH)$ of the distinguish triangle associated to e_v, we obtain

$$R\mathcal{H}om_X(\mathcal{E},\mathcal{O}_X(dH)) \to R\mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \Phi_{S,d}(\mathcal{E})$$

Hence,

$$\mathcal{H}^{-1}(\Phi_{S,d}(\mathcal{E})) \to \mathcal{H}om_X(\mathcal{E},\mathcal{O}_X(dH)) \to \mathrm{Hom}_X(S,\mathcal{E})^\vee \otimes S^\vee \otimes \mathcal{O}_X(dH) \to \mathcal{H}^0(\Phi_{S,d}(\mathcal{E})) \to \mathcal{E}xt^1(\mathcal{E},\mathcal{O}_X(dH)) \to 0$$

Under the assumption that $\mathrm{coker}(e_v)$ is torsion sheaf, we have $\mathcal{H}^{-1}(\Phi_{\mathcal{S},d}(\mathcal{E})) = 0$. Since $\mathcal{E}$ is a torsion free sheaf, $\mathrm{Ext}^p(\mathcal{E},\mathcal{O}_X) = 0$ for all $p \geq 2$.

So,

$$\Phi_{\mathcal{S},d}(\mathcal{E}) = \mathcal{H}^0(\Phi_{\mathcal{S},d}(\mathcal{E})).$$

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

Assume that $\mathbf{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$,

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

Assume that $\mathbf{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$, then

Case $d = 0$:

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

Assume that $\mathbf{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$, then

Case $d = 0$: $v(\Phi_{\mathcal{S},0}(\mathcal{E})) = v$ iff $\mathcal{S} = \mathcal{O}_X$ and $v_0 = v_2$.

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

Assume that $\mathbf{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$, then

Case $d = 0$: $v(\Phi_{\mathcal{S},0}(\mathcal{E})) = v$ iff $\mathcal{S} = \mathcal{O}_X$ and $v_0 = v_2$.

Case $d = 1$:

The Mukai vector of $\Phi_{\mathcal{S},d}(\mathcal{E})$.

Explicitly computations show that

$$v(\Phi_{\mathcal{S},d}(\mathcal{E})) = D \circ R_{\mathcal{S}}(v) \otimes \mathcal{O}_X(dH).$$

Implying that $d = 0$ or $d = 1$.

Assume that $\mathbf{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$, then

Case $d = 0$: $v(\Phi_{\mathcal{S},0}(\mathcal{E})) = v$ iff $\mathcal{S} = \mathcal{O}_X$ and $v_0 = v_2$.

Case $d = 1$: $v(\Phi_{\mathcal{S},1}(\mathcal{E})) = (v_0, v_1 H, v_2)$ iff $v(\mathcal{S}) = (2, 1, g/2)$ and $2v_2 = (2g-2)v_1 - v_0(g/2 - 1)$.

when $\Phi_{\mathcal{S},d}(\mathcal{E})$ is slope-stable?

Key points:

i) $\phi(\mathcal{E})$ is torsion free? Yes ↗ $Ext^1(S,\mathcal{E}) = Ext^2(S,\mathcal{E}) = 0$ and ↘ Coker ev is supp. on isolated points.

Lemma

ii) In addition:

$\phi(\mathcal{E})$ is μ-stable? Yes if Ker(ev) is μ-stable!

when $\Phi_{\mathcal{S},d}(\mathcal{E})$ is slope-stable?

Key points:

i) $\phi(\mathcal{E})$ is torsion free? Yes ↗ $Ext^1(S,\mathcal{E}) = Ext^2(S,\mathcal{E}) = 0$ and ↘ Coker ev is supp. on isolated points.

Lemma

ii) In addition:

$\phi(\mathcal{E})$ is μ-stable? Yes if $Ker(ev)$ is μ-stable!

Again under ass.

$Ext^1(S,\mathcal{E}) = Ext^2(S,\mathcal{E}) = 0$ & $Hom(S,\mathcal{E}) \neq 0$

+ $v(\phi_{S,d}(\mathcal{E})) = v(\mathcal{E})$ & $c_1(\mathcal{E}) = H$

when $\Phi_{\mathcal{S},d}(\mathcal{E})$ is slope-stable?

Key points:

i) $\phi(\mathcal{E})$ is torsion free? Yes ↗ $Ext^1(S,\mathcal{E}) = Ext^2(S,\mathcal{E}) = 0$ and ↘ Coker ev is supp. on isolated points.

Lemma

ii) In addition:

$\phi(\mathcal{E})$ is μ-stable? Yes if Ker(ev) is μ-stable!

Again under ass.

$Ext^1(S,\mathcal{E}) = Ext^2(S,\mathcal{E}) = 0$ & $Hom(S,\mathcal{E}) \neq 0$

Lemma

+ $v(\phi_{S,d}(\mathcal{E})) = v(\mathcal{E})$ & $c_1(\mathcal{E}) = H$

$\Rightarrow$ Coker ev is supp. on isol. points & Ker(ev) is μ-stable.

Main Results when $\operatorname{rk} \mathcal{S} = 1$

Theorem (Faenzi, Menet, P)

Let X be a projective K3 surface with $H \in \mathrm{Pic}(X)$ and $H^2 = 2(g-1)$. Let $r \geq 1$ be an integer with $r^2 \leq g < (r+1)^2$. Then, $\Phi_{\mathcal{S},d}$ is a well-defined involution on $M(r, H, r)$.

Birational involutions

Relaxing conditions on g and r, the map is a birational involution!

Corollary

Assuming $r \geq 2$ and $\dim M(v) \geq 2$. Then, $\Phi_{\mathcal{S},d}$ defines a birational involution on $M(r,1,r)$.

Birational involutions

Relaxing conditions on g and r, the map is a birational involution!

Corollary

Assuming $r \geq 2$ and $\dim M(v) \geq 2$. Then, $\Phi_{\mathcal{S},d}$ defines a birational involution on $M(r,1,r)$.

Example ($r = 1$)

Beauville's involution.

$$M(v) = M(1,H,1) = X^{[g-1]}$$

$$\ni \quad \mathcal{I}_Z(H) \Rightarrow \text{taking dual: } \mathcal{E}xt^1(\mathcal{I}_Z(H), \mathcal{O}_X) \simeq \omega_Z$$

But $Z \subset X$ of length 1 or 2

$$\Rightarrow \omega_Z \simeq \mathcal{O}_Z.$$

Birational involutions

Relaxing conditions on g and r, the map is a birational involution!

Corollary

Assuming $r \geq 2$ and $\dim M(v) \geq 2$. Then, $\Phi_{\mathcal{S},d}$ defines a birational involution on $M(r,1,r)$.

Example ($r = 1$)

Beauville's involution.

$M(v) = M(1,H,1) = X^{[g-1]}$

$\ni$

$\mathcal{J}_Z(H) \Rightarrow$ taking dual: $\mathcal{E}xt^1(\mathcal{J}_Z(H), \mathcal{O}_X) \simeq \omega_Z$

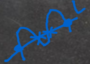
But $Z \subset X$ of length 1 or 2

$\Rightarrow \omega_Z \simeq \mathcal{O}_Z$.

If $g=3 \Rightarrow X \subset \mathbb{P}^3$ quartic $\Rightarrow Z \subset X$ length 2-subscheme

$\Rightarrow h^0(\mathcal{J}_Z(H)) = 2$

$\Rightarrow$ Let $L \subset \mathbb{P}^3$ spanned by Z:



Birational involutions

Relaxing conditions on g and r, the map is a birational involution!

Corollary

Assuming $r \geq 2$ and $\dim M(v) \geq 2$. Then, $\Phi_{\mathcal{S},d}$ defines a birational involution on $M(r, 1, r)$.

Example ($r = 1$)

Beauville's involution.

$M(v) = M(1, H, 1) = X^{[g-1]}$

$\cup$

$\mathcal{I}_Z(H) \Rightarrow$ taking dual: $Ext^1(\mathcal{I}_Z(H), \mathcal{O}_X) \simeq \omega_Z$

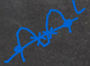
But $Z \subset X$ of length 1 or 2

$\Rightarrow \omega_Z \simeq \mathcal{O}_Z$.

If $g = 3 \Rightarrow X \subset \mathbb{P}^3$ quartic $\Rightarrow Z \subset X$ length 2-subscheme

$\Rightarrow h^0(\mathcal{I}_Z(H)) = 2$

$\rightarrow$ let $L \subset \mathbb{P}^3$ spanned by Z:



$\rightsquigarrow Z'$.

$\Rightarrow$ Coker$(e_{\mathcal{I}_Z(H)})$ is $\mathcal{O}_{Z'}$

$\Phi(\mathcal{I}_Z(H)) \simeq \mathcal{I}_{Z'}(H)$

Birational involutions

Relaxing conditions on g and r, the map is a birational involution!

Corollary

Assuming $r \geq 2$ *and* $\dim M(v) \geq 2$*. Then,* $\Phi_{\mathcal{S},d}$ *defines a birational involution on* $M(r,1,r)$*.*

Example ($r = 1$)

Beauville's involution.

Example Ex: $g=2$?

There exist two involutions on $M(2,1,2)$ for $g \geq 5$.

Involutions when $\operatorname{rk} S > 1$

Involutions when $\operatorname{rk} \mathcal{S} > 1$

Consider $\mathcal{S}$ be a spherical bundle of Mukai vector $v(\mathcal{S}) = (2, 1, g/2)$.

Involutions when $\operatorname{rk}\mathcal{S} > 1$

Consider $\mathcal{S}$ be a spherical bundle of Mukai vector $v(\mathcal{S}) = (2, 1, g/2)$.

Theorem

If $g \equiv 2 \mod 4$, then $\Phi_{\mathcal{S},1}$ is a birational involution on $S^{[\frac{g+2}{4}]}$.

Involutions when $\operatorname{rk}\mathcal{S} > 1$

Consider $\mathcal{S}$ be a spherical bundle of Mukai vector $v(\mathcal{S}) = (2, 1, g/2)$.

Theorem

If $g \equiv 2 \mod 4$, then $\Phi_{\mathcal{S},1}$ is a birational involution on $S^{[\frac{g+2}{4}]}$.

Theorem

If $g \geq 10$ and $4|(g+2)$, then, $\Phi_{\mathcal{S},1}$ is a regular involution on $M(3, 1, \frac{g+2}{4})$.

Involutions when $\operatorname{rk}\mathcal{S} > 1$

Consider $\mathcal{S}$ be a spherical bundle of Mukai vector $v(\mathcal{S}) = (2, 1, g/2)$.

Theorem

If $g \equiv 2 \mod 4$, then $\Phi_{\mathcal{S},1}$ is a birational involution on $S^{[\frac{g+2}{4}]}$.

Theorem

If $g \geq 10$ and $4|(g+2)$, then, $\Phi_{\mathcal{S},1}$ is a regular involution on $M(3, 1, \frac{g+2}{4})$.

Theorem

If $g \geq 2$ and $v = (v_0, 1, g-1-\frac{v_0}{2}(g/2-1))$ is an integral Mukai vector with $3 \leq v_0 \leq 3(g-1)$, then $\Phi_{\mathcal{S},1}$ is a birational involution on $M(v)$.

Anti-symplectic involutions

Theorem

The involution $\Phi_{\mathcal{S},d}$ on $M(r, H, r)$ such that $r \geq 1$, $r^2 \leq g < (r+1)^2$ and $H^2 = 2(g-1)$ is anti-symplectic.

Anti-symplectic involutions

Theorem

The involution $\Phi_{\mathcal{S},d}$ on $M(r,H,r)$ such that $r \geq 1$, $r^2 \leq g < (r+1)^2$ and $H^2 = 2(g-1)$ is anti-symplectic.

To prove this: $\phi^*_{S,d} \circlearrowright H^2(M(v),\mathbb{Z})$

following sketch from beginning!

Anti-symplectic involutions

Theorem

The involution $\Phi_{\mathcal{S},d}$ on $M(r,H,r)$ such that $r \geq 1$, $r^2 \leq g < (r+1)^2$ and $H^2 = 2(g-1)$ is anti-symplectic.

To prove this: $\phi^*_{S,d} \curvearrowright H^2(M(v),\mathbb{Z})$ ← endowed by the BB-quadratic form.

following sketch from Gosgunder!

$\Rightarrow$ Lattice theory: $H^2(M(v),\mathbb{Z})^{\phi^*_{S,d}}$ and $\left(H^2(M(v),\mathbb{Z})^{\phi^*_{S,d}}\right)^{\perp}$

谢谢

(thanks)

Coming soon on ArXiv...

(we hope)

谢谢

(thanks)