Small scale formations for the incompressible Boussinesq equation

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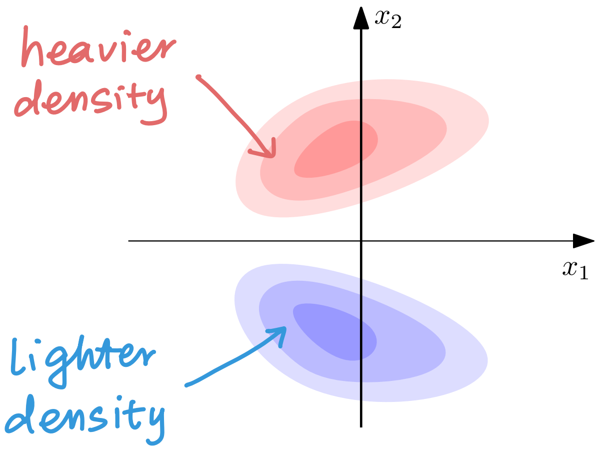
Partial Differential Equations in Fluid Dynamics

BIRS-IASM, Hangzhou

2D Boussinesq equation without density diffusivity

- $\rho(x,t)$: density of incompressible fluid.
- $u(x,t)$: velocity field of fluid.
- The spatial domain Ω is either the plane $\mathbb{R}^2$, the torus $\mathbb{T}^2$, or the strip $\mathbb{T}\times[-\pi,\pi]$.
- Throughout this talk, we consider the 2D Boussinesq equation without density diffusivity:

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0, \\ u_t + u\cdot\nabla u = -\nabla p - \rho e_2 + \nu\Delta u, \\ \nabla\cdot u = 0, \end{cases}$$



- We'll discuss the viscous case $\nu > 0$, and the inviscid case $\nu = 0$.
- Goal: In both cases, we'll prove that solution can have small scale formation (infinite-in-time growth of Sobolev norms) as $t\to\infty$.

The viscous case: global well-posedness and upper bounds

When $\nu > 0$, global-wellposedness of regular solutions is known:

- For $\Omega = \mathbb{R}^2$: global regularity by Hou–Li '05 in the space $(u, \rho) \in H^m \times H^{m-1}$ for $m \geq 3$, and Chae '06 in $H^m \times H^m$ for $m \geq 3$.
- For bounded Ω: global regularity by Lan–Pan–Zhao '11 in $H^3 \times H^3$, and Hu–Kukavica–Ziane '13 in $H^m \times H^{m-1}$ for $m \geq 2$.

Upper bounds for the global solution:

- Ju '17: For bounded Ω, $\|\rho\|_{H^1} \lesssim e^{Ct^2}$;
- Kukavica–Wang '20: For bounded Ω, $\|\rho\|_{H^1} \lesssim e^{Ct}$ and $\|u\|_{W^{2,p}} \leq C_p$; for $\mathbb{R}^2$, $\|\rho\|_{H^1} \lesssim e^{Ct^{(1+\beta)}}$.
- Kukavica–Massatt–Ziane '21: For bounded Ω, $\|\rho\|_{H^2} \leq C_\epsilon e^{\epsilon t}$, $\|u\|_{H^3} \leq C_\epsilon e^{\epsilon t}$.

What about lower bounds?

- Note that the above estimates all deal with the upper bounds of solutions.
- **Question.** What about lower bounds? Can solutions have small scale formation as $t \to \infty$?
- Lower bound by Brandolese–Schonbek '12: in $\mathbb{R}^2$, if ρ_0 does not have mean zero, $\|u(t)\|_{L^2} \sim (1+t)^{1/4}$. (This is due to potential energy converting to kinetic energy, and does not imply growth in higher derivatives)
- We are not aware of any examples of infinite-in-time growth of $\|\rho(t)\|_{H^m}$ in the literature.

Small scale formation in the viscous case

Theorem (Kiselev–Park–Y. '22, preprint)

Let $\nu > 0$, $\Omega = \mathbb{T}^2$. If the smooth initial data (ρ_0, u_0) satisfies the following

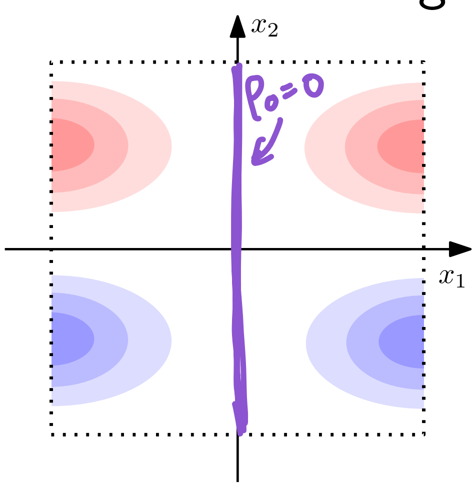
- *Symmetry assumptions:* ρ_0 *is even-odd,* u_{01} *is odd-even,* u_{02} *is even-odd.*
- *Sign assumptions:* $\rho_0 \geq 0$ *for* $x_2 \geq 0$*, and* $\rho_0 = 0$ *on the* x_2*-axis.*

Preserved for all time!

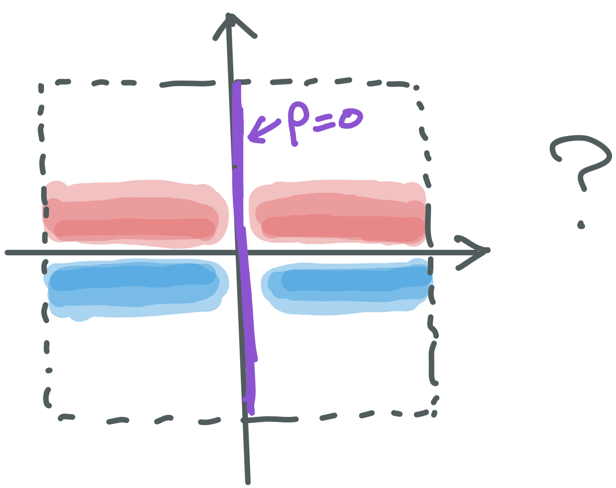
Then the global-in-time smooth solution satisfies

$$\limsup_{t\to\infty} t^{-\frac{1}{6}} \|\rho(t)\|_{\dot{H}^1} = +\infty.$$

Remark: Under these assumptions one can show $\|\rho(t)\|_{\dot{H}^1}$ has a refined sub-exponential upper bound $\exp(Ct^\alpha)$ for some $\alpha \in (0,1)$, so the growth is somewhere between algebraic and sub-exponential.



Asymptotics as $t \to \infty$? (Still open!)



Evolution of potential energy

- Define the potential energy $E_P(t) := \int_{\mathbb{T}^2} \rho x_2 dx$, and kinetic energy $E_K(t) := \int_{\mathbb{T}^2} |u|^2 dx$.
- It's well-known that the total energy is decreasing in time:

$$\frac{d}{dt}(E_P(t) + E_K(t)) = -\nu \|\nabla u(t)\|_{L^2}^2.$$

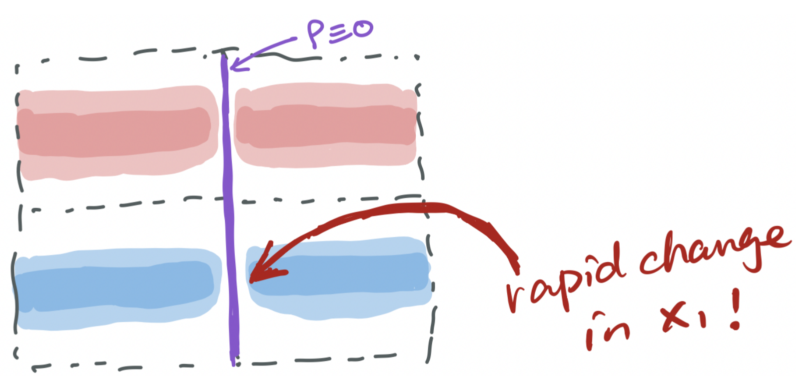
  This implies that $\int_0^\infty \|\nabla u(t)\|_{L^2}^2 dt < C(\nu, \rho_0, u_0)$.
- Since the two equations are coupled by the gravity force, we'll track the evolution of potential energy $E_P(t)$ itself.
- A quick computation gives $\frac{d}{dt} E_P(t) = \int_{\mathbb{T}^2} \rho u_2 dx$, which is uniformly bounded.
- Let's take one more time derivative:

$$\frac{d^2}{dt^2} E_P(t) = \underbrace{\sum_{i,j=1}^{2} \int_{\mathbb{T}^2} \overbrace{((-\Delta)^{-1} \partial_2 \rho)}^{\text{bounded}} \partial_i u_j \, \partial_j u_i dx}_{=:A(t)} - \underbrace{\nu \int_{\mathbb{T}^2} \nabla \rho \cdot \nabla u_2 dx}_{=:B(t)} - \|\partial_1 \rho\|_{\dot{H}^{-1}}^2.$$

- Let's take one more time derivative:

$$\frac{d^2}{dt^2}E_P(t) = \underbrace{\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}((-\Delta)^{-1}\partial_2\rho)\,\partial_i u_j\,\partial_j u_i dx}_{=:A(t)} - \nu\underbrace{\int_{\mathbb{T}^2}\nabla\rho\cdot\nabla u_2 dx}_{=:B(t)} - \|\partial_1\rho\|^2_{\dot{H}^{-1}}.$$

- Since $\int_0^\infty \|\nabla u(t)\|^2_{L^2}dt < \infty$, this implies $\int_0^\infty A(t)dt < \infty$.
- Suppose $\limsup_{t\to\infty}\|\nabla\rho\|_{L^2} < \infty$, we have $\int_0^t B(s)ds \lesssim t^{1/2}$.
- This implies $\int_0^t \|\partial_1\rho\|^2_{\dot{H}^{-1}}ds \lesssim t^{1/2}$, so $\|\partial_1\rho\|^2_{\dot{H}^{-1}}$ needs to decay to zero like $t^{-1/2}$ as $t\to\infty$.
- Key observation (by a Fourier argument): If $\|\partial_1\rho(t)\|_{\dot{H}^{-1}} \ll 1$ and $\rho\equiv 0$ on x_2 axis, we have $\|\rho\|_{\dot{H}^1} \gg 1$. More precisely, $\|\rho\|_{\dot{H}^1} \gtrsim \|\partial_1\rho(t)\|^{-1}_{\dot{H}^{-1}}$.



- This contradicts our assumption $\limsup_{t\to\infty}\|\nabla\rho\|_{L^2} < \infty$. (A more careful argument gives us algebraic growth in time).

Inviscid 2D Boussinesq equation

- In the inviscid case $\mu = 0$, let us work with the variables ρ and vorticity ω:

$$\begin{cases} \rho_t + u \cdot \nabla \rho = 0, \\ \omega_t + u \cdot \nabla \omega = -\partial_1 \rho, \end{cases}$$

  where u can be recovered from the Biot-Savart law $u = \nabla^\perp (-\Delta)^{-1} \omega$.

- Whether smooth initial data can lead to a blow-up in $\mathbb{T}^2$ or $\mathbb{R}^2$ is an outstanding open question.
- It is well-known that away from the axis of symmetry, the 3D axisymmetric Euler equation is closely related to 2D Boussinesq:

$$\begin{cases} D_t(ru^\theta) = 0, \\ D_t\left(\frac{\omega^\theta}{r}\right) = r^{-4}\partial_z (ru^\theta)^2, \end{cases}$$

  where $D_t := \partial_t + u^r \partial_r + u^z \partial_z$ is the material derivative, and (u^r, u^z) can be recovered from ω^θ / r by a similar Biot-Savart law.

Blow-up for inviscid 2D Boussinesq and 3D Euler

In the presence of boundary, or for non-smooth initial data, there are many exciting developments on finite-time blow-up:

- Luo–Hou '14: convincing numerical evidence for blow-up at the boundary for 3D axisymmetric Euler
- Elgindi–Jeong '20: blow-up in domain with a corner
- Elgindi '21: blow-up for $C^{1,\alpha}$ solutions for 3D Euler
- Chen–Hou '21: blow-up for $C^{1,\alpha}$ solutions with boundary
- Wang–Lai–Gómez-Serrano–Buckmaster '22: numerics for approximate self-similar blow-up solution using physics-informed neural networks.
- Chen–Hou '22: stable nearly self-similar blowup for smooth solutions (combination of analysis + computer-assisted estimates)

Question: Can one construct solutions with infinite-in-time growth for more general class of initial data?

Infinite-in-time growth in a strip

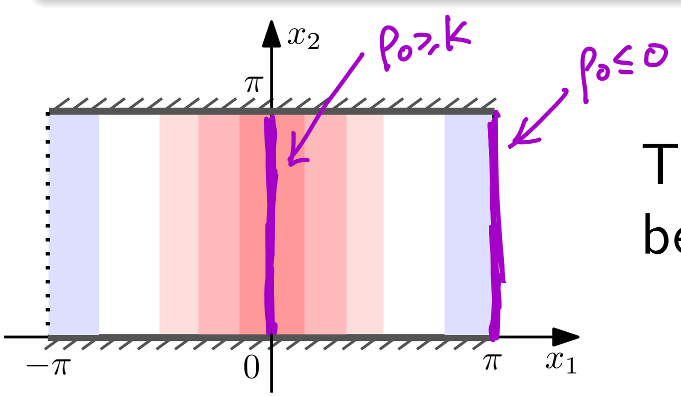
Theorem (Kiselev–Park–Y. '22, preprint)

Let $\Omega = \mathbb{T} \times [0,\pi]$. Let $\rho_0 \in C^\infty(\Omega)$ be even in x_1, and $\omega_0 \in C^\infty(\Omega)$ be odd in x_1, with $\int_{[0,\pi]\times[0,\pi]} \omega_0 dx \geq 0$. Assume that there exists $k_0 > 0$ such that $\rho_0 \geq k_0 > 0$ on $\{0\} \times [0,\pi]$, and $\rho_0 \leq 0$ on $\{\pi\} \times [0,\pi]$. Then the solution satisfies the following during its lifespan:

$$\|\omega(t)\|_{L^p(\Omega)} \gtrsim t^{3-\frac{2}{p}},$$

$$\|u(t)\|_{L^\infty(\Omega)} \gtrsim t,$$

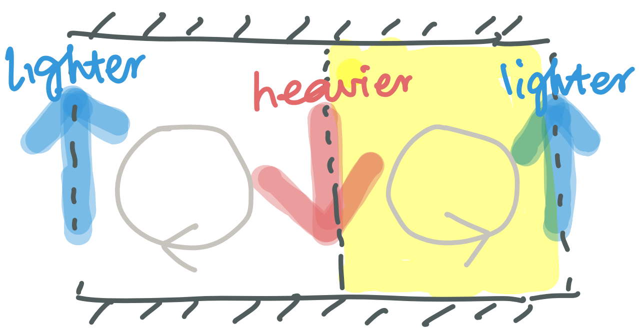
$$\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty(\Omega)} \gtrsim t^2.$$



The proof is a soft argument, based on an interplay between various monotone and conservative quantities.

Monotonicity of vorticity integral

- Let Q be the right half of the strip. Simple but useful observation:



$$\frac{d}{dt}\int_Q \omega dx = \int_Q -u\cdot\nabla\omega dx \overset{\to 0}{} - \int_Q \partial_1\rho dx$$
$$= \int_0^\pi \underbrace{\rho(0,x_2,t)}_{\geq k_0} dx_2 - \int_0^\pi \underbrace{\rho(\pi,x_2,t)}_{\leq 0} dx_2$$
$$\geq k_0\pi.$$

- Since $\int_{\partial Q} u\cdot dl = \int_Q \omega dx \geq k_0\pi t$, we have $\|u(t)\|_{L^\infty}$ grows at least linearly.
- On the other hand, $\|u\|_{L^2}$ is bounded for all times by energy conservation.
- Combining the boundedness of $\|u\|_{L^2(Q)}$ and linear growth of $\int_{\partial Q} u\cdot dl$, we know u must change rapidly in a small neighborhood of ∂Q, leading to super-linear growth of ∇u (and ω).

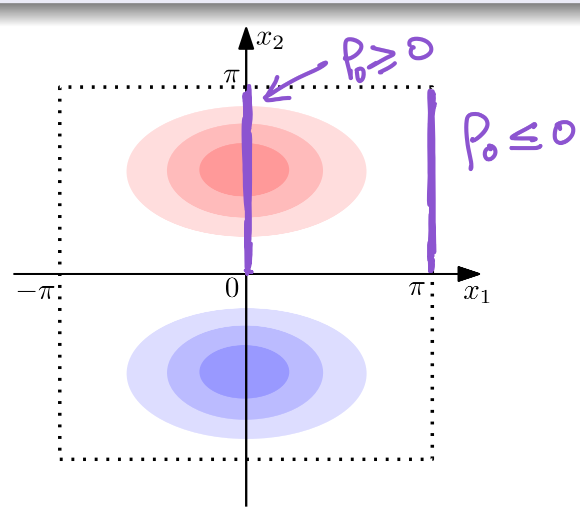
Infinite-in-time growth in $\mathbb{T}^2$

- To our best knowledge, there has been no blow-up / infinite-in-time growth results in $\mathbb{T}^2$.
- In $\mathbb{T}^2$, we obtain infinite-in-time growth for a large class of initial data satisfying certain symmetry/sign conditions:

Theorem (Kiselev–Park–Y. '22, preprint)

Let $\rho_0 \in C^\infty(\mathbb{T}^2)$ be even-odd, and $\omega_0 \in C^\infty(\mathbb{T}^2)$ be odd-odd. Assume $\rho_0 \geq 0$ on $\{0\} \times [0, \pi]$ with $k_0 := \sup_{x_2 \in [0,\pi]} \rho_0(0, x_2) > 0$, and $\rho_0 \leq 0$ on $\{\pi\} \times [0, \pi]$. Then the solution satisfies the following during its lifespan:

$$\sup_{\tau \in [0,t]} \|\nabla \rho(\tau)\|_{L^\infty(\mathbb{T}^2)} \gtrsim t^{1/2}. \tag{1}$$



3D axisymmetric Euler in an annular cylinder

Using a similar idea, we obtain infinite-in-time growth for the 3D axisymmetric Euler equation in an annular cylinder

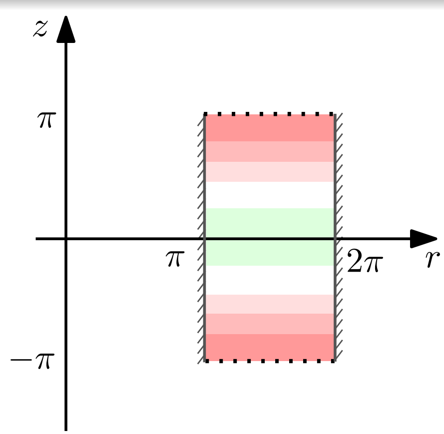
$$\Omega = \{(r,\theta,z) : r \in [\pi, 2\pi]; \theta \in \mathbb{T}, z \in \mathbb{T}\}.$$

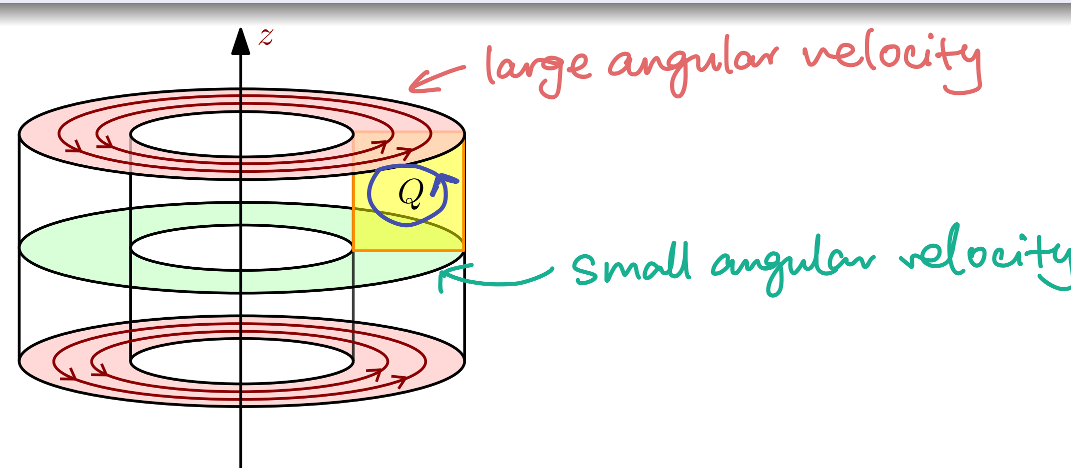
Theorem (Kiselev–Park–Y. '22, preprint)

Let $u_0^\theta \in C^\infty(\Omega)$ be even in z, $\omega_0^\theta \in C^\infty(\Omega)$ odd in z, with $\int_0^\pi \int_\pi^{2\pi} \omega_0^\theta drdz \geq 0$. Assume there exists $k_0 > 0$ such that $u_0^\theta \geq k_0 > 0$ on $z = \pi$, and $|u_0^\theta| \leq \frac{1}{8} k_0$ on $z = 0$. Then the solution to axisymmetric 3D Euler satisfies

$$\|\omega^\theta(t)\|_{L^p(\Omega)} \gtrsim t^{3-\frac{2}{p}} \quad \text{and} \quad \|u(t)\|_{L^\infty(\Omega)} \gtrsim t$$

during the lifespan of the solution.





Thank you for your attention!