On the construction of Lagrangian multiforms for infinite and finite dimensional integrable hierarchies

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Lagrangian Multiform Theory and Pluri-Lagrangian Systems
BIRS Workshop

• Pleasure to acknowledge collaboration and numerous discussions with M. Dell'Atti, F. Nijhoff, A.A. Singh, D. Sleigh, M. Stoppato, M. Vermeeren, B. Vicedo, C.Q. Zhang.

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• Today's talk based on following works

- VC, M. Dell'Atti, A.A. Singh, *Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems*, arXiv:2307.07339

- VC, M. Stoppato, B. Vicedo *Classical Yang-Baxter equation, Lagrangian multiforms and ultralocal integrable hierarchies*, arXiv:2201.08286

- VC, M. Stoppato, *Multiform description of the AKNS hierarchy and classical r-matrix*, J. Phys. A54 (2021)

- VC, M. Stoppato, *Hamiltonian multiform description of an integrable hierarchy*, J. Math. Phys. 61, 123506 (2020)

Plan

1. Variational criterion for integrability: Lagrangian multiforms
2. Review of some standard integrability frameworks
3. How to construct a Lagrangian multiform?
4. Conclusions and Perspectives

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1. Variational criterion for integrability: Lagrangian multiforms

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3. How to construct a Lagrangian multiform?

4. Conclusions and Perspectives

Important notational remark: L for “Lax stuff”, $\mathscr{L}$ for “Lagrangian stuff”

NB: in this talk, everything is for **continuous theories**: ODEs/PDEs or finite/infinite dimensional systems.

1. Variational criterion for integrability: Lagrangian multiforms

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• Idea of **Lagrangian multiforms** originally proposed in [Lobb, Nijhoff '09] as a variational framework to encode multidimensional consistency and integrability.

Then many contributions mainly by "Leeds and Berlin schools" until this workshop where hopefully it will inspire more people around the world.

Practical implementation

Step 1. Replace Lagrangian density $\mathscr{L}[\mathbf{q}]$ and action

$$S[\mathbf{q}] = \int_a^b \mathscr{L}[\mathbf{q}]\, dt$$

by a **collection of Lagrangians** $\mathscr{L}_k$ assembled into **a 1-form** $\mathscr{L}[\mathbf{q}]$ and an action:

$$S[\mathbf{q}, \Gamma] = \int_\Gamma \mathscr{L}[\mathbf{q}] = \int_\Gamma \mathscr{L}_1[\mathbf{q}]\, dt_1 + \cdots + \mathscr{L}_N[\mathbf{q}]\, dt_N$$

depending not only on $\mathbf{q}$ (and higher jets) but also on **a curve** Γ in the multi-time space $\mathbb{R}^N$.

1. Variational criterion for integrability: Lagrangian multiforms

Step 2. Propose a generalised variational principle with two ingredients:

(a) There exist (nontrivial) critical configurations $\mathbf{q}(t_1, \ldots, t_N)$ of $S[\mathbf{q}, \Gamma]$ which arise by imposing the **action principle for "arbitrary"** curves Γ

(b) On solutions or **on-shell**, the **value of** $S[\mathbf{q}, \Gamma]$ **is stationary w.r.t. (local) variations of** Γ.

1. Variational criterion for integrability: Lagrangian multiforms

• Consequences of (a): set of equations called **multi-time or multiform Euler-Lagrange (E-L) equations**. Two essential types of equations:

1. **Standard E-L eqs for each Lagrangian** $\mathscr{L}_k$ and its associated time t_k:

$$\frac{\partial \mathscr{L}_k}{\partial \mathbf{q}} - \frac{d}{dt_k}\frac{\partial \mathscr{L}_k}{\partial \mathbf{q}_{t_k}} = 0\,,$$

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2. **New E-L eqs**:

$$\frac{\partial \mathscr{L}_k}{\partial \mathbf{q}_{t_j}} = 0\,, \quad j \neq k\,, \quad \frac{\partial \mathscr{L}_k}{\partial \mathbf{q}_{t_k}} = \frac{\partial \mathscr{L}_j}{\partial \mathbf{q}_{t_j}}\,, \quad \forall j,k\,.$$

$\rightarrow$ Constraints on allowed Lagrangians.

1. Variational criterion for integrability: Lagrangian multiforms

• Consequence of (b). The **closure relation**: **on shell** we must have

$$d\left(\sum_{k=1}^{N} \mathscr{L}_k[\mathbf{q}]\, dt_k\right) = 0\,.$$

In components,

$$\frac{\partial \mathscr{L}_k}{\partial t_j} - \frac{\partial \mathscr{L}_j}{\partial t_k} = 0\,.$$

1. Variational criterion for integrability: Lagrangian multiforms

Integrable classical field theories

- All above ideas and results generalise to field theories (in $1+1$ and even $2+1$ dimensions). Focus on $1+1$ case.

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Step 1. For $1+1$ dimensional theories, replace action for a volume form

$$S[u] = \iint_D \mathscr{L}[u] dx \wedge dt$$

with an action for a 2-form

$$\mathcal{S}[u, \sigma] = \iint_\sigma \sum_{i<j} \mathscr{L}_{ij}[u]\, dt_i \wedge dt_j \,.$$

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σ is a 2D surface in an (infinite) multi-time space $\mathcal{M}$ with coordinates $(t_1, t_2, t_3, \dots)$: countable numbers of "times".

1. Variational criterion for integrability: Lagrangian multiforms

- Remark: **we need a 2-form**

$$\sum_{i<j} \mathscr{L}_{ij}[u]\, dt_i \wedge dt_j$$

and not just the naive extension

$$\sum_{j=1}^{\infty} \mathscr{L}_j[\mathbf{q}]\, dt_j$$

of the 1-form $\mathscr{L}_1[\mathbf{q}]\, dt_1 + \cdots + \mathscr{L}_N[\mathbf{q}]\, dt_N$ used in the finite dimensional case, as could be wrongly inferred from

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$\to \mathscr{L}_{ij}[u]$ suggests that the proper Hamiltonian counterpart should be a **covariant Hamiltonian** H_{ij}. This leads to the notion of **Hamiltonian multiform** [Caudrelier, Stoppato '20]

1. Variational criterion for integrability: Lagrangian multiforms

Step 2. Same generalised variational principle as before with same two consequences:

(a) **multi-time Euler-Lagrange equations** for each $\mathscr{L}_{ij}$.
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$$\delta d\mathscr{L} = 0$$

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(a) **multi-time Euler-Lagrange equations** for each $\mathscr{L}_{ij}$.
Remark: using the variational bicomplex, these can be written generally as

$$\delta d\mathscr{L} = 0$$

(b) **Closure relation**: $d\mathscr{L} = 0$ **on-shell**.

In components,

$$\partial_{t_k}\mathscr{L}_{ij} + \partial_{t_j}\mathscr{L}_{ki} + \partial_{t_i}\mathscr{L}_{jk} = 0$$

1. Variational criterion for integrability: Lagrangian multiforms

Two main problems

• **Why is it a good criterion for integrability?** Meaning that we could (should?) substitute it to other more well-known criteria.
→ Matter of taste or real "added value" (cf Suris's talk).

• **How to construct all the Lagrangian coefficients $\mathscr{L}_j$ or $\mathscr{L}_{ij}$?**
→ Amounts to a classification problem of integrable hierarchies. Lagrangian multiforms philosophy: the equations for the Lagrangians themselves can serve this purpose in principle.

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$\rightarrow$ Amounts to a classification problem of integrable hierarchies. Lagrangian multiforms philosophy: the equations for the Lagrangians themselves can serve this purpose in principle.
• Several (partial) results by brute force, use of variational symmetries or discrete to continuum limits.

1. Variational criterion for integrability: Lagrangian multiforms

• Our more modest approach: shed some light on these two questions simultaneously by trying to understand how Lagrangian multiform theory relates to two well established integrability criteria:

- Lax pairs;

- Hamiltonian structures and Liouville integrability.

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- Hamiltonian structures and Liouville integrability.

• First, review how these two aspects are related and inject this understanding into Lagrangian multiforms theory.
$\rightarrow$ Double reward: sheds new light on relation between integrability criteria and gives means to construct new integrable hierarchies (in field theories), e.g by coupling separate ones together relatively easily.

Lax pairs

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• Less obvious but far reaching fact: **Lax equation is Hamiltonian**.
There exist a Poisson bracket $\{\ ,\ \}$ and a function H such that

$$\frac{dL}{dt} = [M(L)\,,\,L] = \{L\,,\,H\}\,.$$

Key notion: integrable **hierarchies**

• The Hamiltonians $H_k = \mathrm{Tr}(L^k)$ satisfy $\{H_j, H_k\} = 0$
$\rightarrow$ can consider a **hierarchy of commuting Hamitonian flows**

$$\partial_{t_k} L = \{L\,,\, H_k\}$$

imposed simultaneously on L.

• Under suitable conditions, can be written back in Lax form to get a **hierarchy of Lax equations:**

$$\partial_{t_k} L = [M_k(L)\,,\, L]\ .$$

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• Link between these two pictures of integrability: there is a systematic way to construct a Poisson bracket $\{\ ,\ \}_R$ on an appropriate phase space for L such that

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$$\partial_{t_k} L = \{L\,,\, H_k\}_R = \frac{1}{2}\,[R(\nabla H_k(L))\,,\, L]\ .$$

$\rightarrow$ Semenov-Tian-Shansky's **Lie dialgebra theory**.

Key ingredients: In finite dimensions first

• Fix $\mathfrak{g}$ a (matrix) Lie algebra, $\mathfrak{g}^*$ its dual space. With $R:\mathfrak{g}\to\mathfrak{g}$ a **solution of the modified CYBE**

$$[R(X),R(Y)]-R\left([R(X),Y]+[X,R(Y)]\right)=-[X,Y]\,,\ \forall\, X,Y\in\mathfrak{g}$$

the bracket

$$[X,Y]_R=\frac{1}{2}\left([R(X),Y]+[X,R(Y)]\right)$$

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$$\{f, g\}_R(\xi) = (\xi, [\nabla f(\xi)\,, \nabla g(\xi)]_R)\,, \quad f, g \in C^\infty(\mathfrak{g}^*)\,,$$

• Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using an *ad*-invariant bilinear form $\langle\, ,\rangle$.

Then Semenov-Tian-Shansky's theorem implies that if we have a collection of independent Ad**-invariant functions** H_k then we have a hierarchy of commuting Hamiltonian flows which take the Lax form

$$\partial_{t_k} L = \{L, H_k\}_R = [M_k, L] \,, \quad M_k = \frac{1}{2} R \nabla H_k(L) \,, \ \{H_j; H_k\}_R = 0 \,.$$

and the natural choice of phase space for L is a coadjoint orbit of the group G_R in $\mathfrak{g}^*$:

$\rightarrow L = \mathrm{Ad}^{R*}_{\varphi} \cdot \Lambda$, roughly think of "$L = \varphi \, \Lambda \, \varphi^{-1}$".

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- Commutativity of flows ensured by zero curvature (ZC) eqs

$$\partial_{t_k} M_j - \partial_{t_j} M_k + [M_k\,,\, M_j] = 0$$

themselves ensured by modified CYBE.

NB: ZC eqs will always be in the background for us.

3. How to construct a Lagrangian multiform?

Taking this on board **[Caudrelier, Dell'Atti, Singh '23]**

• Parametrise $L = \mathrm{Ad}^{R*}_{\varphi} \cdot \Lambda$, field $\varphi \in G_R$ contains the dynamical degrees of freedom.

• Define

$$\mathscr{L}[\varphi] = \sum_{k=1}^{N} \mathscr{L}_k[\varphi]\, dt_k \tag{1}$$

with

$$\mathscr{L}_k[\varphi] = \left(L\,,\, \partial_{t_k}\varphi \cdot_R \varphi^{-1} \right) - H_k(L)\,. \tag{2}$$

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Remark: Kinetic part looks a bit complicated at first but in group coordinates, each $\mathscr{L}_k$ is of familiar form

$$\mathbf{p} \cdot \partial_{t_k} \mathbf{q} - H_k$$

Theorem

The standard Euler-Lagrange equations associated with the Lagrangian coefficients $\mathscr{L}_k$ take the form of compatible Lax equations

$$\partial_{t_k} L = \frac{1}{2}[R\nabla H_k(L), L]\,, \qquad k = 1, \dots, N\,. \tag{3}$$

The remaining multi-time Euler-Lagrange equations for the Lagrangian 1*-form* (1) *are trivially satisfied.*
The closure relation holds: on solutions of (3) *we have*

$$\partial_{t_k} \mathscr{L}_j - \partial_{t_j} \mathscr{L}_k = 0\,, \qquad j, k = 1, \dots, N\,.$$

Examples

• Open Toda chain, achieved in two different ways: different Lie algebra decompositions and r-matrices
$\rightarrow$ one is skew-symmetric, the other not!

• Comment: Our construction gives natural canonical variables which are neither the usual q, p Toda variables nor the Flaschka variables.

• (Rational) Gaudin model: detailed account of construction for this example in Anup's talk later today.

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The following **identity** *holds*

$$\frac{\partial \mathscr{L}_k}{\partial t_\ell} - \frac{\partial \mathscr{L}_\ell}{\partial t_k} + \Upsilon_k^m \, P_{mn} \, \Upsilon_\ell^n = \{H_k, H_\ell\}_R \, .$$

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R Lie-Poisson bracket of Hamiltonians, controlling involutivity

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R Lie-Poisson bracket of Hamiltonians, controlling involutivity

→ **Corollary: equivalence of closure relation and involutivity of Hamiltonians**

Infinite dimensional case: same ideas and structures apply!

• Hamiltonian/Lax pair relation for Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy first observed in FNR [Flaschka, Newell, Ratiu '83], without knowledge of r-matrix.

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- Presence of spectral parameter λ: work with loop algebras.

$$L(\lambda)=\sum_{j=0}^{\infty}L_j\lambda^{-j}\,,\quad L_j=\begin{pmatrix}a_j & b_j\\ c_j & -a_j\end{pmatrix}\in \mathrm{sl}(2,\mathbb{C})\,,\quad L_0=-i\sigma_3\,.$$

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• **Hierarchy of Hamiltonian flows/Lax equations** reads

$$\partial_{t_k} L(\lambda) = \{L\,,\, H_k\}_R = [M_k(\lambda), L(\lambda)]\,, \quad M_k(\lambda) = P_+(\lambda^k L(\lambda))$$

$$M_k(\lambda) = \lambda^k L_0 + \cdots + L_k \ \ \text{(Lax matrix for the } t_k \text{ flow)}$$

• t_1, t_2, t_3 flows give **NLS and mKdV systems** for $b_1 = q$, $c_1 = r$

$$\text{NLS:} \qquad i\partial_{t_2} q + \frac{1}{2}\partial_{t_1}^2 q - q^2 r = 0, \qquad -i\partial_{t_2} r - \frac{1}{2}\partial_{t_1}^2 r - q r^2 = 0$$

$$\text{mKdV:} \quad \partial_{t_3} q + \frac{1}{4}\partial_{t_1}^3 q - \frac{3}{2} q r \partial_{t_1} q = 0, \quad \partial_{t_3} r + \frac{1}{4}\partial_{t_1}^3 r - \frac{3}{2} q r \partial_{t_1} r = 0$$

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- As before, our goal was to construct an entire Lagrangian multiform for the AKNS hierarchy of Lax equations

$$\partial_{t_k} L(\lambda) = [P_+(\lambda^k L(\lambda))\,,\, L(\lambda)] \tag{4}$$

$\to$ done in [Caudrelier, Stoppato '21] thanks to some inspiration.

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- Nijhoff's idea of compounding a hierarchy using formal series

$$\partial_\mu \equiv \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} \partial_{t_k}\,, \quad \iota_\mu \frac{1}{\mu - \lambda} = \sum_{k=00}^{\infty} \frac{\lambda^k}{\mu^{k+1}}$$

allows us to write (4) compactly as a **generating Lax equation for integrable hierarchy**

$$\partial_\mu L(\lambda) = \left[\iota_\mu \frac{1}{\mu - \lambda} L(\mu), L(\lambda)\right].$$

3. How to construct a Lagrangian multiform?

• This leads to the idea of using a **generating Lagrangian multiform**

$$\mathscr{L}(\lambda,\mu)=\sum_{i,j=0}^{\infty}\frac{\mathscr{L}_{ij}}{\lambda^{i+1}\mu^{j+1}}$$

• Then, inspired by **[Zakharov, Mikhailov '80]** and precursor work **[Sleigh, Nijhoff, Caudrelier '19]**, we proposed the following formula for $\mathscr{L}(\lambda,\mu)$:

3. How to construct a Lagrangian multiform?

- $\mathscr{L}(\lambda,\mu) = K(\lambda,\mu) - V(\lambda,\mu)$ with

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- $\phi(\lambda) = \mathbb{1} + \sum_{j=1}^{\infty}\frac{\phi_j}{\lambda^j}$: group element containing the fields/phase space coordinates

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Theorem

$\mathscr{L}(\lambda,\mu)$ is a Lagrangian multiform for the AKNS hierarchy equations i.e.

$$\delta d\mathscr{L} = 0 \Leftrightarrow \partial_\mu L(\lambda) = \left[\iota_\mu \frac{1}{\mu-\lambda} L(\mu), L(\lambda)\right],$$

and $d\mathscr{L} = 0$ on these equations (closure relation). In generating form, the latter is equivalent to

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- Componentwise

$$\delta \mathscr{L}_{jk} = 0 \Leftrightarrow \partial_{t_j} M_k(\lambda) - \partial_{t_k} M_j(\lambda) + [M_k(\lambda), M_j(\lambda)] = 0 \quad j,k \geq 0$$

where $\mathscr{L}_{ij}$ easily obtained by taking residues

$$\mathscr{L}_{ij} = \operatorname{res}_\lambda \operatorname{res}_\mu \lambda^i \mu^j \mathscr{L}(\lambda,\mu)$$

Example: Nonlinear Schrödinger and modified KdV in Ablowitz-Kaup-Newell-Segur hierarchy for fields q, r. Times $t_1(=x)$, t_2, t_3. Notations

$$\partial_{t_i}^n q \equiv q_{ii\dots i}\,, \quad i=1,2,3$$

$$\text{NLS:} \qquad q_2 - \tfrac{i}{2}q_{11} + iq^2 r = 0 \qquad r_2 + \tfrac{i}{2}r_{11} - iqr^2 = 0,$$

$$\text{mKdV:} \quad q_3 + \tfrac{1}{4}q_{111} - \tfrac{3}{2}qrq_1 = 0 \quad r_3 + \tfrac{1}{4}r_{111} - \tfrac{3}{2}qrr_1 = 0\,.$$

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Lagrangians

$$\mathscr{L}_{12} = \frac{1}{2}(rq_2 - qr_2) + \frac{i}{2}q_1r_1 + \frac{i}{2}q^2r^2$$

$$\mathscr{L}_{13} = \frac{1}{2}(rq_3 - qr_3) - \frac{1}{8}(r_1q_{11} - q_1r_{11}) - \frac{3}{8}qr(rq_1 - qr_1)$$

3. How to construct a Lagrangian multiform?

Key observations for a large generalisation:

$\rightarrow$ Beyond a Lagrangian multiform for a single hierarchy.

1. The potential term in $\mathscr{L}(\lambda,\mu)$ has a characteristic form

$$\mathrm{Tr}_{12}\left(r_{12}(\lambda,\mu)L_1(\lambda)L_2(\mu)\right)$$

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$\to$ How about replacing this particular r-matrix with another (skew-symmetric) r-matrix?

Remark: this "avatar" of the r-matrix is related to linear map R used before via

$$(RX)(\lambda)=\oint\frac{d\mu}{2i\pi}\,\mathrm{Tr}_2\left(r_{12}(\lambda,\mu)X_2(\mu)\right)$$

with $X_2(\mu)=\mathbb{1}\otimes X(\mu)$, $r_{12}(\lambda,\mu)=r_{ab}(\lambda,\mu)I^a\otimes I^b$.

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$\rightarrow$ How about considering other elements in the loop algebra to construct different phase spaces and even considering other Lie algebras than sl_2?

• Careful implementation of these observations involves using the **Lie algebra of $\mathfrak{g}$-valued adèles** associated with a Lie algebra $\mathfrak{g}$ [Semenov-Tian-Shansky '08].

3. How to construct a Lagrangian multiform?

• Careful implementation of these observations involves using the **Lie algebra of $\mathfrak{g}$-valued adèles** associated with a Lie algebra $\mathfrak{g}$ [Semenov-Tian-Shansky '08].

• In a nutshell, with $\lambda_a = \lambda - a$ for $a \in \mathbb{C}$ and $\lambda_\infty = \frac{1}{\lambda}$, we work with tuples $\boldsymbol{X}(\boldsymbol{\lambda}) = (X^a(\lambda_a))_{a \in \mathbb{C}P^1}$ where all but finitely many of the formal Laurent series $X^a(\lambda_a) \in \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_a)\!)$ are Taylor series in λ_a.

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- This framework allows for flexibility to work "locally" around point in $\mathbb{C}P^1$ while keeping the language of formal series with coefficients in a Lie algebra $\mathfrak{g}$.

3. How to construct a Lagrangian multiform?

Schematic implementation of the generalisation

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\mathscr{L}(\lambda,\mu) & \rightarrow & \mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu}) = \text{collection of } \mathscr{L}^{a,b}(\lambda_a,\mu_b)
\end{array}$$

Main results

Theorem

The generating Lax equation

$$\mathcal{D}_{\boldsymbol{\mu}}\boldsymbol{L}_1(\boldsymbol{\lambda}) = \left[\operatorname{Tr}_2\left(\boldsymbol{\iota}_{\boldsymbol{\lambda}}\boldsymbol{\iota}_{\boldsymbol{\mu}} r_{12}(\lambda,\mu)\boldsymbol{L}_2(\boldsymbol{\mu})\right), \boldsymbol{L}_1(\boldsymbol{\lambda})\right]. \tag{5}$$

is variational: it derives from the multiform EL eqs for $\mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu})$.
The flows (5) on the Lie algebra of $\mathfrak{g}$-valued adèles commute as a consequence of the CYBE

$$\left[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)\right] + \left[r_{12}(\lambda,\mu), r_{23}(\mu,\nu)\right] - \left[r_{13}(\lambda,\nu), r_{32}(\nu,\mu)\right] = 0.$$

3. How to construct a Lagrangian multiform?

Theorem

The closure relation in generating form

$$\mathcal{D}_{\boldsymbol{\nu}}\mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu})+\mathcal{D}_{\boldsymbol{\mu}}\mathscr{L}(\boldsymbol{\nu},\boldsymbol{\lambda})+\mathcal{D}_{\boldsymbol{\lambda}}\mathscr{L}(\boldsymbol{\mu},\boldsymbol{\nu})=0$$

holds as a consequence of the CYBE equation.

$\rightarrow$ **CYBE cast in variational framework for the first time**

If interested in models corresponding to specific times then:

- Extract **elementary Lagrangians** as

$$\mathscr{L}^{a,b}_{m,n} := \operatorname{res}^{\lambda}_{a} \operatorname{res}^{\mu}_{b} \mathscr{L}^{a,b}(\lambda_a, \mu_b)\lambda^{-m-1}d\lambda\, \mu^{-n-1}d\mu$$

- Compute **elementary Lax matrices** $V^a_m(\lambda)$ similarly from

$$\boldsymbol{V}(\lambda; \boldsymbol{\mu}) := \mathrm{Tr}_2\left(\boldsymbol{\iota}_{\boldsymbol{\mu}} r_{12}(\lambda, \mu)\boldsymbol{L}_2(\boldsymbol{\mu})\right)$$

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- Then Euler-Lagrange eqs for $\mathscr{L}^{a,b}_{m,n}$ equivalent to zero curvature equation for times t^a_m, t^b_n

$$\partial_{t^b_n}V^a_m(\lambda) - \partial_{t^a_m}V^b_n(\lambda) + \left[V^a_m(\lambda), V^b_n(\lambda)\right] = 0$$

$\rightarrow$ **Full integrable hierarchy in variational form.**

3. How to construct a Lagrangian multiform?

Procedure to get examples.

Choose:

(*i*) a skew-symmetric r-matrix (rational or trig for us),

(*ii*) an effective divisor $\mathcal{D} := \sum_{a\in S} N_a a$, with support given by a finite subset $S \subset \mathbb{C}P^1$,

(*ii*) a Lie algebra $\mathfrak{g}$ which for simplicity we take to be either gl_N or sl_N,

(*iv*) a $\mathfrak{g}$-valued rational function $F(\lambda) \in R_\lambda(\mathfrak{g})$ with poles divisor $(F)_\infty = \mathcal{D}$, *i.e.* with a pole of order N_a at each point $a \in S$.

3. How to construct a Lagrangian multiform?

Recovering the original AKNS example.

Fix the data as

$$S = \{\infty\}\,, \quad N_\infty = 0\,, \quad \mathfrak{g} = \mathrm{sl}_2\,, \quad F(\lambda) = -i\sigma_3\,,$$

and choose the rational r-matrix $r_{12}(\lambda, \mu) = \frac{P_{12}}{\mu - \lambda}$.

Sine-Gordon hierarchy

For the hierarchy of the sine-Gordon equation (in light-cone coords)

$$u_{xy} + \sin u = 0\,,$$

we fix $S = \{0, \infty\}$, $N_0 = 1 = N_\infty$, $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C})$,

$$F(\lambda) = \frac{i}{2}\left(\frac{1}{\lambda}\sigma_+ + \sigma_- - \sigma_+ - \lambda\sigma_-\right)$$

and we choose the trigonometric r-matrix

$$r_{12}^{\mathrm{trig}}(\lambda, \mu) = \frac{1}{2}\left(P_{12}^+ - P_{12}^- + \frac{\mu + \lambda}{\mu - \lambda} P_{12}\right)$$

3. How to construct a Lagrangian multiform?

We can derive all elementary Lagrangians. We find

$$\mathscr{L}_{\mathrm{sG}} \equiv \mathscr{L}_{00}^{0,\infty} = -\frac{1}{4}u_x u_y - \frac{1}{2}\cos u$$

$$\mathscr{L}_{\mathrm{mKdV}} \equiv \mathscr{L}_{01}^{\infty,\infty} = \frac{1}{4}u_x u_z + \frac{1}{16}u_x^4 - \frac{1}{4}u_{xx}^2 - \frac{i}{4}\partial_x\left(\frac{1}{6}u_x^3 + iu_x u_{xx}\right)$$

$$\begin{aligned}\mathscr{L}_{\mathrm{mixed}} \equiv \mathscr{L}_{01}^{0,\infty} \quad = \quad & -\frac{1}{4}u_y u_z - \frac{1}{2}u_{xx}(u_{xy} + \sin u) + \frac{1}{4}u_x^2 \cos u \\ & -\frac{i}{4}\partial_y\left(\frac{1}{6}u_x^3 + iu_x u_{xx}\right)\end{aligned}$$

Recover the results of [Suris '16]. Can also derive all relevant Lax matrices and zero curvature equations.

3. How to construct a Lagrangian multiform?

Hierarchies of Zakharov-Mikhailov type

Correspond to Lax matrices of Zakharov-Shabat type: rational Lax matrices with prescribed pole structures.

• In our setup, choose the following data

$$S = \{a_1, \dots, a_P\} \subset \mathbb{C}\,, \quad P > 0\,, \quad \mathfrak{g} = \mathrm{gl}_N\,,$$

$$F(\lambda) = -\sum_{i=1}^{P} \sum_{r=0}^{n_i} \frac{A_{ir}}{(\lambda - a_i)^{r+1}}\,.$$

• Each $A_{ir} \in \mathrm{gl}_N$ is a non-dynamical constant matrix.

• r-matrix can be the rational (original Zakharov-Mikhailov case) or trigonometric (new models). Even in rational case, obtain full hierarchy, not just a single model/level.

3. How to construct a Lagrangian multiform?

Most famous example: Faddeev-Reshetikhin version of Principal chiral model

- 2 simple poles $a, b = -a$ in S,

$$F(\lambda) = -\frac{A}{(\lambda - a)} - \frac{B}{(\lambda + a)}\,.$$

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- Lowest elementary Lax matrices for times $t^a_{-1} \equiv \xi$, $t^{-a}_{-1} \equiv \eta$

$$V^a_{-1}(\lambda) = \frac{\phi A \phi^{-1}}{\lambda - a} \equiv \frac{J_0}{\lambda - a}\,, \quad V^b_{-1}(\lambda) = \frac{\psi B \psi^{-1}}{\lambda + a} \equiv \frac{J_1}{\lambda + a}$$

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- Zero curvature equations

$$\partial_\eta J_0 + \frac{1}{2a}\,[J_0, J_1] = 0\,, \quad \partial_\xi J_1 + \frac{1}{2a}\,[J_0, J_1] = 0\,.$$

- We get the lowest elementary Lagrangian as

$$\mathscr{L}^{ab}_{-1-1} = \mathrm{Tr}\left(\phi^{-1}\partial_\eta\phi A - \psi^{-1}\partial_\xi\psi B - \frac{\phi A\phi^{-1}\,\psi B\psi^{-1}}{2a}\right).$$

3. How to construct a Lagrangian multiform?

Coupling models/hierarchies

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• Procedure produces new models/hierarchies that are automatically integrable. NB: not the same as taking combination of flows within a hierarchy.

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• Example: **couple nonlinear Schrödinger to Faddeev-Reshetikhin model**

$$S = \{a, -a, \infty\}\,, \quad a \in \mathbb{C}^\times\,, N_a = N_b = 1\,, \quad N_\infty = 0\,, \quad \mathfrak{g} = \mathrm{sl}_2\,,$$

$$F(\lambda) = -i\alpha\sigma_3 + \frac{A}{\lambda - a} + \frac{B}{\lambda + a} \equiv \alpha F^{AKNS}(\lambda) + F^{FR}(\lambda)\,,$$

where A, B are constant sl_2 matrices.

• α couples the two theories: $\alpha = 0$ gives a pure FR theory while sending α to infinity produces a pure AKNS hierarchy.

3. How to construct a Lagrangian multiform?

- Equations of motion at lowest level in the hierarchy:

$$\alpha\partial_x L_1 + i\alpha^2[\sigma_3, L_2] + i\alpha[J_0, \sigma_3] = 0\,,$$

$$\alpha\partial_t L_1 - \alpha\partial_x L_2 + \alpha^2[L_1, L_2] - ia\alpha[J_0, \sigma_3] - i\alpha[\sigma_3, J_1] + \alpha[J_0, L_1] = 0\,.$$

$$\partial_t J_0 + \frac{1}{2a}\,[J_0, J_1] + \alpha\,[J_0, V_{NLS}(a)] = 0\,,$$

$$\partial_x J_1 + \frac{1}{2a}\,[J_0, J_1] - \alpha\,[U_{NLS}(-a), J_1] = 0\,,$$

Nonlinear Schrödinger part
Faddeev-Reshetikhin part
Coupling between them

4. Conclusions and Perspectives

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- Found new integrable hierarchies (trigonometric ZM) and a mechanism to couple hierarchies very easily.

4. Conclusions and Perspectives

- Some open questions: from specific to general

- Systems such as Calogero-Moser or Ruijsenaars models do not seem to fit in our construction but they are known to be derivable from Hamiltonian reduction ideas. Could we use this to derive Lagrangian multiforms for these systems? Compare with known results.

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- closure relation equivalence to $\{H_i, H_j\} = 0$ also established in field theory [Vermeeren '21].

$\rightarrow$ Non covariant nature of Hamiltonians H_i in field theory suggests that **covariant Hamiltonians** $\mathcal{H}_{ij}$ should play a role. For $1+1$ field theories, closure relation equivalent to

$$\partial_{t_k}\mathcal{H}_{ij} + \partial_{t_j}\mathcal{H}_{ki} + \partial_{t_i}\mathcal{H}_{jk} = 0$$

but open problem: covariant Poisson bracket formulation in terms of $\mathcal{H}_{ij}$?

- In a sense, our construction is "sufficient". Could we imagine a sort of converse under (mild) assumptions *i.e.* prove that if the multi-time EL eqs and closure relation are imposed on a given Lagrangian form, then necessarily the algebraic structures we used appear naturally? Related to general classification problem. Also, to my knowledge, one could pose the same problem in Hamiltonian framework? It is obvious that a full converse is impossible (i.e. classical r-matrix is not a necessary structure) since there exists integrable systems with **dynamical** r**-matrices** which are not captured by our construction.

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- Another big challenge: develop the geometry of Lagrangian/ Hamiltonian multiforms to establish a **Liouville theorem in infinite dimensions**?

4. Conclusions and Perspectives

• Ulterior motive: **quantisation** of integrable (field) theories **via Feynman path integrals**: develop a Lagrangian counterpart of impressive developments of the Quantum Inverse Scattering Method/Algebraic Bethe Ansatz.

→ Maybe not just a reformulation of known results but could open the way to the quantization of non ultralocal theories which conventional QISM framework cannot treat. Yet another big challenge.

It would be very interesting to see what role Andrew Kels's results play in this programme.

Could the interplay with certain gauge theories (BF or Chern-Simons) be useful? (cf. Benoit Vicedo's talk yesterday)

THANK YOU!