Multiplicities for Strongly Tempered Spherical Varieties and BSV Duality

Lei Zhang (National University of Singapore)
Joint with Chen Wan (Rutgers at Newark)

Arthur packets, BIRS
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Notation

- k: a number field
- $\mathbb{A}$: the ring of its adeles
- G: a connected (split) reductive group defined over k.

$$\text{e.g. } \mathrm{GL}_{\mathfrak{n}},\ \mathrm{GSO}(V_{\mathfrak{n}}),\ \mathrm{GU}(V_{\mathfrak{n}}),\ \mathrm{GSp}_{2n},\ \widetilde{\mathrm{Sp}}_{2n},\ E_7.$$

- H: a closed subgroup of G
- $\pi = \otimes_v \pi_v$: an automorphic representation representation of $G(\mathbb{A})$
- $F := k_v$: a local field of characteristic 0

Example. (Gan-Gross-Prasad model)
$G = \mathrm{SO}(n+1) \times \mathrm{SO}(n)$ and $H = \mathrm{SO}(n)^{\triangle}$

Problems

Period integral: Let φ_π be an automorphic form of $G(\mathbb{A})$. Define the period integral over H

$$\mathcal{P}_H(\varphi) := \int^{\star}_{H(k)\backslash H(\mathbb{A})} \varphi_\pi(g)\chi(g)\,\mathrm{d}g.$$

Global Problem: Establish an identity (or non-vanishing equivalence) of the period integral.

For instance,

- $\mathcal{P}_H(\varphi)$ is zero unless its global Arthur parameter is of certain type.
- $\mathcal{P}_H(\varphi)$ is not zero iff $m(\pi_v, \chi_v^\vee) \neq 0$ for all v and $L(\frac{1}{2}, \pi, \rho_X) \neq 0$.

Local multiplicity: Define the local multiplicity of π to be

$$m(\pi_v, \chi_v) := \dim \mathrm{Hom}_{H(F)}(\pi_v, \chi_v).$$

Local Problem: Give the multiplicity formulas for local Arthur packets.

Spherical subgroups

Definition

H is called a spherical subgroup of G if the action of H on the flag variety of G has an open orbit.

Example

- *Symmetric subgroups: $H = G^\sigma$ for some involution σ of G. e.g., $(\mathrm{GL}_n, \mathrm{O}_n)$, $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$.*
- *Whittaker models*
- $(G, H) = (\mathrm{SO}_{n+1} \times \mathrm{SO}_n, \mathrm{SO}_n^{\triangle})$*: Gan-Gross-Prasad models*

Remark

1. *Gan–Gross–Prasad conjectures*
2. *Prasad Conjecture: $(G(E), G(F))$ where E is a quadratic extension of F.*
3. *Ben-Zvi–Sakellaridis–Venkatesh: duality between hyperspherical Hamiltonian varities*

Strongly tempered spherical subgroups

Assumptions:

1. H is strongly tempered;
   - If H is reductive, all the matrix coefficients of tempered representations of G are integrable on $H/Z_{G,H}$ where $Z_{G,H} = Z_G \cap H$;
   - If (G, H) is of Whittaker-induction type, the reductive part (G°, H°) is strongly tempered.
2. No Type N spherical root (cf. $(\mathrm{GL}_n, \mathrm{O}_n)$).

Additional assumption:
$H(F)\backslash G(F)/B(F)$ has a unique rational open orbit.

Such family of spherical subgroups are expected to enjoy the same properties with the Gan–Gross–Prasad models, i.e., the analogy of local and global Gan–Gross–Prasad Conjectures holds.

Strongly tempered spherical subgroups

	G (quasi-split)	$(H, \mathrm{triv} \otimes \psi)$	$\rho_X : {}^LG \to \mathrm{GL}(V)$
1	$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$(\wedge^2 \otimes std_2) \oplus std_4 \oplus std_4^\vee$
2	$\mathrm{GU}_4 \times \mathrm{GU}_2$	$(\mathrm{GU}_2 \times \mathrm{GU}_2)^\circ$	$(\wedge^2 \otimes std_2) \oplus std_4 \oplus std_4^\vee$
3	$\mathrm{GSp}_6 \times \mathrm{GSp}_4$	$(\mathrm{GSp}_4 \times \mathrm{GSp}_2)^\circ$	$\mathrm{Spin}_7 \otimes \mathrm{Spin}_5$
4	GU_6	$\mathrm{GU}_2 \ltimes U_{[3^2]}$	$\wedge^3$
5	GL_6	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\wedge^3$
6	GSp_{10}	$\mathrm{GL}_2 \ltimes U_{[5^2]}$	Spin_{11}
7	$\mathrm{GSp}_6 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\mathrm{Spin}_7 \otimes std_2$
8	$\mathrm{GSO}_8 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[4^2]}$	$\mathrm{HSpin}_8 \otimes std_2$
9	GSO_{12}	$\mathrm{GL}_2 \ltimes U_{[6^2]}$	HSpin_{12}
10	E_7	$\mathrm{PGL}_2 \ltimes U$	ω_7

Remark

1. *Classification: Bravi–Pezzini, Gan–Wang*
2. *Addition Assumption fails for the simply connceted groups. e.g.,* $(\mathrm{SL}_2, \mathrm{GL}_1)$.

Example: Ginzburg–Rallis model

Let $G = \mathrm{GL}_6$ and consider the unipotent subgroup associated to $[3, 3]$:

$$\mathcal{O} = [3^2] \to \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{pmatrix},\ U_{[3^2]} = \left\{ n = \begin{pmatrix} I_2 & A & B \\ 0 & I_2 & C \\ 0 & 0 & I_2 \end{pmatrix} : A, B, C \in M_{2\times 2} \right\}$$

and the non-degenerated character $\psi_{\mathcal{O}}(n) = \psi(\mathrm{tr}(A + C))$.

The Levi subgroup M of $U_{[3^2]}$ is $\{\mathrm{diag}(a_1, a_2, a_3) : a_i \in \mathrm{GL}_2\}$.
The stabilizer $M_{\mathcal{O}}$ of M acting on $\psi_{\mathcal{O}}$ is $\{\mathrm{diag}(a, a, a)\} \cong \mathrm{GL}_2$.
Then $B(F)\backslash \mathrm{GL}_6(F)/\mathrm{GL}_2^{\triangle} \ltimes U_{[3^2]}$ has finitely many double cosets.

- Ginzburg–Rallis model: $(\mathrm{GL}_6, \mathrm{GL}_2^{\triangle} \ltimes U_{[3^2]}, \psi_{\mathcal{O}})$. Refer to Wan's wroks.
- The associated period integral of automorphic forms is related to $L(\frac{1}{2}, \pi, \wedge^3)$.

Global conjecture: Ichino-Ikeda type formula

Conjecture (Wan-Z (2021))

Let G and H be in the above table, π be an irreducible cuspidal automorphic representation of generic A-parameter. Then

$$|\int_{Z_{G,H}(\mathbb{A})H(k)\backslash H(\mathbb{A})} \phi(h)\xi^{-1}(h)\,\mathrm{d}h|^2$$
$$=\frac{1}{|S_\phi|}\cdot\frac{C_{H/Z_{G,H}}}{\Delta_{H/Z_{G,H}}(1)}\cdot\lim_{s\to 1}\frac{\Delta_G(s)}{L(1,\pi,Ad)}\cdot L(\frac{1}{2},\pi,\rho_X)\cdot\Pi_{v\in S}I^{\sharp}_{H_v}(\phi_v).$$

Remark.

1. Refined Gan-Gross-Prasad Conjecture: Ichino-Ikeda, N. Harris, Y. Liu, H. Xue, ect.
2. Lapid-Mao

Pure inner forms: $Im(H^1(F, H/Z_{G,H}) \to H^1(F, G/Z_{G,H}))$

	G	H	G_D
1	$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$
3	$\mathrm{GSp}_6 \times \mathrm{GSp}_4$	$(\mathrm{GSp}_4 \times \mathrm{GSp}_2)^\circ$	$\mathrm{GSp}_3(D) \times \mathrm{GSp}_2(D)$
4	GU_6	$\mathrm{GU}_2 \ltimes U_{[3^2]}$	$\mathrm{GU}_{4,2}$
5	GL_6	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\mathrm{GL}_3(D)$
6	GSp_{10}	$\mathrm{GL}_2 \ltimes U_{[5^2]}$	$\mathrm{GSp}_5(D)$
7	$\mathrm{GSp}_6 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\mathrm{GSp}_3(D) \times \mathrm{GL}_1(D)$
8	$\mathrm{GSO}_8 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[4^2]}$	$\mathrm{GSO}_4(D) \times \mathrm{GL}_1(D)$
9	GSO_{12}	$\mathrm{GL}_2 \ltimes U_{[6^2]}$	$\mathrm{GSO}_6(D)$
10	$E_{7,ad}$	$\mathrm{PGL}_2 \ltimes U$	$E_{7,4}$

Pure inner forms of $(\mathrm{GU}_4 \times \mathrm{GU}_2, (\mathrm{GU}_2 \times \mathrm{GU}_2)^\circ)$:

$$(\mathrm{GU}_{2,2} \times \mathrm{GU}_{2,0}, (\mathrm{GU}_{2,0} \times \mathrm{GU}_{0,2})^\circ),\ (\mathrm{GU}_{3,1} \times \mathrm{GU}_{1,1}, (\mathrm{GU}_{1,1} \times \mathrm{GU}_{2,0})^\circ),$$
$$(\mathrm{GU}_{3,1} \times \mathrm{GU}_{2,0}, (\mathrm{GU}_{2,0} \times \mathrm{GU}_{1,1})^\circ),\ (\mathrm{GU}_{4,0} \times \mathrm{GU}_{2,0}, (\mathrm{GU}_{2,0} \times \mathrm{GU}_{2,0})^\circ).$$

Local Langlands correspondence

- $\phi : W'_F \to {}^L G/Z_{G,H}$ is a tempered Langlands parameter of $G/Z_{G,H}$
- $S_\phi := Z_\phi / Z_\phi^\circ$ the component group of $Z_\phi = \mathrm{Cent}_{\widehat{G/Z_{G,H}}}(\phi)$
- $\Pi_\phi = \cup_{\alpha \in H^1(F, G/Z_{G,H})} \Pi_\phi(G_\alpha)$: the Vogan L-packet, a finite set of tempered representations of G_α

LLC: In our case, there is a *canonical* bijection between

$$\Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi).$$

And we have a decomposition of the set of irreducible representations of G_α:

$$\cup_{\alpha \in H^1(F, G/Z_{G,H})} \mathrm{Irr}_{\mathrm{temp}}(G_\alpha) = \cup_\phi \Pi[\phi], \qquad \pi \longleftrightarrow (\phi, \chi).$$

Multiplicity One Theorem

Addition Assumption: $H(F)\backslash G(F)/B(F)$ has a unique rational open orbit.

Theorem (Wan, Wan-Zhang)

Assume that the local Langlands correspondences hold.
For all models in Table except Models 5–10 for $F = \mathbb{R}$, if ϕ is a tempered L-parameter, then

$$\sum_{\pi \in \Pi_\phi(G)} \dim(\chi_\pi) m(\pi) + \sum_{\pi_D \in \Pi_\phi(G_D)} \dim(\chi_D) m(\pi_D) = 1$$

Conjecture (Wan-Zhang)

With the notation above, the unique $(H, 1 \otimes \xi)$-distinguished element in the Vogan packet $\Pi[\phi]$ is the one associated to the character ω_{ϕ,ρ_X}.

Local conjecture: multiplicity formula for generic L-packets

Definition (cf. Ben-Zevi–Sakallaridis–Venkatesh)

A symplectic representation ρ of ${}^L G$ is called **anomaly free** *if*

- *it has a decomposition*

$$\rho|_{{}^L T} = \Lambda \oplus \Lambda^\vee,$$

  where T is a maximal split torus of G;
- *there exist a character χ of ${}^L T$ and a character θ of ${}^L G$ such that* $\det(\Lambda) = \chi^2 \cdot \theta|_{{}^L T}$.

For an extended endoscopic triple $(G', s, {}^L\eta)$, denote $\rho_{s,{}^L\eta,-}$ to be the symplectic representation of ${}^L G'$ on $V_{s,-}$, where $V_{s,-}$ is the eigenspace of $\rho(s)$ with eigenvalue -1,

Definition

A symplectic representation ρ of ${}^L G$ is called **anomaly free under endoscopy** *if for any $(G', s, {}^L\eta)$ of G, the symplectic representation $\rho_{s,{}^L\eta,-}$ of ${}^L G'$ is anomaly free.*

Distinguished character

Assumption: $X = G/H$ is strongly tempered and without Type N spherical root.
Conjecturally, the representation ρ_X of ${}^LG_X$ is symplectic and anomaly free under endoscopy.

Let $\phi' \colon W'_F \to {}^LG_X$ be tempered.
For $s \in \mathrm{Cent}_{\hat{G}_X}(\phi')$, there exists $(G', s, {}^L\eta)$ of G such that

$$\phi' = {}^L\eta \circ \phi_0 \qquad \text{for some } L\text{-parameter } \phi_0 \text{ of } G'.$$

Then we take

$$\omega_{\phi',\rho_X}(s) = \theta \circ \phi_0(-1)\epsilon(\frac{1}{2}, \rho_{X,s,{}^L\eta,-} \circ \phi_0) \in \{\pm 1\}.$$

Conjecture

ω_{ϕ',ρ_X} is independent of the choice of $(G', s, {}^L\eta)$ and the lifting, and is a character of $S_{\phi'}$.

Epsilon Dichotomy Conjecture

- Let $\phi \colon W_F' \to {}^L G$ be tempered.
- For a lifting $\phi' \colon W_F' \to {}^L G_X$ of ϕ, denote by $\widetilde{\phi}' \colon S_{\phi'} \to S_\phi$ the induced map.
- I_ϕ is the set of all liftings ϕ' such that $\omega_{\phi',s}$ is trivial on $\ker \widetilde{\phi}'$.

Conjecture (Wan-Zhang)

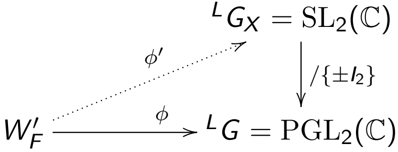
For $\pi \in \mathrm{Irr}_{temp}(G)$ with central character trivial on $Z_G \cap H$, one has

$$\dim \mathrm{Hom}_H(\pi, \mathrm{triv} \otimes \psi) = \sum_{i \in I_\phi} \left\langle \mathrm{Ind}_{\widetilde{\phi}'(S_{\phi'})}^{S_\phi} \omega_{\phi',\rho_X}, \chi_\pi \right\rangle .$$

Remark

The above conjecture can be easily extended to the generic L-parameters.

Example: $(\mathrm{SL}_2(\mathbb{Q}_5), \mathrm{GL}_1(\mathbb{Q}_5^\times))$



$\exists \phi' \iff \omega_\pi(-I_2) = id.$

- $|S_\phi| \in \{1, 2, 4\}$
- $\rho_X = \mathbb{C}^2 \oplus \mathbb{C}^2$: 2 copies of standard representations of $\mathrm{SL}_2(\mathbb{C})$
- $\rho_{X,s,{}^L\eta,-} = \rho_X$ (where $s = \pm I_2$) or trivial, then $\theta = 1$
- $\omega_{\phi',\rho_X}(s) = 1$ for all s, then $\mathrm{Ind}_{\tilde{\phi}'(S_{\phi'})}^{S_\phi} 1 = \mathbb{C}[S_\phi]$
- $|I_\phi| = |F^\times/(F^\times)^2|/|S_\phi|$

Then we have

$$\dim \mathrm{Hom}_{\mathrm{GL}_1}(\pi, \mathbb{C}) = |I_\phi| \times \langle \mathbb{C}[S_\phi], \chi_\pi \rangle = |I_\phi|.$$

Endoscopy Type

Theorem (Wan-Zhang)

Assume the local Langlands correspondences holds.
If F is non-Archimedean and ϕ is of endoscopy type, then Epsilon Dichotomy Conjecture holds.

Remark.

- Conjectures also hold for the generic L-parameters.

Non-unique models:

$$\dim \mathrm{Hom}_H(\pi, \mathrm{triv} \otimes \psi) = \sum_{i \in I_\phi} \left\langle \mathrm{Ind}_{\widetilde{\phi}'(S_{\phi'})}^{S_\phi} \omega_{\phi', \rho_X}, \chi_\pi \right\rangle .$$

Questions:

1. Explicate S_ϕ and $S_{\phi'}$;
2. Enumerate all the possibilities of $\widetilde{\phi}'(S_\phi)$;
3. How to formulate the multiplicity formulas for the non-tempered Arthur packets (analogy of local GGP)?

BSV dual: Dual of Whittaker induction

Assumption: $H = \mathrm{PGL}_2 \ltimes U$

G	$\hat{G}$	ρ_X	Φ_X
PGL_6	SL_6	$\wedge^3$	{2,3}
$\mathrm{GSO}_8 \times \mathrm{GL}_2/\mathrm{GL}_1$	$S(\mathrm{GSpin}_8 \times \mathrm{GL}_2)$	$\mathrm{HSpin}_8 \otimes \mathrm{std}_2$	{1,1,3}
PGSO_{12}	Spin_{12}	HSpin_{12}	{3,5}
$E_{7,ad}$	$E_{7,sc}$	ω_7	{5,9}
PGSp_{10}	Spin_{11}	Spin_{11}	{3,5}
$\mathrm{GSp}_6 \times \mathrm{GL}_2/\mathrm{GL}_1$	$S(\mathrm{GSpin}_7 \times \mathrm{GL}_2)$	$\mathrm{Spin}_7 \otimes \mathrm{std}_2$	{1,1,3}

- $S(\mathrm{GSpin}_n \times \mathrm{GL}_2) = \{(g,h) : \lambda(g)\det(h) = 1\}$

Period integrals on $\hat{G}(\mathbb{A})$:

$$\int^{\star}_{\hat{G}(k)Z_{\hat{G}}(\mathbb{A})\backslash \hat{G}(\mathbb{A})} \varphi_\pi(g)\Theta_X(g)\,\mathrm{d}g$$

for an irreducible discrete automorphic representation π of $\hat{G}(\mathbb{A})$ with trivial central character.

Non-tempered Global Arthur parameters

G	$H = H_0 \times U$	$\hat{G}$
PGL_6	$\mathrm{PGL}_2 \ltimes U_{[3^2]}$	SL_6
$\mathrm{GSO}_8 \times \mathrm{GL}_2/\mathrm{GL}_1$	$\mathrm{PGL}_2 \ltimes U_{[4^2]}$	$S(\mathrm{GSpin}_8 \times \mathrm{GL}_2)$
PGSO_{12}	$\mathrm{PGL}_2 \ltimes U_{[6^2]}$	Spin_{12}
E_7	$\mathrm{PGL}_2 \ltimes U$	$E_{7,sc}$
GSp_{10}	$\mathrm{PGL}_2 \ltimes U_{[5^2]}$	Spin_{11}
$\mathrm{GSp}_6 \times \mathrm{GL}_2/GL_1$	$\mathrm{PGL}_2 \ltimes U_{[3^2]}$	$\mathrm{Spin}_7 \otimes std_2$

Define the embedding

$$\iota_X : H_0(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to G(\mathbb{C})$$

such that

- the Lie algebra of $\iota_X(\mathrm{SL}_2(\mathbb{C}))$ is the $\mathfrak{sl}_2$-triples of the nilpotent orbits (b^2);
- $H_0(\mathbb{C})$ commutes with $\iota_X(\mathrm{SL}_2(\mathbb{C}))$.

Global Conjecture

Conjecture (BSV, Mao–Wan–Zhang)

1. *The period integral is nonzero is nonzero only if the Arthur parameter of π factors through $\iota_X : H_0(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to G(\mathbb{C})$.*
2. *If π is a lifting of a global Arthur packet Π of $\mathrm{SL}_2(\mathbb{A})$, then*

$$\left| \int_{\hat{G}(k) Z_{\hat{G}}(\mathbb{A}) \backslash \hat{G}(\mathbb{A})} \varphi_\pi(g) \Theta_X(g) \, \mathrm{d}g \right|^2 \approx \frac{\prod_{i \in \Phi_X} L(i, \Pi, \mathrm{Ad})}{L(1, \Pi, \mathrm{Ad})}.$$

Remark

- *The above integral depends on the automorphic realization of π in the L^2-space.*
- *The above conjecture also implies that the local–global principles holds.*

Relative trace formula

- Relative trace formula for $\hat{G}(\mathbb{A})$:

$$I(f) = \int_{N(k)\backslash N(\mathbb{A})} \int_{\hat{G}(k)Z_{\hat{G}}(\mathbb{A})\backslash \hat{G}(\mathbb{A})} K_f(g,n)\Theta_X(g)\xi_N(n)\, dg\, dn.$$

- Kuznetsov trace formula for $\mathrm{SL}_2(\mathbb{A})$

$$J(f') = \int_{N'(k)\backslash N'(\mathbb{A})} \int_{N'(k)\backslash N'(\mathbb{A})} K_{f'}(n_1,n_2)\psi(n_1)^{-1}\psi(n_2)\, dn_1\, dn_2.$$

Theorem (Mao–Rallis (97), Mao–Wan–Zhang)

Over the p-adic places, Fundamental Lemma and Smooth Transfer hold for the above relative trace formulas.

Local Multiplicity formula?

G	$H = H_0 \times U$	$\hat{G}$
PGL_6	$\mathrm{PGL}_2 \ltimes U_{[3^2]}$	SL_6
GSp_{10}	$\mathrm{PGL}_2 \ltimes U_{[5^2]}$	Spin_{11}
$\mathrm{GSO}_8 \times \mathrm{GL}_2/\mathrm{GL}_1$	$\mathrm{PGL}_2 \ltimes U_{[4^2]}$	$S(\mathrm{GSpin}_8 \times \mathrm{GL}_2)$
PGSO_{12}	$\mathrm{PGL}_2 \ltimes U_{[6^2]}$	Spin_{12}
E_7	$\mathrm{PGL}_2 \ltimes U$	$E_{7,sc}$
$\mathrm{GSp}_6 \times \mathrm{GL}_2/GL_1$	$\mathrm{PGL}_2 \ltimes U_{[3^2]}$	$\mathrm{Spin}_7 \otimes std_2$

Let ϕ be an Arthur parameter of $\hat{G}(\mathbb{A})$ and $\Pi(\phi)$ be its global Arthur packets. Assume that ϕ factor through ι_X, "equivalently",

$$\phi \text{ is of } (\Pi, b) \text{ type.}$$

Questions:

1. How to parametrize the residual representations in $\Pi(\phi)$?
2. How to explicate the local Arthur packet $\Pi(\phi_v)$ over each place?
3. How to eastablish the local multiplicity formulas for the non-tempered local Arthur packets?

Thank You!


数 学 高 等 研 究 院
INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS
ZHEJIANG UNIVERSITY
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