Matrix-valued Orthogonal Polynomials & Non-commutative Integrable Systems

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1. Introduction

Orthogonal Polynomials $\longleftrightarrow$ Integrable Systems

Origin: ① Inverse scattering of Toda equation (Moser, Deift, ⋯)

② Quantum field theory (2-dimensional gravity, matrix model, ⋯)

(1975)

FINITELY MANY MASS POINTS ON THE LINE UNDER THE INFLUENCE OF AN EXPONENTIAL POTENTIAL -- AN INTEGRABLE SYSTEM

Jürgen Moser*

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1. Analogue of the Toda Lattice for Finitely Many Mass Points

We consider the analogue of the Toda lattice [8] where only a finite number of mass points are admitted which move freely on the real axis. Denoting the position of the mass points by x_k, $k = 1,\ldots,n$, we form the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})} \tag{1.1}$$

Moser's original work: solving Toda eq. by continued fraction & Hankel determinant

(1990)

MATRIX MODELS OF TWO-DIMENSIONAL GRAVITY AND TODA THEORY

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Gerasimov et al showed that partition functions of several different matrix models are τ-functions for different integrable systems

1. Introduction

Definition: We call polynomials $\{P_n(x)\}_{n\in N}$ are orthogonal with respect to weight $\omega(x)$, if they satisfy

$$\int_{\mathbb{R}} P_n(x)\, P_m(x)\, \underline{\omega(x)}\, dx = h_n \delta_{n,m}, \quad h_n > 0.$$

non-negative weight function

Deformed orthogonal polynomials:

$$\omega(x) \longrightarrow \omega(x;t)$$

$$P_n(x) \longrightarrow P_n(x;t)$$

$$\int_{\mathbb{R}} P_n(x;t)\, P_m(x;t)\, \omega(x;t)\, dx = h_n(t)\, \delta_{n,m}$$

1. Introduction

Orthogonal Polynomials $\xleftrightarrow[\text{is introduced}]{\text{if time variable}}$ Integrable Systems

polynomials	wave function
recurrence relation	spectral problem
ladder operator	Lax operator
normalization factor	τ- function

(1995 - 2000)

Group factorization, moment matrices and Toda lattices*

M. Adler† P. van Moerbeke‡

Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials

M. Adler* P. van Moerbeke†

Adler and van Moerbeke have done a serial of works regarding with orthogonal polynomials, Fay identities, Virasoro constraints, ...

1. Introduction

Orthogonal Polynomials $\longleftrightarrow$ Integrable Systems

generalized orthogonal polynomials

Bi-orthogonal polynomials	2d-Toda theory	(Adler & van Moerbeke Ann. Math, 1999)
Cauchy bi-orthogonal polynomials	Toda equation of CKP type	(C. Li & SHL, JNS, 2019)
Multiple orthogonal polynomials	Multi-component Toda	(Adler & van Moerbeke CMP, 2010)
Skew-orthogonal polynomials	Pfaff lattice	(Adler & van Moerbeke IMRN, 1999 & Duke Math J. 2002)
Partial-skew-orthogonal polynomials	Toda equation of BKP type	(X. Chang, Y. He, X. Hu & SHL CMP, 2018)
...	...	

2. Matrix-valued orthogonal polynomials & Quasi-determinants

Definition. (of matrix-valued orthogonal poly.):

We call $\{P_n(x)\}_{n\in\mathbb{N}}$ are matrix-valued orthogonal polynomials, if there exists a matrix-valued function $W(x)$, s.t.

$$\int_{\mathbb{R}} P_n(x)\, W(x)\, P_m^T(x)\, dx = h_n \delta_{n,m},$$

where h_n is (usually) a positive definite matrix.

Krein: Infinite J-matrices and a matrix moment problem, 1949 $\longrightarrow$ Translated by W. van Assche, arXiv: 1606.07754

Application of matrix-valued orthogonal polynomials

① Birth & Death Process

② Aztec diamond problem

③ Approximation theory (e.g.: A. Doliwa, Integrability & geometry of the Wynn recurrence, Numer. Algo. 2022)

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JEMS

Maurice Duits · Arno B. J. Kuijlaars

The two-periodic Aztec diamond and matrix valued orthogonal polynomials

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MATRIX VALUED ORTHOGONAL POLYNOMIALS ARISING FROM GROUP REPRESENTATION THEORY AND A FAMILY OF QUASI-BIRTH-AND-DEATH PROCESSES*

F. ALBERTO GRÜNBAUM† AND MANUEL D. DE LA IGLESIA‡

In the paper by Duits & Kuijlaars, matrix-valued orthogonal polynomials were used to analyze the asymptotics of Aztec diamond problem

In the paper by Grünbaum et al, matrix-valued OPs were used to formulate stochastic process such as quasi-birth-and death process

Algebraic description (Gelfand et al, 05'):

R: division ring, $\quad R[x] = \left\{ \sum_{i=0}^{+\infty} a_i x^i, \ a_i \in R \right\}$

Inner product $\langle \cdot , \cdot \rangle$: $R[x] \times R[x] \longrightarrow R$

$$\left\langle \sum_{i=0}^{+\infty} a_i x^i, \sum_{i=0}^{+\infty} b_i x^i \right\rangle \longmapsto \sum_{i,j=0}^{+\infty} a_i m_{i+j} b_j^*$$

$$\langle\langle f, g \rangle\rangle_R = \int f(x)^\dagger \, d\mu(x) \, g(x), \quad \|f\|_R = (\mathrm{Tr}\langle\langle f, f \rangle\rangle_R)^{1/2},$$

$$\langle\langle f, g \rangle\rangle_L = \int g(x) \, d\mu(x) \, f(x)^\dagger, \quad \|f\|_L = (\mathrm{Tr}\langle\langle f, f \rangle\rangle_L)^{1/2},$$

Different notations. In the paper "The analytic theory of matrix orthogonal polynomials" by D. Damanik, A. Pushnitski & B. Simon, the authors used left/right to indicate the place of involution in the inner product.

Algebraic description (Gelfand et al, 05'):

R: division ring, $R[x] = \left\{ \sum_{i=0}^{+\infty} a_i x^i,\ a_i \in R \right\}$

Inner product $\langle \cdot, \cdot \rangle$: $R[x] \times R[x] \longrightarrow R$

$$\left\langle \sum_{i=0}^{+\infty} a_i x^i, \sum_{i=0}^{+\infty} b_i x^i \right\rangle \longmapsto \sum_{i,j=0}^{+\infty} a_i m_{i+j} b_j^*$$

Orthogonal relation: $\langle P_n(x), P_m(x) \rangle = h_n \delta_{n,m}$

$\Big\updownarrow$ $\longleftarrow$ Monic polynomials

$\langle P_n(x), x^i \underset{\substack{\uparrow \\ \text{unity in } R}}{\mathbb{1}} \rangle = 0, \quad 0 \le i \le n-1$

Closed form for matrix-valued orthogonal polynomials

$$\langle P_n(x), x^i \mathbb{1} \rangle = 0,$$

$$\Big\downarrow \quad P_n(x) = x^n \cdot \mathbb{1} + a_{n,n-1} x^{n-1} + \cdots + a_{n,0}, \quad a_{n,i} \in R$$

$$m_{n+i} + \sum_{j=0}^{n-1} a_{n,j} m_{i+j} = 0$$

(a linear system with non-commutative coefficients)

$$\Big\downarrow$$

Facts: (i) The existence & uniqueness of $P_n(x)$ is that $(m_{i+j})_{i,j=0}^{n-1}$ is invertible;

(ii) The solution of this linear system could be given by a quasi-determinants.

Closed form for matrix-valued orthogonal polynomials

$$m_{n+i} + \sum_{j=0}^{n-1} a_{n,j} m_{i+j} = 0, \quad i = 0, 1, \cdots, n-1 \quad \text{has a solution}$$

$$a_{n,i} = -e_{i+1} \begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{pmatrix}^{-1} \begin{pmatrix} m_n \\ m_{n+1} \\ \vdots \\ m_{2n-1} \end{pmatrix}$$

$$:= \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & \cdots & m_n & m_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} & m_{2n-1} \\ & e_{i+1} & & & \boxed{0} \end{vmatrix}$$

$$\left(\begin{vmatrix} A & b \\ c & \boxed{d} \end{vmatrix} \triangleq d - cA^{-1}b \right)$$

Closed form for matrix-valued orthogonal polynomials

$$P_n(x) = \left| \begin{matrix} m_0 & m_1 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & \cdots & m_n & m_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} & m_{2n-1} \\ \mathbb{1} & x\cdot\mathbb{1} & \cdots & x^{n-1}\cdot\mathbb{1} & \boxed{x^n\cdot\mathbb{1}} \end{matrix} \right|$$

$$h_n = \left| \begin{matrix} m_0 & m_1 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & \cdots & m_n & m_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} & m_{2n-1} \\ m_n & m_{n+1} & \cdots & m_{2n-1} & \boxed{m_{2n}} \end{matrix} \right| \longrightarrow$$

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Non-commutative integrability, paths and quasi-determinants

Philippe Di Francesco[a b], Rinat Kedem[c]

Quasi Hankel determinants have been widely used in different contexts, such as non-commutative integrable system, combinatorics, …

Recurrence relation for matrix-valued orthogonal polynomials

Prop.: For monic matrix-valued orthogonal polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$,

we have

$$x P_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x),$$

where

$$a_n = \langle x P_n(x), P_n(x)\rangle \cdot h_n^{-1}, \qquad b_n = h_n h_{n-1}^{-1}$$

Quasi-determinants give explicit expressions for these recurrence coefficients a_n. That is,

$$a_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ m_{n+1} & m_{n+2} & \cdots & \boxed{m_{2n+1}} \end{vmatrix} h_n^{-1} + \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} & \\ m_1 & m_2 & \cdots & m_n & e_{n-1}^T \\ \vdots & \vdots & & \vdots & \\ m_{n-1} & m_n & \cdots & m_{2n-2} & \\ m_n & m_{n+1} & \cdots & m_{2n-1} & \boxed{0} \end{vmatrix}$$

Matrix-valued orthogonal polynomials & integrable systems

Ref: SHL, Matrix OPs, NC Toda & BT, accepted by Sci. China Math, 2023.

① Continuous evolution $W(x) \longrightarrow W(x;t) = W(x)\exp\left(\sum_{i=1}^{+\infty} t_i x^i\right)$

$$\partial_{t_1} P_n(x;t) = -b_n P_{n-1}(x;t) \Longrightarrow \begin{cases} \partial_{t_1} a_n = b_{n+1} - b_n \\ \partial_{t_1} b_n = a_n b_n - b_n a_{n-1} \end{cases}$$

lower triangular part

$$\partial_{t_k} \Phi = -(L^k)_- \Phi \Longrightarrow \partial_{t_k} L = [L, (L^k)_-]$$

8 The non-Abelian Toda lattice

The non-Abelian Toda lattice is a Hamiltonian system which describes the evolution of a system of particles $X_1, \ldots, X_N$ in the space of invertible $n \times n$ matrices. There is a standard version, introduced by A.M. Polyakov [10], which generalises the classical Toda lattice. There is also an *indefinite* version, in which the potential has the opposite sign, as considered by Popowicz [40,41]. As in the scalar case [35,37], it is the indefinite version which is relevant to our setting.

Writing $A_i = X_{i+1}X_i^{-1}$ and $B_i = \dot{X}_i X_i^{-1}$, the Hamiltonian is given by

$$H = \operatorname{tr}\left(\frac{1}{2}\sum_{i=1}^{N} B_i^2 - \sum_{i=1}^{N-1} A_i\right),$$

N. O'Connell (Prob. Theory Related Fields, 19') showed that the diffusion interacting on positive definite matrices is related to NC Toda.

Matrix-valued orthogonal polynomials & integrable systems

② Discrete evolution $\quad W(x) \longrightarrow W(x;k) = x^k W(x)$

Christoffel transformation

$$P_n(x;l) = x P_{n-1}(x;l+2) - A_n^l P_{n-1}(x;l+1)$$

Geronimus transformation

$$x P_n(x;l+1) = P_{n+1}(x;l) + B_n^l P_n(x;l)$$

$$\Rightarrow \begin{cases} B_{n-1}^{l+1} - A_n^l = B_n^{l-1} - A_{n+1}^{l-1} \\ (B_n^{l+1} - A_{n+1}^l) B_n^l = B_{n+1}^{l-1} (B_n^l - A_{n+1}^{l-1}) \end{cases}$$

More examples connecting matrix-valued polynomials with non-commutative integrable systems

Commutative version:

Hankel (Orthogonal polynomials / Toda)

- Grammian (Bi-orthogonal polynomials / KP/Reductions of KP)
- Toeplitz (Orthogonal polynomials in the unit circle (OPUC) / mkdV/relativistic Toda)
- block Hankel (θ-deformed orthogonal polynomials / lattice Gelfand-Dickey)

More examples connecting matrix-valued polynomials with non-commutative integrable systems

Example 1 (Ref.: SHL, Y. Shi, G. Yu & J. Zhao, Matrix-valued Cauchy bi-orthogonal polynomials and a novel noncommutative integrable lattice, arXiv: 2212.14512)

Matrix-valued Cauchy bi-orthogonal polynomials

$$\langle f(x), g(x) \rangle = \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{1}{x+y} f(x) W_1(x) W_2^T(y) g^T(y)\, dx\, dy$$

$$h_n = \begin{vmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n} \\ \vdots & \vdots & & \\ m_{n,0} & m_{n,1} & \cdots & \boxed{m_{n,n}} \end{vmatrix}, \qquad m_{i,j} = \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{x^i y^j}{x+y} W_1(x) W_2^T(y)\, dx\, dy$$

More examples connecting matrix-valued polynomials with non-commutative integrable systems

Example 2 (Ref.: C. Gilson, SHL & Y. Shi, Matrix-valued θ-deformed bi-orthogonal polynomials, non-commutative Toda theory and Bäcklund transformation, arXiv:2305.17962)

Matrix-valued θ-deformed orthogonal polynomials

$$\langle f(x), g(x) \rangle = \int_{\mathbb{R}} f(x) W(x) g(x^\theta)\, dx, \quad \theta \in \mathbb{Q}_+$$

$$h_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_\theta & m_{1+\theta} & \cdots & m_{n+\theta} \\ \vdots & \vdots & & \vdots \\ m_{n\theta} & m_{1+n\theta} & \cdots & \boxed{m_{n+n\theta}} \end{vmatrix} \Longrightarrow$$

Corresponding combinatorical interpretation of such quasi-determinant is unknown!

More examples connecting matrix-valued polynomials with non-commutative integrable systems

Example 3 (Ref.: B. Wang & SHL, On non-commutative leapfrog map. arXiv: 2310.01993.)

Matrix-valued Laurent polynomials

$$\langle f(x), g(x) \rangle = \int_{\mathbb{R}} f(x) W(x) g(x^{-1})\, dx$$

$$h_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_{-1} & m_0 & \cdots & m_{n-1} \\ \vdots & \vdots & & \vdots \\ m_{-n} & m_{1-n} & \cdots & \boxed{m_0} \end{vmatrix} \Longrightarrow$$

This quasi-Toeplitz determinant has been used in the description of NC map, which is related to a discrete evolution of nc cross ratio in a projective line.

Concluding Remarks

	Commutative	Non-commutative
τ-function	determinants / Pfaffian	quasi-determinants / quasi-Pfaffian?
wave function	Orthogonal polynomials	Matrix-valued orthogonal polynomials
Lax matrix	Matrix with commutative variables	Matrix with non-commutative variables
Connections with subjects in statistical mechanism	random matrices, six-vertex models, Brownian models, Schur measure, ...	Aztec diamond model

Thanks!