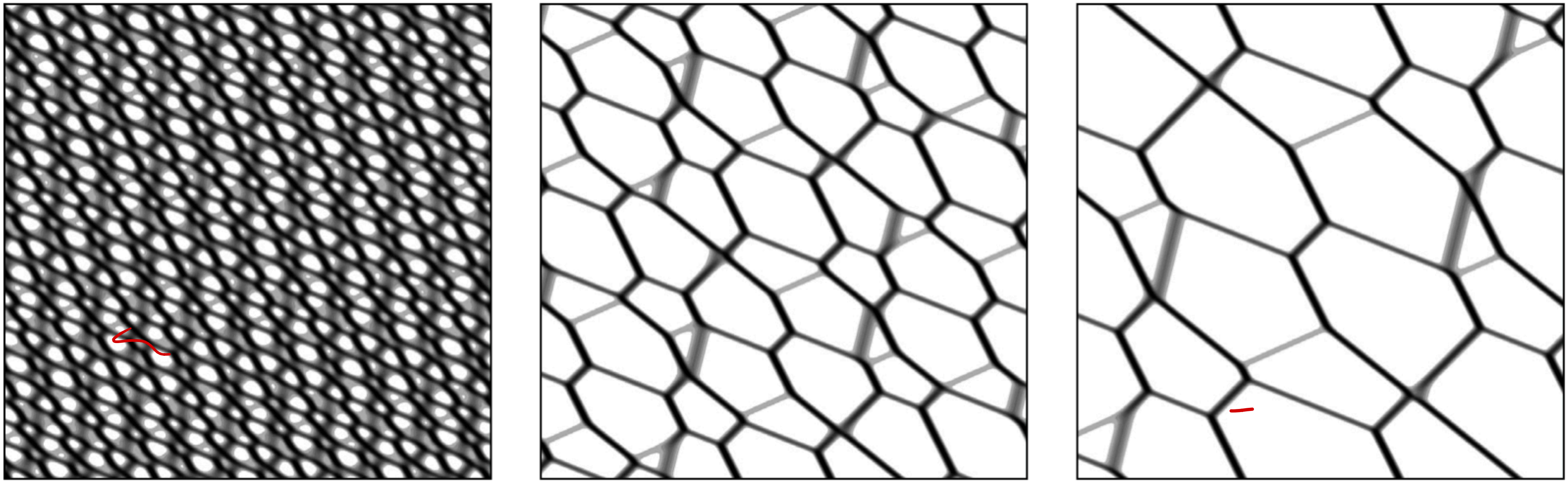
KP solitons, the Riemann theta functions and their applications to soliton gases

Yuji Kodama (SDUST & OSU)

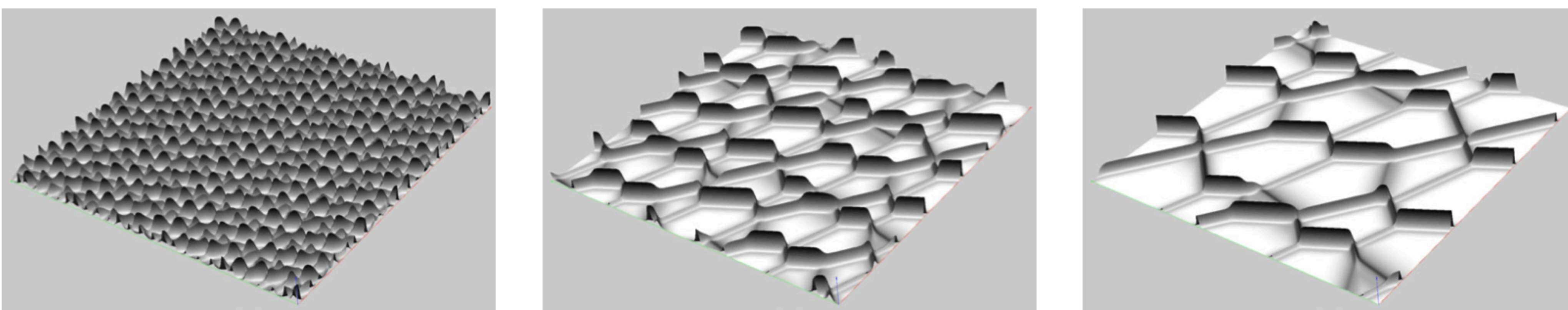
IASM-BIRS Workshop,

Hangzhou, Oct. 26, 2023



(a) (b) (c)

Fig. 7. Level lines for the solutions of the KP-II equation for (a) $\varepsilon = 10^{-2}$, (b) $\varepsilon = 10^{-10}$, and (c) $\varepsilon = 10^{-18}$. The horizontal axis is $-60 \le x \le 60$, and the vertical axis is $0 \le y \le 120$; $t = 0$. The light color corresponds to the lowest values of u, and the dark color, to the highest values of u.



(a) (b) (c)

Abenda - Grinevich (2017)

1. The Riemann theta functions

- M-theta function (Prym theta funct.)
- Vertex operators

2. KP solitons

- τ-functions on $Gr^{TNN}(N,M)$
- τ-functions as an M-theta functs.
  (Normalization and singular curves)

3. Applications

- KP soliton gas 1 (Phaseshifts)
- KP soliton gas 2 (spatial patterns)
- Solitons on quasi-periodic background.

  (elliptic solitons etc)

1. The Riemann theta functions

$$\vartheta_g(z;\Omega) = \sum_{m \in \mathbb{Z}^g} \exp 2\pi i \left(\frac{1}{2} m^T \Omega m + m^T z \right)$$

$z \in \mathbb{C}^g$,

$$\Omega \in \mathcal{H}_g := \left\{ \begin{array}{l} g \times g \text{ symmetric matrix} \\ \operatorname{Im} \Omega > 0 \end{array} \right\}$$

C : smooth compact Riemann surface

$\exists \{ a_1, \dots, a_g, b_1, \dots, b_g \} \subset H_1(C; \mathbb{Z})$

Canonical homological cycles

$\exists \{ \omega_1, \dots, \omega_g \}$: normalized holomorphic differentials

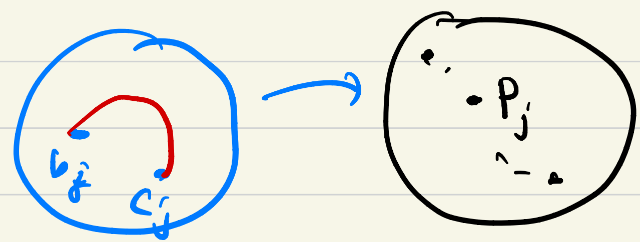
$$\oint_{a_i} \omega_j = \delta_{ij} \quad , \quad \oint_{b_i} \omega_j = \Omega_{ij}$$

Mumford (1984) constructed a theta function on a singular curve $\tilde{C}$ of $g=0$ with singular pts $S=\{p_1,\dots,p_g\}$. Assume these pts are ordinary double pts (nodes). Then $\exists$ the normalization

$$\pi : \mathbb{P} \longrightarrow \tilde{C}$$

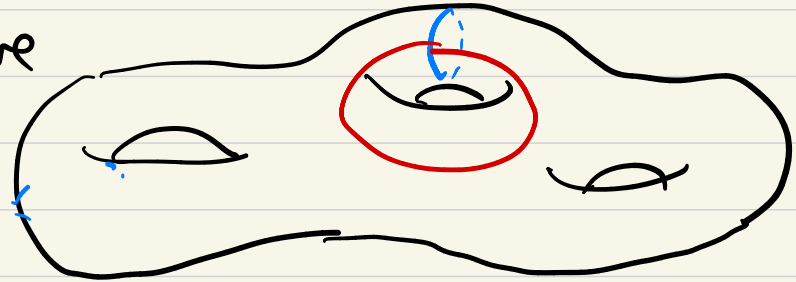
with

$$\pi^{-1}(p_j)=\{b_j,c_j\}$$



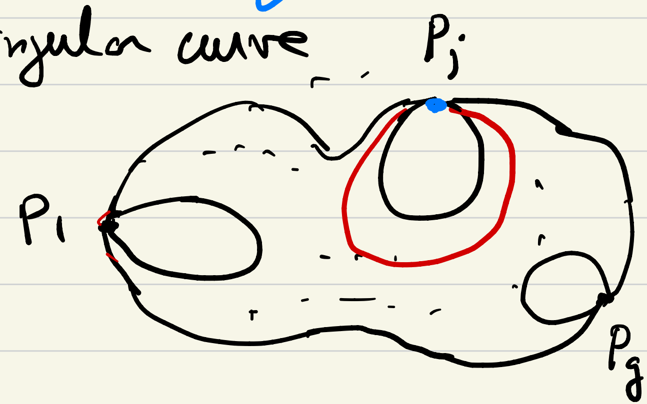
$$\pi(b_j)=\pi(c_j)=p_j$$

smooth curve

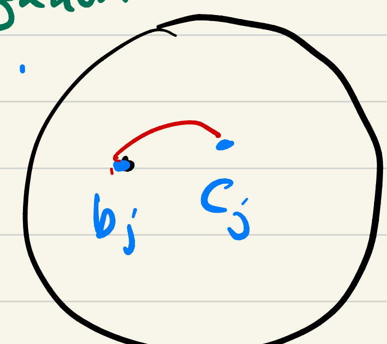


Pinch

Singular curve



desingularization



$$\omega_j \longrightarrow \tilde{\omega}_j = \left(\frac{1}{z-b_j} - \frac{1}{z-c_j} \right) dz$$

$$\left(\text{e.g.} \quad y^2 = \prod_{j=1}^{2g} (x-\lambda_j) \,, \qquad \omega_j = \frac{p_j(x)}{y} dx \right)$$

In the limit (Pinch), (Kella, 2011, Ichikawa 2023)

$$\boxed{\begin{aligned} &\mathrm{Im}\, \Omega_{jj} \longrightarrow \infty \qquad 1 \le j \le g \\ &\Omega_{jk} \to \hat{\Omega}_{jk} = \frac{1}{2\pi i} \int_{c_j}^{b_j} \hat{\omega}_k = \frac{1}{2\pi i} \log \frac{(b_j-b_k)(c_j-c_k)}{(b_j-c_k)(c_j-b_k)} \end{aligned}}$$

Before taking limits, consider shifts

$$z_j \to z_j - \frac{1}{2}\Omega_{jj}$$

$$\vartheta_g(z;\Omega) = \sum_{m\in\mathbb{Z}^g} \exp 2\pi i\left(\frac{1}{2}\sum_{j=1}^{g} \underline{m_j(m_j-1)\Omega_{jj}} + \sum_{j<k} m_j m_k \Omega_{jk} + \sum_{j=1}^{g} m_j z_j \right)$$

Then taking the limits $\underline{\Omega_{jj} \to +i\infty}$,

$$\vartheta_g \longrightarrow \tilde{\vartheta}_g(z;\tilde{\Omega}) = \sum_{m \in \{0,1\}^g} \exp 2\pi i \left(\sum_{j<k} m_j m_k \tilde{\Omega}_{jk} + \sum_{j=1}^{g} m_j z_j \right)$$

$$= \underbrace{1 + \sum_{j=1}^{g} e^{2\pi i z_j} + \cdots + e^{2\pi i \sum_{j<k} \tilde{\Omega}_{jk}} \, e^{2\pi i \sum_{j=1}^{g} z_j}}_{2^g \text{ terms.}}$$

$m = (0, \cdots, 0)$ $\qquad$ $(0 \cdots 0, 1, 0 \cdots 0)$ $\qquad$ $(1, 1, \cdots, 1)$

Remarks

① This is the Hirota g-soliton solution of the KP equation where

$$2\pi i z_j = \varphi_j = \sum_{n=1} (p_j^n - q_j^n) t_n$$

Also note this is the KP solitons corresponding to the lowest dimensional irreducible element of $Gr(g, 2g)$

② $\tilde{\vartheta}_g$ can be written in the form,

$$\hat{\vartheta}_g = \prod_{j=1}^{g} \left(1 + \hat{V}_j[\tilde{\Omega}]\right) \cdot 1$$

where $\hat{V}_j[\tilde{\Omega}] = \exp 2\pi i \left(z_j + \sum_{\substack{k=1 \\ k \neq j}}^{g} \hat{\Omega}_{jk} \frac{\partial}{\partial z_k} \right)$

Note:

$$\hat{V}_j[\tilde{\Omega}] \cdot \hat{V}_k[\tilde{\Omega}] = e^{2\pi i \tilde{\Omega}_{jk}} :V_j[\tilde{\Omega}] V_k[\tilde{\Omega}]:$$

$$\Downarrow$$

$$\frac{(b_j - b_k)(c_j - c_k)}{(b_j - c_k)(c_j - c_k)}$$

③ $\tilde{\vartheta}_g = \det\left(\delta_{j,k} + \frac{b_j - c_j}{b_j - c_k} e^{\pi i (z_j + z_k)}\right)$

the Grammian form

④ Considering double covers of singular curve, (Prym variety), BKP

$$\tilde{\Omega}^B_{jk} = \frac{1}{2\pi i} \int_{c_j}^{b_j} \left[\left(\frac{1}{z-b_\ell} - \frac{1}{z-c_\ell} \right) - \left(\frac{1}{z+b_\ell} - \frac{1}{z+c_\ell} \right) \right] dz$$

$\tilde{\vartheta}_g$ becomes a Pfaffian.

$\pi^{-1}(P_j^{\pm}) = \{\pm b_j, \pm c_j\}$

2. KP solitons

KP equation $(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$

$$u = 2\partial_x^2 \ln \tau.$$

<u>Theorem</u> Let $\{f_j : j = 1, \cdots, N\}$ be a set of indep. sols of the linear systems $\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2}$, $\frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3}$.

Then $\boxed{\tau = Wr(f_1, \cdots, f_N)}$

gives a solution of the KP equation.

Let $A=(a_{ij})\in Gr^{TNN}(N,M)$. [irreducible]

- No zero collumn
- No row having just pivot

Take $f_i=\sum_{j=1}^{M}a_{ij}e^{\xi_j}$, $\xi_j=k_jx+k_j^2y+k_j^3t$

Then the τ-function can be expressed as

$$\tau_A:=\det(AE^T),\quad E=\begin{pmatrix} e^{\xi_1} & \cdots & e^{\xi_M} \\ k_1e^{\xi_1} & \cdots & k_Me^{\xi_M} \\ \vdots & \vdots & \vdots \\ k_1^{N-1}e^{\xi_1} & \cdots & k_M^{N-1}e^{\xi_M} \end{pmatrix}$$

The Binet-Cauchy lem. gives

$$\tau_A = \sum_{I \in \mathcal{M}(A)} \Delta_I(A) E_I, \qquad I = \{ i_1, \dots, i_N \}$$

$$\mathcal{M}(A) = \left\{ I \in \binom{[M]}{N} \;\middle|\; \Delta_I(A) > 0 \right\}$$

$\Delta_I(A)$ = the minor corresp. to the column $\{i_1, \dots, i_N\}$.

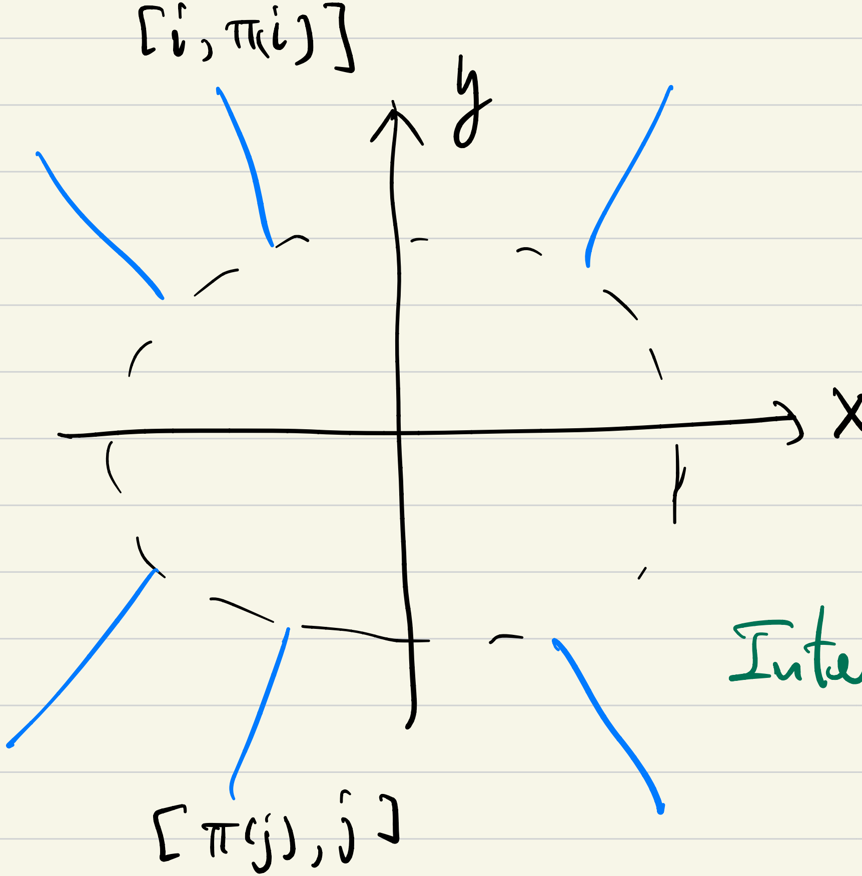
$$E_I = Wr\left(e^{\xi_{i_1}}, \dots, e^{\xi_{i_N}}\right) = \prod_{k<l} (\kappa_{i_k} - \kappa_{i_l}) e^{\xi_{i_1} + \dots + \xi_{i_N}}$$

With the order $(\kappa_1 < \kappa_2 < \dots < \kappa_M)$ $\quad E_I > 0$.

Lemma: Each $A \in Gr^{TNN}(N,M)$ can be parametrized by a derangement $\pi \in S_M$.

Theorem: Each KP soliton has the following properties: Let π be a derangement in S_M.

- $\forall\ \pi(i) > j$ (excedance) $\exists$ a soliton of type $[i, \pi(i)]$ in $y \gg 0$
- $\forall\ \pi(i) < j$ (anti-excedance) $\exists$ a soliton of type $[\pi(i), i]$ in $y \ll 0$.



$\exists$ N solitons in $y \gg 0$

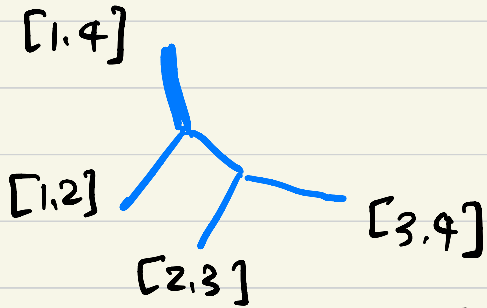
$\exists$ $M-N$ solitons in $y \ll 0$

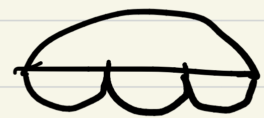
Interaction patterns consist of X and Y shapes

Examples

$Gr(N,4)$

- $N=1$ $\quad A=(1,*,*,*),$ $\quad \dim = 3$



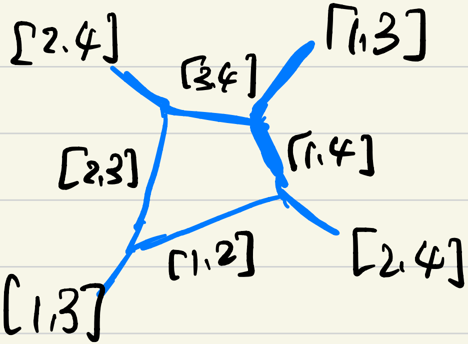


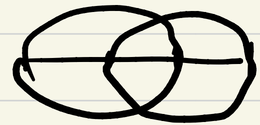
$\pi = (4123)$

$(g=3)$

- $N=2$

(a) $\quad A = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$ $\quad \dim = 4$

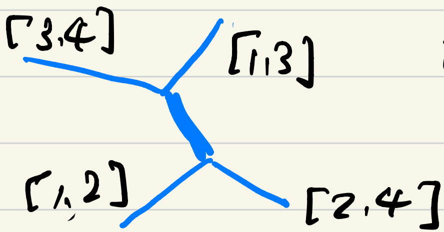




$\pi = (3412)$

$(g = 3)$

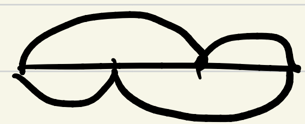
(b)



$$A = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

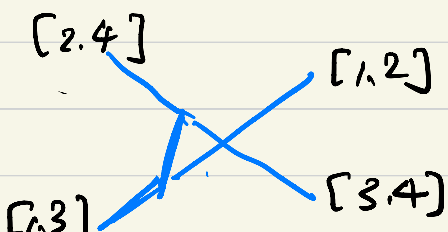
dim $= 3$

$(g = 3)$



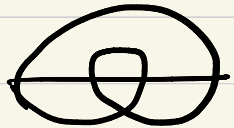
$\pi = (3142)$

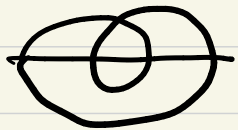
(c)

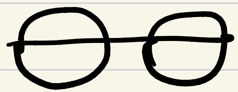


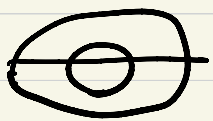
$\pi = (2413)$

dim $= 3$ $(g = 3)$

(d)  $\pi = (4312)$ $(g=3)$

(e)  $\pi = (3421)$ $(g=3)$

(f)  $\pi = (2143)$ $(g=2)$

(g)  $\pi = (4321)$ $(g=2)$

- $N=3$  $\pi = (2341)$ $(g=3)$

Let $I_0 \in \mathcal{M}(A)$ be the lexico. graphical min. of $\mathcal{M}(A)$.

Fine dividing τ_A by $\Delta_{I_0}(A) E_{I_0}$ $(\Delta_{I_0}(E) = 1)$,

$$\tau_A \hat{=} \tilde{\tau}_A = 1 + \sum_{I \in \mathcal{M}(A) \setminus \{I_0\}} \Delta_I(A) \frac{E_I}{E_{I_0}}$$

Theorem:

$$\boxed{\tilde{\tau}_A(x, y, t) = \tilde{\vartheta}_{\tilde{g}}(Z ; \tilde{\Omega})}$$

where

- $\tilde{g} = \left| \{ I \in \mathcal{M}(A) \mid |I \cap I_0| = N-1 \} \right|$

  $= \# \{ \text{nonzero elements in } A \} - \underset{\text{pivots}}{N}$

- $2\pi i z_j = \varphi_j \qquad 1 \le j \le \tilde{g}$

$$A = \begin{array}{c} \begin{array}{cccccccc} & 1 & & 2 & & \cdots & n_1 & \\ & & & & n_1+1 & \cdots & & n_1+n_2 \end{array} \\ \begin{pmatrix} 1 & * & 0 & * & \cdots & \cdots & * & 0 \\ & & 1 & 0 & * & \cdots & & * \\ & & & & \ddots & & & \\ & & & & \cdots & 1 & * \cdots & * \end{pmatrix} \\ \begin{array}{cccccccc} & & & & & & & \tilde{g} \end{array} \end{array}$$

- $\varphi_{\hat{g}_{k-1}+\ell} = \xi_{j_\ell^{(k)}} - \xi_{i_k} \qquad 1 \le \ell \le n_k$

$$A = \begin{array}{cccccccl}
\Big(0 & \cdots & 0 & 1 & 0 & * & \cdots \Big) & \leftarrow \ell\text{th row} \\
 & & & \uparrow & & \uparrow & & \\
 & & & i_k & & j_\ell^{(k)} & &
\end{array}$$

- Singular pts $\{p_1, \cdots, p_{\tilde{q}}\}$

$$\pi^{-1}(p_{\hat{g}_{k-1}+\ell}) = \{\underline{K_{i_k}}, \underline{K_{j_\ell^{(k)}}}\}$$

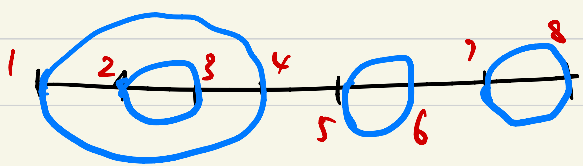
- $$e_{j\ell} := e^{2\pi i c_{j\ell}}$$
$$= \frac{(b_j - b_\ell)(c_j - c_\ell)}{(b_j - c_\ell)(c_j - b_\ell)}$$

$$\begin{cases} j = \hat{g}_{k-1} + m \\ \ell = \hat{g}_{k'-1} + m' \end{cases}$$

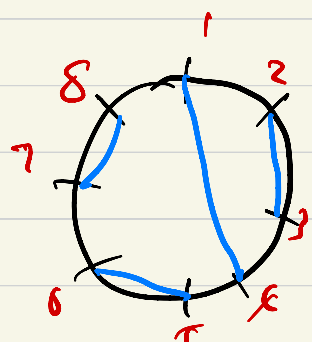
Example: Hirota 4-soliton

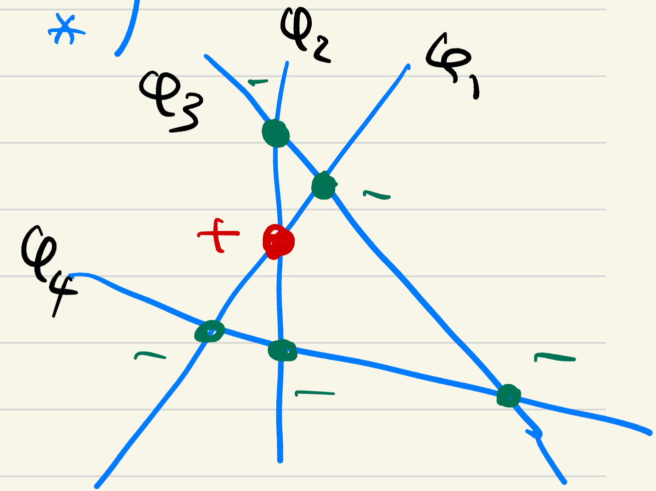
$$A = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & * & & & & \\ & 1 & * & 0 & & & & \\ & & & & 1 & * & 0 & 0 \\ & & & & & & 1 & * \end{pmatrix} \end{array} \in Gr^{TNN}(4,8)$$

$$\begin{cases} \varphi_1 = \xi_4 - \xi_1, & \varphi_3 = \xi_6 - \xi_5 \\ \varphi_2 = \xi_3 - \xi_2, & \varphi_4 = \xi_8 - \xi_7 \end{cases}$$



or





b	κ_1	κ_2	κ_5	κ_7
c	κ_4	κ_3	κ_6	κ_8

Example:

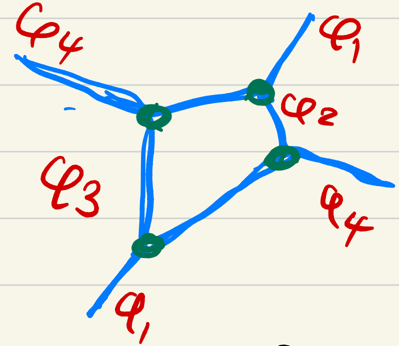
$$g = 3$$

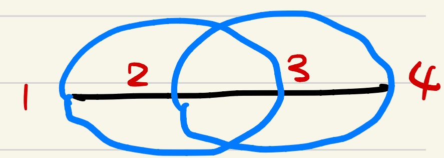
$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \end{array}$$

$$\varphi_1 + \varphi_4 = \varphi_2 + \varphi_3$$

$$\varphi_1 = \xi_3 - \xi_1, \quad \varphi_2 = \xi_4 - \xi_1, \quad \varphi_3 = \xi_3 - \xi_2, \quad \varphi_4 = \xi_4 - \xi_2$$

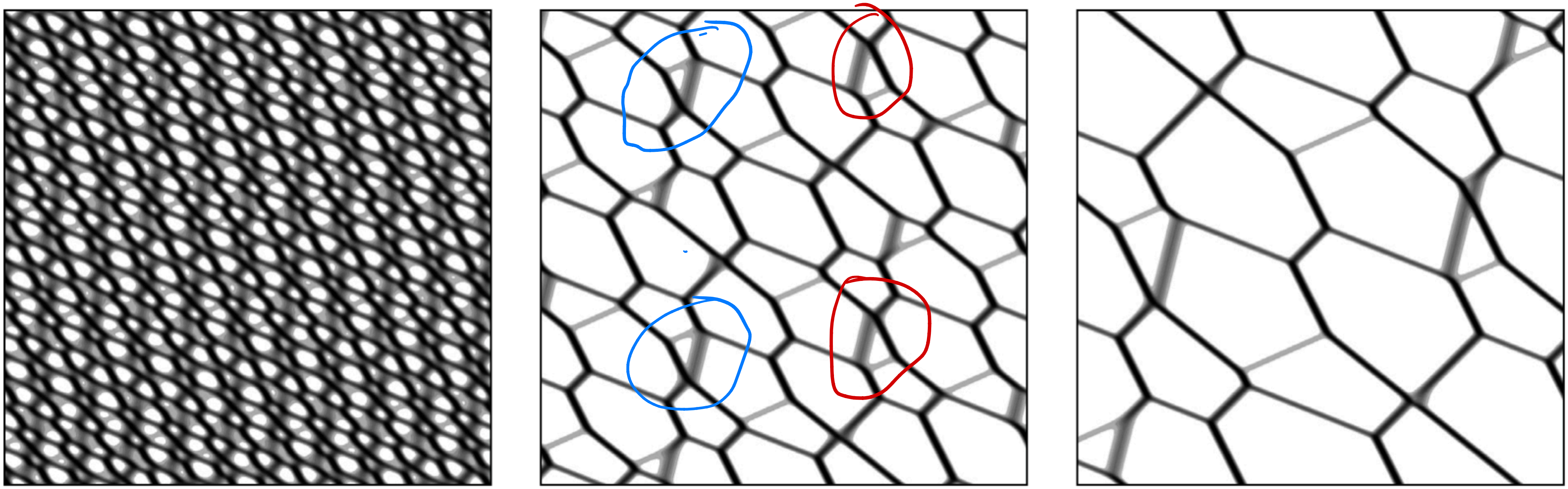
b	κ_1	κ_1	κ_2	κ_2
c	κ_3	κ_4	κ_3	κ_4





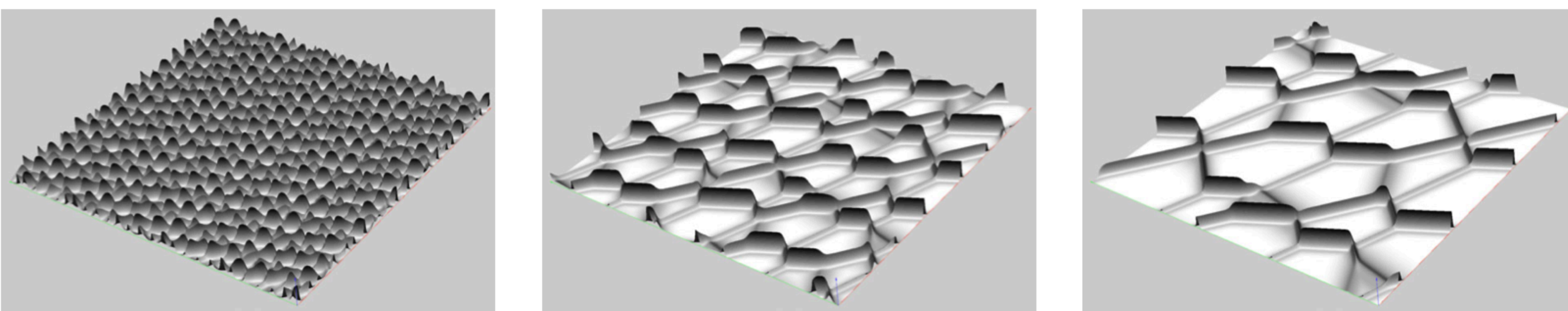
$(3, 4, 1, 2)$

$$\Rightarrow \quad C_{12} = C_{13} = C_{24} = C_{34} = 0 \qquad C_{14} \neq 0, \quad C_{23} \neq 0$$



(a) (b) (c)

Fig. 7. Level lines for the solutions of the KP-II equation for (a) $\varepsilon = 10^{-2}$, (b) $\varepsilon = 10^{-10}$, and (c) $\varepsilon = 10^{-18}$. The horizontal axis is $-60 \le x \le 60$, and the vertical axis is $0 \le y \le 120$; $t = 0$. The light color corresponds to the lowest values of u, and the dark color, to the highest values of u.

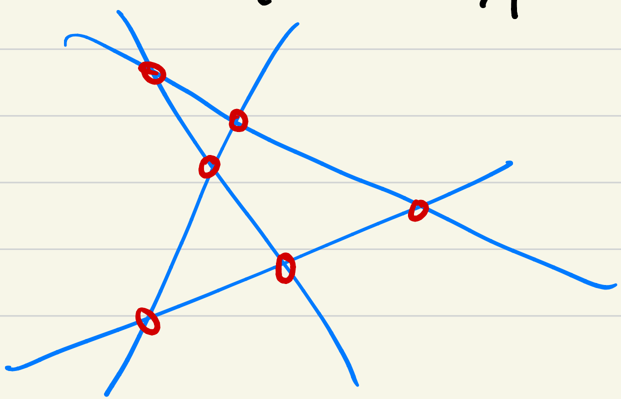


Abenda - Grinevich 2017, $(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$

Possible models of KP soliton gas

① Hirota g-soliton.

There are $C_g = \frac{1}{g+1}\binom{2g}{g}$ soliton solutions of this type



Give random phase shift and assign a proper weight for each C_{jk}.

② Give random permutation, with proper measure (e.g. Schur measure?).
Then determine the most likely pattern generated by KP solitons.

All $\mathrm{Im}\,\Omega_{jj}$ are large. (but finite)

Observe the patterns from $Gr(k, g)$ for $1 \le k \le g-1$. (Flag?)

③ KP solitons on quasi-periodic background. (Nakayashiki, Kakei, Zhang et al.)

Consider only for some $\Omega_{jj} \to +i\infty$.

$$\hat{\vartheta}_g^{(n)}(z;\tilde{\Omega}) = \prod_{j=1}^{n} \underbrace{(1+\hat{V}_j[\hat{\Omega}])}_{\text{soliton}} \underbrace{\vartheta_{g-n}(z^{(n)};\tilde{\Omega}^{(n)})}_{\text{Quasi-periodic}}$$

$$\begin{cases} \bar{\Omega}^{(n)} : & (g-n)\times(g-n) \text{ period matrix.} \\ z^{(n)} = (z_{n+1}, \dots, z_g) \end{cases}$$