Lagrangian Multiform Theory and Pluri-Lagrangian Systems (23w5043)

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1 Overview of the Field

This was the first ever workshop on a new subject of mathematical physics, called Lagrangian multiform theory, (sometimes also referred to as pluri-Lagrangian systems). It comprises a novel variational approach to integrable systems, which breaks with conventional variational calculus in that it captures the phenomenon of multidimensional consistency (i.e. the coexistence of infinite hierarchies of equations) within a single Lagrangian framework. This approach, which was first proposed in 2009 in a pioneering paper by Lobb and Nijhoff, [1], and differs from all conventional variational approaches in the following aspects:

1. the Lagrangians are no longer scalar objects (or volume forms), but differential p-forms or difference p-forms L in a space of independent variables of arbitrary dimension. Here p is related to the dimensionality of the equations (i.e. we have Lagrangian 1-forms for ordinary differential or difference equations; 2-forms for partial differential or difference equations with two independent variables, etc.);
2. the action $S[u(\boldsymbol{x});\sigma] = \int_\sigma \mathsf{L}$ is not only a functional of the field variables $u(\boldsymbol{x})$, but also of the (hyper)surfaces σ in the space of independent variables $\boldsymbol{x}$ of arbitrary dimension, over which the Lagrangian p-form is integrated, and the action is critical w.r.t. variations/deformations of those surfaces on solutions of the Euler-Lagrange (EL) equations;
3. There is now an overdetermined set of generalised EL equations that describes a compatible set of equations which forms the multidimensionally consistent set of equations of an (in principle infinite) hierarchy of discrete or continuous integrable equations;
4. Most dramatically, the Lagrangians (i.e. the components of the Lagrangian p-form L) are no longer external input, chosen for the sake of modelling certain physical systems, but are to be solved self-consistently from the entire set of generalised EL equations.

As a consequence, not only is this the first variational principle that produces an entire hierarchy of compatible equations (for one and the same field variable) from one coherent framework, but the variational equations determine the relevant Lagrangians of all the integrable equations themselves. As such this is the first variational theory that predicts the form of the Lagrangian from the variational principle itself. Thus, this aspect offers a totally new perspective for a novel foundational theory of physics.

The Lagrangian multiform theory has important associations with antecedents in various areas of mathematics and physics:

- Baxter's notion of Z-invariance in the statistical mechanics of exactly solvable models (through the Yang-Baxter and star-triangle relations);
- the theory of pluri-harmonic functions in complex function theory;
- Noether's notion of variational symmetries.

One of the key instances in the discovery of the Lagrangian multiform structure was that of the *closure relation*: the observation in the paper [1] that Lagrangians $\mathcal{L}_{ij}$ for integrable quadrilateral lattice equations, associated with directions labeled by i and j, when embedded in a three- or higher-dimensional space with directions $i, j, k, \cdots$, obey a difference closure relation of the form

$$\Delta_i \mathcal{L}_{jk} + \Delta_j \mathcal{L}_{ki} + \Delta_k \mathcal{L}_{ij} = 0$$

when using the solutions of the corresponding quadrilateral lattice equation, where Δ_i denotes a forward difference operator (see below). This relation suggests that the Lagrangians $\mathcal{L}_{ij}$ are not isolated variational objects, but indeed components of 'Lagrangian 2-form' L that is closed, but only on solutions of the quad equation (i.e., only 'on-shell'). A key example where this occurs is the so-called lattice potential Korteweg-de Vries equations (otherwise named H1 equation in the famous classification of Adler, Bobenko and Suris), which takes the form

$$\mathcal{L}_{ij} = u(T_i u - T_j u) - (p_i - p_j) \ln(T_i u - T_j u) \ ,$$

where $u = u(n_i, n_j, n_k)$ is a dependent variable, i.e. a function of discrete independent variables n_i, n_j, n_k, (and where T_i, T_j denote elementary shifts of the variables n_i, n_j), each of which is associated with a 'lattice parameter' p_i, p_j, p_k. (The difference operator Δ_i used above can be written in the shift operator T_i by $\Delta_i = T_i - \mathrm{id}$.) Consequently, the corresponding action functional

$$S[u(\boldsymbol{n}; \sigma] = \sum_{\sigma} \mathsf{L} = \sum_{\sigma_{ij} \in \sigma} \sum_{i<j} \mathcal{L}_{ij} \delta_i \wedge \delta_j \ ,$$

which is now to be considered to be a functional of both the field variables u as well as of the quad-surface σ over which the discrete Lagrangian 2-form L (in a notation similar to [70], where δ_i denote discrete differentials) is to be summed, is invariant under elementary deformations (i.e. lattice flips) of the surface, provided the action is evaluated 'on-shell', i.e. on solution of the quad equation:

$$(T_i u - T_j u)(T_i T_j u - u) = p_i - p_j \ .$$

This property was established for several examples of the ABS list, as well as for the equations in the lattice Gel'fand-Dikii hierarchy, cf. [2]. The corresponding continuous analogue of the closure relation reads:

$$\partial_{p_i} \mathcal{L}_{jk} + \partial_{p_j} \mathcal{L}_{ki} + \partial_{p_k} \mathcal{L}_{ij} = 0$$

for continuous Lagrangian 2-forms

$$\mathsf{L} = \sum_{i<j} \mathcal{L}_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j$$

where the action is that of an integrable two-dimensional field theory embedded in a higher dimensional space:

$$S[u(\boldsymbol{x}); \sigma] = \int_{\sigma} \mathsf{L} \ .$$

In a separate development, Costello et al., cf [43]-[45], formulated a way to generate integrable field theories through a 4-dimensional Chern-Simons (CS) action:

$$S[\boldsymbol{A}] = \int_{\mathcal{M}} \omega \wedge CS_3 \ , \quad CS_3 = \mathrm{tr}\left(\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \tfrac{2}{3} \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \right) \ ,$$

integrated over a manifold $\mathcal{M}$ which combines $\mathbb{CP}^1$ as 2D space of the spectral variable, and the real 1+1-dimensional configuration space of the independent spatial/temporal variables. Here ω is a complex-valued (meromorphic) 1-form in the space of spectral parameter, and where the gauge field $\boldsymbol{A}$ is a connection 1-form, which in a special gauge is related to the zero-curvature Lax pair of those integrable field theories. The 4D CS theory was successful in making a connection with the classical r-matrix formalism, but unlike the multiform theory, was not yet able to make manifest the multidimensional consistency that underlies integrable field theories.

At the same time, in loop quantum gravity (LPG) there are many ideas and results that seem to point to a connection with Lagrangian multiform theory. In particular, natural discretizations (viz. 'perfect discretizations') that appear in the work by Bahr and Dittrich, [36, 37], and the so-called 'Dittrich invariance' (or 'Ditt-invariance') that was highlighted in papers by Rovelli, [38]-[40], seem to get anxiously close to concepts common in integrable systems. This also holds true for ideas concerning (time-)scaling invariance, and scaling-invariant path integrals obtained by sums over surfaces, which may have a parallel with the 'sums over geometries' (in the independent variables) that is suggested by the multiform theory at the quantum level.

A decade and a half since the theory was initiated, Lagrangian multiform theory has attained a level of maturity. Thus, it was the right time now that to have a meeting focused on this particular aspect of integrable systems theory, and to create an opportunity to liaise with experts in areas of physics and applied mathematics, where these new ideas could have an impact. Thus, we tried to engage researchers working in cognate fields in mathematics (e.g. in differential geometry, invariant theory and the theory of moving frames) to help developing the theory further from a theoretical perspective, as well as researchers in areas of physics (e.g. experts in loop quantum gravity and topological field theory), where this novel variational approach may be taken up and perhaps be integrated in the current research in those fields. We chose the IASM of Zhejiang University in Hangzhou (China) as the venue for this workshop, to create an opportunity to the many young and enthusiastic researchers in China to possibly get engaged in this new topic in integrable systems theory and variational calculus.

2 Recent Developments and Open Problems

Since the first paper appeared in 2009 the ideas were immediately taken up by other researchers, notably at the TU Berlin, and the theory has developed rapidly ever since. Highlights of the progress in the subject in the past decade comprise:

- Establishment of the Lagrangian multiform structures for many key examples of discrete and continuous integrable systems, including equations of the ABS (Adler/Bobenko/Suris) lattice systems, lattice KP systems and lattice continuous and discrete Gel'fand-Dikii and Boussinesq systems;
- Establishing of the fundamental system of generalised Euler-Lagrange equations in both the continuous as well as the discrete setting;
- The formulation of variational symmetries and generalised Noether type theorems for the Lagrangian multiform theory;
- The formulation of the Lagrangian multiform structure for non-commuting flows and Lie group actions;
- Connection with classical r-matrices and Poisson-Lie algebras;
- First steps towards a quantum multiform theory in terms of Feynman propagators and path integrals.

However, there are still many open problems and avenues for future work:

- To give a Lagrangian classification of integrable systems based on the multiform and pluri-Lagrangian structure. This would entail the solving of the system of generalised EL equations, viewed as a system of PDEs in the dependent variable;
- To connect, or perhaps even merge, the formulations of Lagrangian multiform theory and those of 4D CS theory (as suggested in the talk by Vicedo, and by recent results on the 3-form multiform structure of the Darboux-KP systems as an infinite-dimensional CS theory, as explained in the talk by Faria Martins);

- To establish connections with the notions of scaling-invariance and perfect discretisations as have appeared in the theory of loop quantum gravity (as explained in the talk by Dittrich);
- To extend of the connection between Lagrangian multiforms and classical r-matrices to the case of discrete Lagrangian multiforms, and possibly to the quantum domain;
- To extend the differential and difference variational bicomplex to include the Lagrangian multiform structure (including the so-called corner equations), perhaps via the variations in the independent variable (which would create a variational tricomplex, as suggested in the talk by Peng);
- To develop the quantum Lagrangian multiform theory for systems beyond the quadratic case, i.e. for nonlinear models, and establish the corresponding path integral framework. This may lead to the definition of a new quantum object (as conjectured in the talks by Nijhoff and Yoo-Kong).

Several of these issues were touched upon in the workshop, but the workshop should primarily be seen as a platform for opening the ground to fruitful collaborations where these longer term issues could be tackled.

3 Presentation Highlights

Here is a brief description of the talks in the order that they were presented.

F Nijhoff gave a brief introduction to Lagrangian multiform theory and pluri-Lagrangian systems. He gave a brief resumé of the basic ideas and founding concepts of Lagrangian multiform theory, and presented some simple examples to illustrate the ideas. He also touched on the pluri-Lagrangian point of view (which is the perspective of the Berlin group at the TU Berlin on multiform theory) pointing out some subtle distinctions.

J Richardson talked about periodic reductions of discrete Lagrangian multiforms. He presented a general framework to perform periodic initial value reductions on discrete 2-dimensional multiforms to derive N-component 1-dimensional multiforms. This was proved generally and then shown for three discrete integrable examples: the discrete potential Korteweg de-Vries equation, the discrete multicomponent Boussinesq equation, and the H3 quad equation.

M Vermeeren discussed the multi-time Euler-Lagrange equations from the point of view of double zeroes. The pluri-Lagrangian approach emphasises the Multi-Time Euler-Lagrange equations, which characterise critical points of the action integral over arbitrary submanifolds. The Lagrangian multiform approach focuses on the exterior derivative of the Lagrangian k-form, which should be zero on solutions. These two approaches are bridged by the fact that the Multi-Time Euler-Lagrange equations can also be obtained by taking variations of the exterior derivative of the Lagrangian k-form. In more pragmatic terms, if the coefficients of the exterior derivative can be written as a product of factors that vanish on a set of equations (or a sum of such products), then this set of equations is sufficient for criticality, i.e. it implies the Multi-Time Euler-Lagrange equations. This property has been central (though sometimes implicit) in the construction of Lagrangian k-forms. In the updated presentation, which explicitly makes use of this property, of the construction of some key examples of continuous and semi-discrete Lagrangian 2-forms were treated. In addition, some cases were highlighted where the equivalence between the Multi-Time Euler-Lagrange equations and the factors of the exterior derivative is nontrivial. This showcases the use of Lagrangian k-forms as a tool to establish surprising connections.

Y Suris in his talk posed the question: what is the pluri-Lagrangian structure good for? The pluri-Lagrangian theory being very satisfying from the aesthetical point of view, a frequently asked question remains: what is the added value of knowing the pluri-Lagrangian structure for a given system? In this talk, a couple examples were presented for a possible answer to this question, one of which was how from the pluri-Lagrangian structure, namely through the so-called corner equations, the commutativity of double-valued maps (e.g. arising from the Bäcklund transforms for Toda type systems) can be understood and proven. Another application was the answer to the question how one can find integrals for a commuting set of Bäcklund transformations, e.g. in two-dimensional systems arising from Lagrangian

2-forms. A third application involved billiards in confocal quadrics viewed as a pluri-Lagrangian system, allowing the derivation of integrals of the billiard map (originally due to Moser) from first principles.

A Kels talked about integrable lattice models of statistical mechanics in connection to Lagrangian multiform theory. He showed how integrable lattice models of statistical mechanics are related to Lagrangian multiform theory/pluri-Lagrangian systems for Adler-Bobenko-Suris (ABS) quad equations. Integrable lattice models of statistical mechanics satisfy a special form of the Yang-Baxter equation called star-triangle relation. A well- known example of such a model is the two-dimensional Ising model. One may take a quasi-classical expansion of the star-triangle relation, and in the leading order one obtains a classical relation that is precisely the closure relation for Lagrangian multiforms. There is also an analogue of invariance of the action functional for Lagrangian multiforms, which is the property of Z-invariance, for which the partition function of the lattice model of statistical mechanics is invariant under certain deformations of a lattice. For such models, the quasi-classical expansion of the partition function leads to an action functional for systems of discrete Laplace-type equations that are related to ABS equations. The lattice models of statistical mechanics thus provide a quantization of these discrete systems in the sense of the path-integral formulation.

C Wu talked about meromorphic solutions of the autonomous Schwarzian differential equations. It has been proved by Ishizaki that the autonomous Schwarzian differential equations, which admit transcendental meromorphic solutions, have six canonical forms. In the talk, based on joint work with Liangwen Liao, Jie Zhang and Donghai Zhao, he presented all of these transcendental meromorphic solutions.

S Yoo-Kong talked about multidimensional consistency and the quantum variational principle in the context of quadratic Lagrangian 1-forms. The multi-time propagator in the continuous case for quadratic Lagrangian 1-forms was presented, including the integrability condition, as well as the path-independent feature as a result of the critical condition of the propagator, on the space independent variables. The two-time harmonic oscillators was given as a toy example.

J Faria Martins gave a derivation of a Lagrange multiform for the Darboux-KP system from a corresponding infinite dimensional Chern-Simons theory. This was a report on recent joint work with Daniel Riccombeni and Frank Nijhoff, [49], on the construction of a Lagrange multiform structure for the Darboux-KP system. The Lagrange multiform is derived from a certain matrix-valued Chern-Simons theory in infinite dimensional space, by judiciously choosing the fields that are allowed. He explained the required elementary Chern-Simons theory used in the construction.

B Dittrich gave an introduction to diffeomorphism symmetry in the discrete and perfect discretizations. Diffeomorphism symmetry plays an important role for general relativity and in particular for quantum gravity. It can be also introduced into other mechanic (then known as reparametrization invariance) or field theoretic systems. The fate of diffeomophism symmetry under discretizations has led to a considerable amount of debate. She explained that diffeomorphism symmetry in the discrete is a very powerful symmetry. However discretizing diffeomorphism symmetric systems typically breaks this symmetry. The symmetry can be restored, e.g. via constructing perfect discretizations which often amounts to solving the continuum dynamics of the system.

A Dzhamay talked about the combinatorics of matrix refactorizations and discrete integrable systems. Many interesting questions in the theory of discrete integrable systems, such as Lax representations, Yang-Baxter maps, and dynamics of discrete Painlevé equations, can be formulated in terms of refactorization transformations of rational matrix functions. One way to better understand such transformations is to study the non-trivial relations that the eigenvectors of these matrix functions must satisfy. In the talk a geometric representation was given of some of these relations and their applications were considered. In particular, it was shown how to encode, in a very natural way, the generating functions of refactorization transformations (i.e., the Lagrangians of the corresponding discrete integrable systems) in some simple cases.

M Hamanaka presented four-dimensional Wess-Zumino-Witten (4D WZW) models and a unified theory of integrable systems. 4D WZW models are analogous to the two dimensional WZW models. The equation of motion of the 4dWZW model is the Yang equation which is equivalent to the anti-self-dual Yang-Mills (ASDYM) equation. It is well known as the Ward conjecture that the ASDYM equations can be reduced to many classical integrable systems, such as the KdV eq. and Toda eq. [Ward, Mason-Woodhouse, etc.]. On the other hand, 4D Chern-Simons (CS) theory has connections to many quantum integrable systems such as spin chains and principal chiral models [Costello-Yamazaki-Witten, et al.]. Furthermore, these two master equations have been derived from a 6dCS theory on a twistor space like a double fibration [Bittleston-Skinner]. This suggests a nontrivial duality between the 4D WZW model and the 4D CS theory. The aim of the talk was to discuss integrability aspects of the 4D WZW model and to construct soliton solutions of it by the Darboux technique. The action density of the solutions was calculated and it was found that the soliton solutions behaves as the KP-type solitons, that is, the one-soliton solution has localized action (energy) density on a 3d hyperplane in 4-dimensions (soliton wall) and the N-soliton solution describes N intersecting soliton walls with phase shifts. It was noted that the Ward conjecture holds mostly in the split signature (+,+,-,-) and then the 4D WZW model describes the open N=2 string theory in the four-dimensional space-time. Hence a unified theory of integrable systems can be proposed in this context with the split signature from Lagrangian viewpoints. This talk was partially based on the collaboration with Shan-Chi Huang, Hiroaki Kanno (Nagoya) and Claire Gilson, Jon Nimmo (Glasgow): [arXiv:2212.11800, 2106.01353, 2004.09248, 2004.01718].

A Kuniba talked about the tetrahedron and 3D reflection equations from quantum cluster algebras perspective. Tetrahedron and 3D reflection equations are natural generalizations of the Yang-Baxter and reflection equations into three dimensions. In the talk a new solution to them was constructed, associated with the quantum cluster algebra defined on the Fock-Goncharov quivers. The key to the construction was to realize a cluster transformation of quantum Y-variables as an adjoint action by using q-Weyl algebras. (This was joint work with Rei Inoue and Yuji Terashima.)

C Zhang introduced a boundary Lagrangian formalism for integrable quad-equations. In this talk, the notion of boundary conditions for integrable quad-graph systems was given for the first time. The boundary conditions are characterized by equations on triangles that are the elementary patterns supporting a boundary for a quad-graph system. The integrability criterion is defined by the so-called boundary consistency. On the basis of this a Lagrangian formalism was provided for integrable boundary conditions. This included a set of Euler-Lagrange equations and a closure relation associated with the boundary consistency. Explicit examples were given for equations in the ABS classification.

Y-y Sun talked about applications of elliptic functions in solving the Boussinesq equation. She first presented a Bäcklund transformation (BT) which connects the continuous to discrete Boussinesq system. This BT is obtained by using elliptic function solutions of the continuous Boussinesq equation. This BT was applied to establish the Lax pair and N-times Darboux transformation for the continuous Boussinesq equation. Starting from an elliptic seed solution, the Darboux transformation is used to construct elliptic soliton solutions. The BT can also be used to construct discrete elliptic seed solution for the lattice Boussinesq system. Furthermore, she established an infinite family of solutions in terms of elliptic functions of the lattice KP system by setting up an elliptic direct linearisation scheme. Through a dimensional reduction of this elliptic direct linearisation scheme, she obtained the elliptic multi-soliton solutions of the lattice Boussinesq system.

P Xenitidis discussed hierarchies of differential-difference equations, their master symmetries and their Lagrangian formulation. Motivated by some observations and the study of well known examples, in this talk some ideas were presented about the Lagrangian formulation of hierarchies of integrable differential-difference equations along with their connection to canonical conservation laws and local symplectic operators. The notion of a master Lagrangian was introduced which can used to generate the Lagrangian for every member in the hierarchy in the same way a master symmetry can be used to generate the whole hierarchy of differential-difference equations. Several hierarchies were considered as examples to demonstrate these ideas.

B Vicedo talked about the connection between gauge theory and integrable systems. He reviewed some recent developments on the study of the gauge-theoretic origin of both finite-dimensional integrable models and (1+1)- dimensional integrable field theories, as proposed by Costello, Witten and Yamazaki.

In particular, he explained how the Zakharov-Mikhailov action naturally arises from 4d Chern-Simons theory in the presence of suitable surface defects. Similarly, he showed how a 1-dimensional action describing the Gaudin model realised on a product of coadjoint orbits naturally arises from 3d mixed BF theory in the presence of suitable line defects. The talk was mainly based on the joint works: arXiv:2201.07300, with J. Winstone, and arXiv:2012.04431 with V. Caudrelier and M. Stoppato.

P Olver presented two new developments for Noether's two theorems. In the first part of his talk, he started by recalling the two well-known classes of partial differential equations that admit infinite hierarchies of higher order generalized symmetries: 1) linear and linearizable systems that admit a nontrivial point symmetry group; 2) integrable nonlinear equations such as Korteweg–de Vries, nonlinear Schrödinger, and Burgers'. He then introduced a new general class: 3) underdetermined systems of partial differential equations that admit an infinite dimensional symmetry algebra depending on one or more arbitrary functions of the independent variables. An important subclass of the latter are the underdetermined Euler–Lagrange equations arising from a variational principle that admits an infinite-dimensional variational symmetry algebra depending on one or more arbitrary functions of the independent variables. According to Noether's Second Theorem, the associated Euler–Lagrange equations satisfy Noether dependencies; examples include general relativity, electromagnetism, and parameter-independent variational principles.
Noether's First Theorem relates strictly invariant variational problems and conservation laws of their Euler–Lagrange equations. The Noether correspondence was extended by her student Bessel-Hagen to divergence invariant variational problems. In the second part of the talk, the role of Lie algebra cohomology in the classification of the latter was highlighted, and the talk concluded with some provocative remarks on the role of invariant variational problems in fundamental physics.

S Li talked about matrix-valued orthogonal polynomials and non-commutative integrable systems. He reported on some recent results in matrix-valued orthogonal polynomials and non-commutative integrable lattices by using the technique of quasi-determinants. Bäcklund transformations for non-commutative integrable systems were also discussed from the perspective of orthogonal polynomials theory.

L Peng discussed discrete Lagrangian multiforms on the difference variational bicomplex. After introducing the prolongation structure for finite difference equations, he defined the difference variational bicomplex and studied its exactness. Similar to its differential counterpart, the difference variational bicomplex offers a convenient framework for exploring discrete variational calculus, inverse problems, symmetry analysis, and more. In particular, its connection with discrete integrable systems that admit Lagrangian multiforms was explained. The talk was based on joint works with Peter Hydon (Kent) and Frank Nijhoff (Leeds).

V Caudrelier talked about the construction of Lagrangian multiforms for infinite and finite dimensional integrable hierarchies. After reviewing the main ideas and ingredients of Lagrangian multiform theory pioneered by Lobb and Nijhoff, he focused on methods to construct Lagrangian multiforms efficiently for field theories in 1+1 dimensions and for finite-dimensional systems. These methods involve three main sources of inspiration: the generating function formalism for hierarchies advocated by Flaschka-Newell-Ratiu and Nijhoff, the insightful construction by Zakharov-Mikhailov of an action for zero-curvature equations of Zakharov-Shabat type and the classical r-matrix/Lie dialgebras theory developed by Semenov-Tian-Shansky. In the case of field theories, the resulting generating Lagrangian multiform contains a huge class of hierarchies and it was shown how many old (AKNS, sine-Gordon) and new (trigonometric Zakharov-Mikhailov, coupled hierarchies) examples can be constructed by fixing a small amount of data (marked points on the Riemann sphere, a Lie algebra and an r-matrix). Based on these multiforms, the talk also explained how some aspects of Lagrangian multiform theory are related to the classical r-matrix theory familiar in Hamiltonian aspects of integrable systems and to the classical Yang-Baxter equation. In particular, the exact relation between the closure relation and the involutivity of Hamiltonians was demonstrated in the case of finite-dimensional systems, and some comment was made on the situation in infinite dimensions. The results support the proposal of using Lagrangian multiforms as a variational criterion for integrability. The talk was based on joint works with M. Dell'Atti, A. Singh, M. Stoppato and B. Vicedo.

X Xu in her talk constructed algebro-geometric solutions to the lattice potential modified Kadomtsev- Petviashvili (lpmKP) equation. A Darboux transformation of the Kaup-Newell spectral problem was employed to generate a Lax triad for the lpmKP equation, as well as to define commutative integrable symplectic maps which generate discrete flows of eigenfunctions. These maps share the same integrals with the finite-dimensional Hamiltonian system associated to the Kaup-Newell spectral problem. The asymptotic behaviours of the Baker-Akhiezer functions was investigated, and their expression in terms of Riemann theta function was obtained. Finally, algebro-geometric solutions for the lpmKP equation were reconstructed from these Baker-Akhiezer functions.

A A Singh presented a Lagrangian multiform for the rational Gaudin model. Gaudin models are a general class of integrable systems associated with quadratic Lie algebras. In his talk, he described the construction of the Lagrangian 1-form for the case of the rational Gaudin model, based on a joint work with V. Caudrelier and M. Dell'Atti. They used the theory of Lie dialgebras, due to Semenov-Tian-Shansky, to construct a general Lagrangian 1-form living on a coadjoint orbit. Lie dialgebras are related to Lie bialgebras, but are more flexible in that they incorporate the case of non-skew-symmetric r-matrices. He illustrated how this construction can be employed in the setting of loop algebras, needed when dealing with Lax matrices with spectral parameters. He also briefly discuss some natural next steps and some possible connections to other recent works with gauge-theoretic flavours.

D-j Zhang talked about the discrete Burgers equation. In his talk he gave a short review on the integrability of the semi-discrete and discrete Burgers equations, which are featured as integrable equations that are linearisable. The continuous and semi-discrete Burgers hierarchies are related to the mKP system via squared eigenfunction symmetry constraints. The semi-discrete Burgers equation acts as a Bäcklund transformation for the continuous Burgers hierarchy. The fully discrete Burgers equation is a simple 3-point lattice equation that are consistent around the cube. It is also the Bianchi identity of the Bäcklund transformation. The Lagrangian of the discrete Burgers equation is not known.

Y Kodama discussed the connection between KP solitons, the Riemann theta functions and their possible applications to soliton gases. He showed that the regular soliton solutions of the KP equation can be explicitly expressed by the Riemann theta functions on singular curves in the sense of Mumford. These solitons can be also obtained by applying vertex operators. In particular, applying the vertex operator to quasi-periodic solutions, we have KP solitons on quasi- periodic background. He also discussed possible models of soliton gases as physical applications of this formulation.

Y Xie discussed the full Kostant-Toda lattice and the flag varieties. In 1967, Japanese physicist Morikazu Toda proposed an integrable lattice model to describe motions of a chain of particles with exponential interactions between nearest neighbors. Since then, the Toda lattice and its generalizations have become the test models for various techniques and philosophies in integrable systems and wide connections are built with many other branches of mathematics. In the talk, a characterization of the singular structure of solutions of the so-called full Kostant-Toda (f-KT) lattices was gi defined on simple Lie algebras in two different ways: through the τ-functions and through the Kowalevski-Painlevé analysis. Fixing the spectral parameters which are invariant under the f-KT flows, a one-to-one correspondence was built between solutions of the f-KT lattices and points in the corresponding flag varieties. (The talk was based on preprint arXiv:2212.03679.)

C Qu in his talk discussed Liouville correspondences between certain integrable systems and their dual integrable systems. It is well-known that integrable systems are related to invariant geometric flows in certain geometries. The talk was mainly concerned with invariant geometric flows in affine-related geometries including several well-known geometries. First, he showed that the special invariant geometric flows in those geometries are related respectively to the well-known integrable systems. Second, the geometric formulations to integrability features of the resulting systems were investigated. Third, the geometric formulations of Miura-transformation and its various extensions are also investigated. This talk was based on the works joint with Peter Olver, Zhiwei Wu and Yun Yang.

N Reshetikhin presented the concept of 'Hybrid Integrable Systems'. These are quantum integrable systems which ride on the background of classical integrable systems. In the talk he introduced the notion of such systems, and gave some examples. Also the connection with well known structures was explained.

D Yang talked about the constrained KP hierarchy and the bigraded Toda hierarchy. The constrained KP hierarchy was introduced by Yi Cheng. In the talk, an extension of the matrix-resolvent approach for studying tau- functions was presented, for the constrained KP hierarchy and the bigraded Toda hierarchy. In particular, expressions of n-point functions of both hierarchies were obtained. He showed that the tau-function of an arbitrary solution to the bigraded Toda hierarchy is a tau-function for the constrained KP hierarchy, which generalizes the Carlet–Dubrovin–Zhang theorem. This was a joint work with Ang Fu and Dafeng Zuo (arXiv:2306.09115).

J Cheng talked about bosonizations and KP integrable systems. He reviewed some typical bosonizations for KP, BKP, CKP and DKP hierarchies. Subsequently he gave some recent results on the applications of bosonizations in symmetries, Darboux transformations and reductions for KP integrable hierarchy.

All in all there were 29 talks, of which 10 talks were online, and the remaining were in-person.

Sandpit (Panel discussion)

A one-hour panel discussion on the Lagrangian formalism took place on Wednesday afternoon with experts working on different aspects of Lagrangian formalism.

Panel members: V. Caudrelier, H. Hamanaka, F. Nijhoff, M. Vermeeren, DJ. Zhang, and Y. Shi (chair/interviewer)

Some of the topics discussed on the various aspects of the Lagrangian formalism are:

- pluri versus multi-form approaches
- continuous versus discrete systems
- commutative versus non-commutative
- classical versus quantum
- mathematics versus physics perspectives

Some of the questions and topics that came up during the discussion and from the relevant talks.

- What use are multi/pluri-Lagrangian systems (key: Suris' talk)
- What is the significance of putting the dependent and independent variables on the same footing? (key: Nijhoff's talk)
- "Lagrangiability" (why are some multidimensional consistent (MDC) systems defying a Lagrangian structure?)
- Connection with 4d (or 6d) CS theory (key: Hamanaka's and Vicedo's talks)
- What does the connection with the statistical models mean? (key: Caudrelier's and Kels's talks)
- Variational multi-complexes, would they need to be invented? (key: Olver's and Peng's talks)
- What are the connections with loop quantum gravity (key: Dittrich's talk)
- What is the current state of quantum multiform theory, and what are the technical difficulties involved? (key: Yoo-Kong's talk)
- What is the physical motivation that gives rise to non-commutative equations? (key: Hamanaka's talk)
- Why Lagrangians has not been found for neither the continuous (key: Olver's talk), not the discrete (key: DJ. Zhang's talk) versions of the Burgers' equation?
- Where is the theory going? What avenues to follow?

Finally, Kodama commented on the ubiquitous/universality of integrable systems, whose works on integrable systems range from physics (experimental/theoretical) to mathematics (applied and pure).

4 Scientific Progress Made

The following areas were covered during the workshop:

1. foundations of the theory: the basic principles and the derivation of multiform Euler-Lagrange equations, in particular through the 'double zero' phenomenon;
2. emergence of (higher) variational symmetries, conservation laws and (multiform) Noether theorem;
3. symmetry reductions (e.g. to integrable maps and Painlevé hierarchies) and connections to differential and difference invariant theory;
4. applications of pluri-Lagrangian structures to some yet unsolved, or less understood areas of integrable systems theory;
5. discrete and continuous differential geometric descriptions and the variational bicomplex;
6. the quantum formulation of the theory and Feynman path integrals and connections with solvable models in statistical mechanics;
7. connections with topological field theory and universal formulations of integrable field theory (e.g. through 4D Chern-Simons theory and 4D Wess-Zunino-Witten actions), and fundamental theories of physics (e.g., loop quantum gravity).

There has been distinct progress in these areas, bringing together various strands of research, and deepening the understanding of the nature and impact of the theory.

5 Outcome of the Meeting

The aim of the meeting was to bring this new field of variational calculus to the fore, present the state-of-the-art, and create an interface with other fields of research. In the latter respect, the workshop was successful in that it brought together experts in integrable systems, in topological field theory (4D Chern-Simons and Wess-Zumino-Witten models) and loop quantum gravity. There were restraints, in the sense that many confirmed participants could not attend in-person (due to the timing of the meeting in teaching season, and due to travel restrictions) and also some of the online participation was restricted by the time zone differences (7 hours with Europe, and 12 hours with the USA and Canada). In terms of diversity there were 9 female participants, among whom 3 speakers and the chair of the sandpit.

Nonetheless, the workshop has been very successful in attracting a number of early-stage researchers (there were 11 participating in-person and 2 online), especially from China, and also in forging new collaborations and novel points of contacts with scientists from cognate areas. The sandpit/panel discussion on the Wednesday afternoon, in which some online participants as well the present (in-person) audience took part, was lively and engaged the newcomers to the field. Some issues that had not been touched upon (like the potential for the theory to better understand/deal with anomalies in quantum field theory) came to the fore through the discussion. Another outcome of the meeting is some new collaborations that were forged during the meeting, and which would not (or not as easily) have come about without the workshop. The workshop was supported by BIRS, the IASM, Zhejiang University and the Academic Island, which offered the venue of the meeting.

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Participants

There were 40 in-person participants, and several online participants, among whom 10 speakers. Those who have been noted as online participants (indicated by a $*$) are listed below, but the list may be incomplete.

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