Multidimensional consistency and quantum variational principle: Quadratic Lagrangian 1-form

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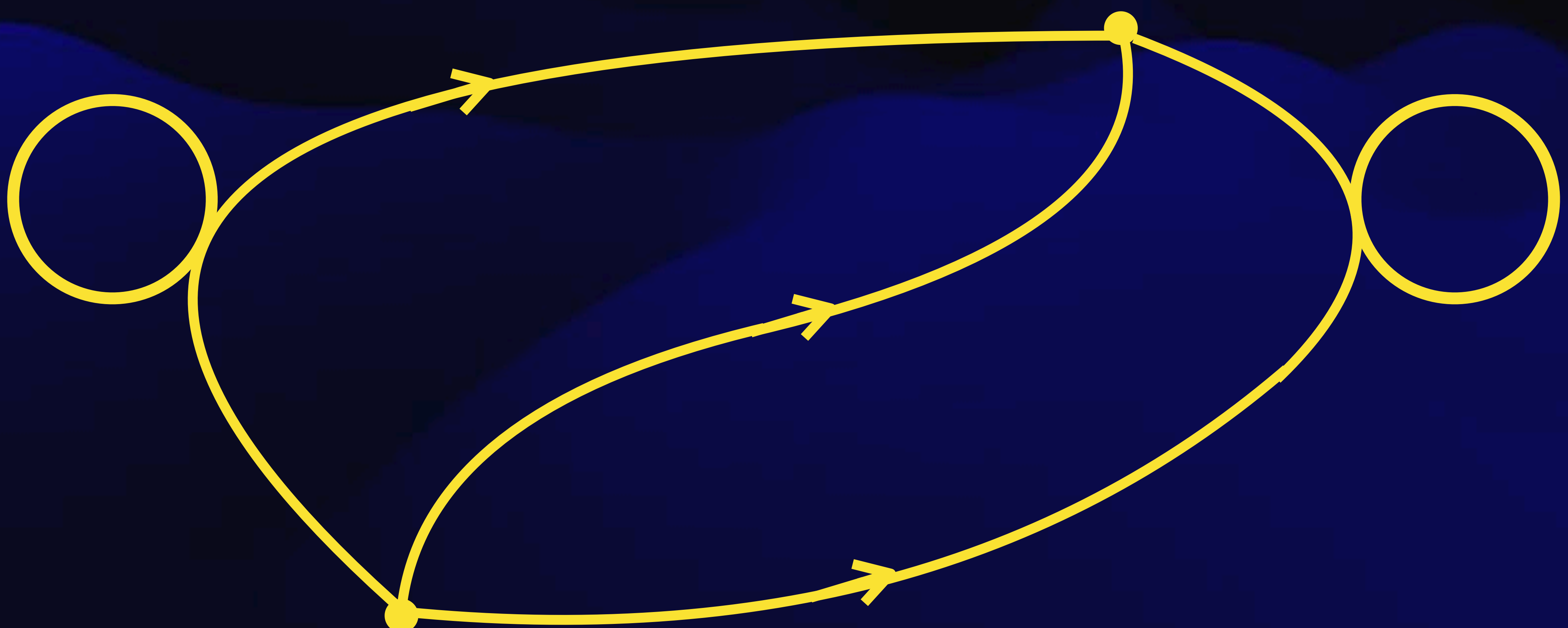

Lagrangian Multiform Theory and Pluri-Lagrangian Systems at IASM, Hangzhou, China

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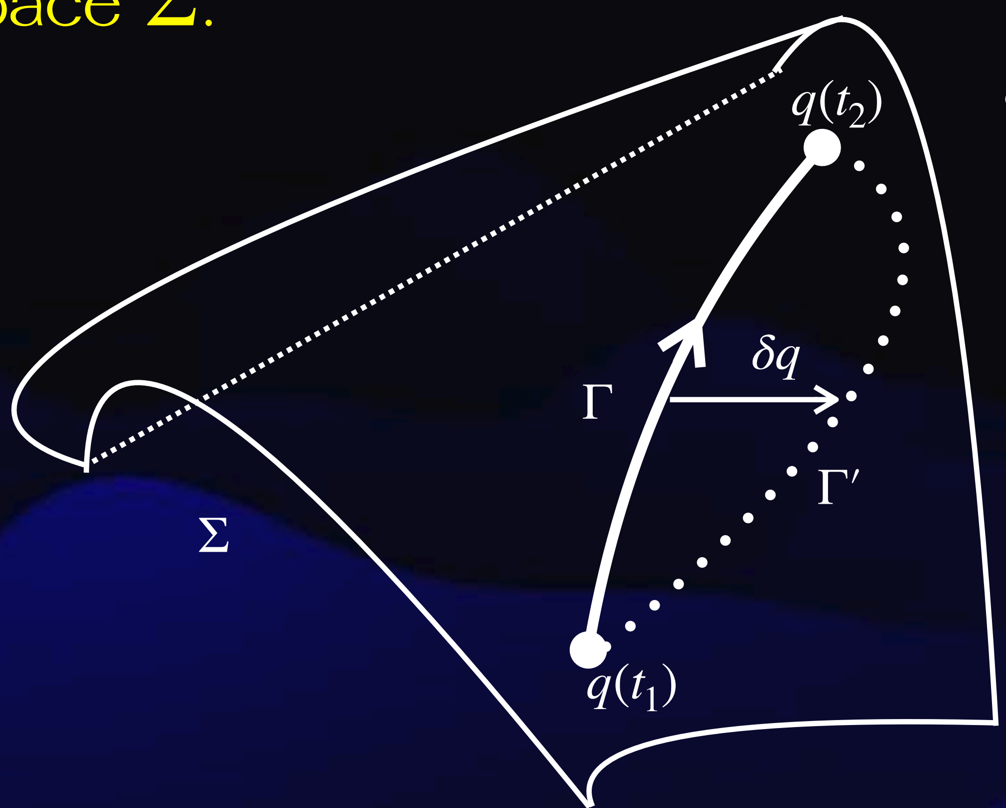
Outline

- The least action principle
- Path integration
- Integrability as Multi-dimensional consistency
- Lagrangian 1-form and closure relation
- Multi-time propagator-construction
- Integrability criteria: Quantum variational principle
- Summary

The least action principle

The statement: The system will evolve in time along the path, that the action functional is "critical", from an initial point to a final point on the configuration space Σ.



$$\delta S = S_{\Gamma'}[q+\delta q] - S_{\Gamma}[q] = \int_{\Gamma'} L dt - \int_{\Gamma} L dt = 0$$

$$\Downarrow$$

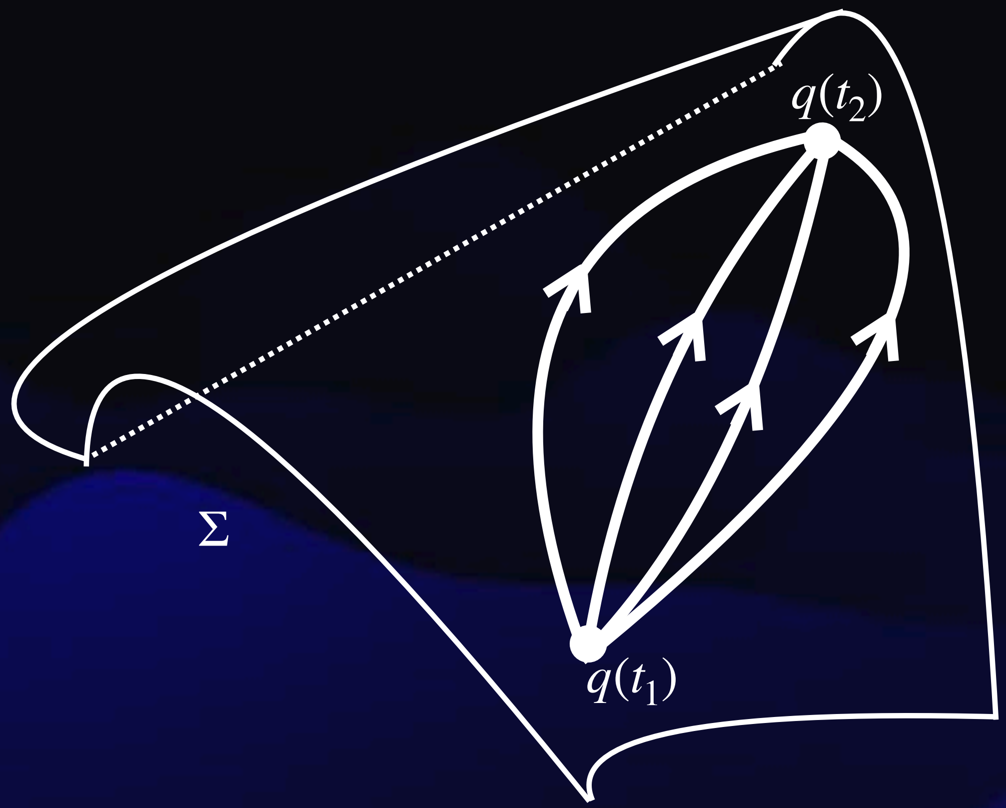
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \ , i = 1,2,..,N$$

The configuration space Σ: $\{q(t) = \{q_1(t), q_2(t), \ldots, q_N(t)\}$, where $q(t)$ is a set of good coordinates and N is a degree of freedom.

Note: There is only one true path called a classical path for a system to evolve from the initial point to the final point.

All possible paths

In quantum mechanics, a particle will take all possible paths from the initial point to the final point at once, with different prob weight $\approx e^{\frac{i}{\hbar}S}$.



The propagator $K(q_2, t_2; q_1, t_1) = \langle q_2 | U(t_2 - t_1) | q_1 \rangle$ is a function that specifies the prob amplitude for a particle to get from the initial point to the final point. Here $U(T) = e^{-\frac{i}{\hbar}\hat{H}T}$ is nothing but the time evolution operator with condition $U^\dagger U = I$.

In the case of "quadratic Hamiltonian (p^2)", one can derive

$$K(q_2, t_2; q_1, t_1) = \int_{q_1}^{q_2} \mathscr{D}[q] e^{\frac{i}{\hbar}S} ,$$

where $\mathscr{D}[q]$ denotes integration over all path q from q_1 to q_2.

Integrability as Multidimensional consistency

Liouville-Arnold theorem: The integrability is defined in terms of the existence of the invariances, implying the existence of Action-Angle variables.

Theorem: If, in a Hamiltonian dynamical system with n degrees of freedom, there are also known n first integrals of motion $F = (F_1, F_2, \ldots, F_n) \in \mathscr{F}(T^*\Sigma)$ which are independent and in involution: $\{F_i, F_j\} = 0$, $i \neq j$, then there exists a canonical transformation to A.A. variables $(\mathbf{I}, \theta)$, where $I_j = I_j(H)$ in which the transformed Hamiltonian is dependent only upon the action variables and the angle variables evolve linearly in time. Thus the equation of motion can be solved in quadratures: $\dot{I}_j = 0$ and $\dot{\theta}_j = \Omega_j(\mathbf{I})$.

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Hamiltonian commuting flows: Since all invariances are defined on cotangent bundle $(F_1, F_2, \ldots, F_n) \in \mathscr{F}(T^*\Sigma)$, we now may treat them as a set of Hamiltonians$(F_1, F_2, \ldots, F_n) = (H_1, H_2, \ldots, H_n)$
which is known as the Hamiltonian hierarchy equipped with the property $\{H_i, H_j\} = 0$ (involution)

Integrability as Multidimensional consistency

From the standard structure, the Hamiltonian can be treated as the "time generator" s.t. $\dot{\circ} = \{\circ, H\}$, where $\circ \in \mathscr{F}(T^*\Sigma)$. Then we, in this context, have a multi-time structure: $\mathbf{t} = (t_1, t_2, \ldots, t_n)$.

Now given $F \in \mathscr{F}(T^*\Sigma)$ and $\dfrac{dF}{dt_k} = \{F, H_k\}$, $\dfrac{dF}{dt_l} = \{F, H_l\}$, we have $\dfrac{d}{dt_l}\dfrac{dF}{dt_k} = \{\{F, H_k\}, H_l\}$ and $\dfrac{d}{dt_k}\dfrac{dF}{dt_l} = \{\{F, H_l\}, H_k\}$.

Next, we consider $\dfrac{d}{dt_l}\dfrac{dF}{dt_k} - \dfrac{d}{dt_k}\dfrac{dF}{dt_l} = \{\{F, H_k\}, H_l\} - \{\{F, H_l\}, H_k\} = \{F, \{H_l, H_k\}\}$ (Jacobi identity was used.)

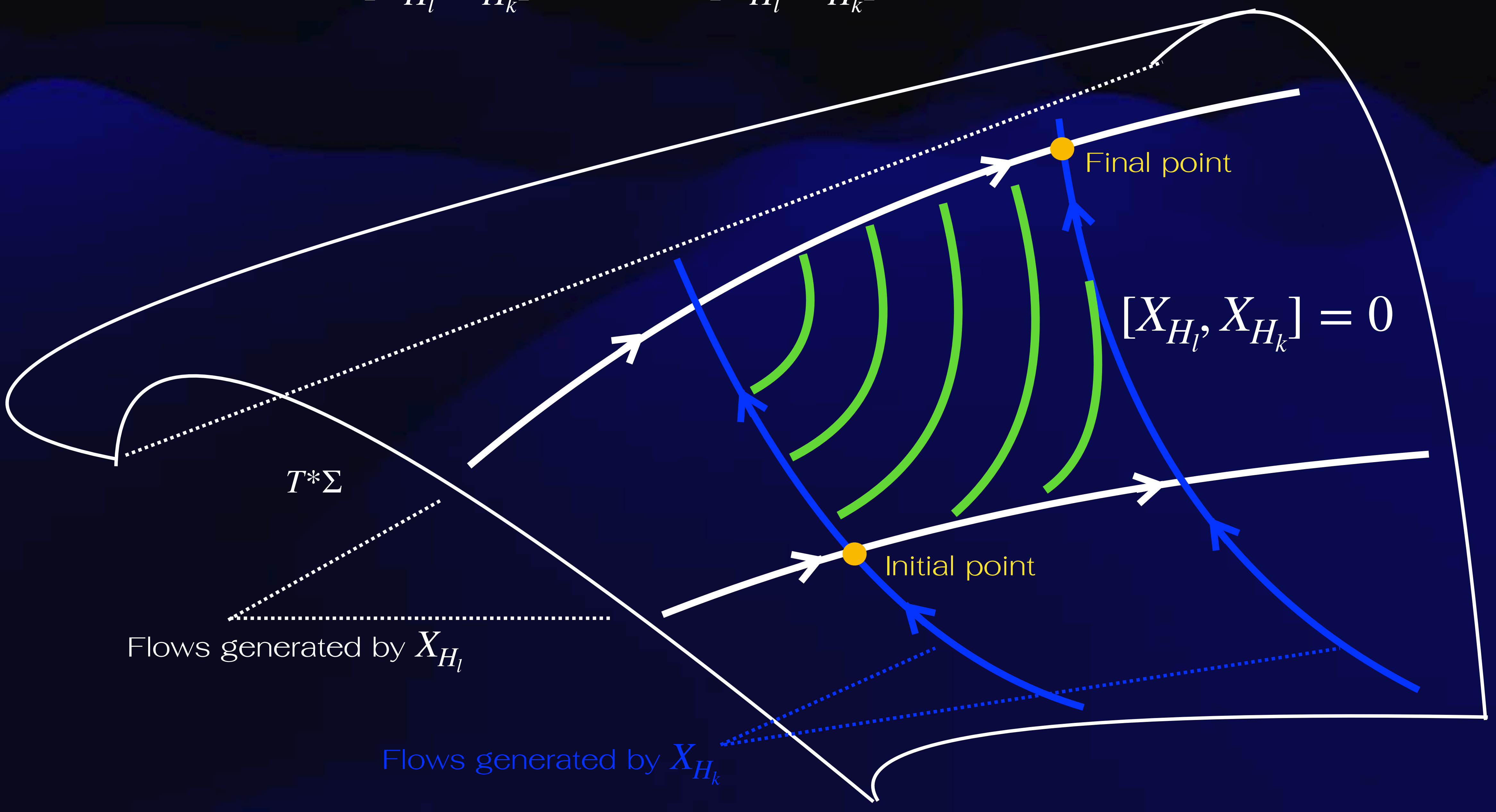
Therefore, $\dfrac{d}{dt_l}\dfrac{dF}{dt_k} - \dfrac{d}{dt_k}\dfrac{dF}{dt_l} = 0$. This means that the order of evolution does not matter.

Integrability as Multidimensional consistency

Equivalently, one can write $\dot{\circ} = \{\circ, H\} = X_H \circ$, where X_H is a Hamiltonian vector field. Then the relation

$$\frac{d}{dt_l}\frac{dF}{dt_k} - \frac{d}{dt_k}\frac{dF}{dt_l} = 0$$

can be re-expressed as $[X_{H_l}, X_{H_k}]F = 0$ or $[X_{H_l}, X_{H_k}] = 0$



Lagrangian 1-from and the closure relation

Given a set of Lagrangians $\{L_1, L_2, \ldots, L_N\}$, an action functional is given by

$$S[q(t)] = \int_\Gamma \mathscr{L} \ ,$$

where $\mathscr{L} = \sum_{i=1}^{N} L_i dt_i$ is a Lagrangian 1-form and Γ is a curve defined on the space of independent variables.

Lagrangian 1-from and the closure relation

Given a set of Lagrangians $\{L_1, L_2, \ldots, L_N\}$, an action functional is given by

$$S[q(t)] = \int_\Gamma \mathscr{L} = \int_\Gamma dS \ ,$$

where $\mathscr{L} = \sum_{i=1}^{N} L_i dt_i$ is a Lagrangian 1-form and Γ is a curve defined on the space of independent variables.

Imposing a critical condition $\delta S[q(t)] = 0$ and applying Stokes' theorem, we obtain

$$\delta S[q(t)] = \oint_{\partial C} dS = \iint_C d^2 S = 0,$$

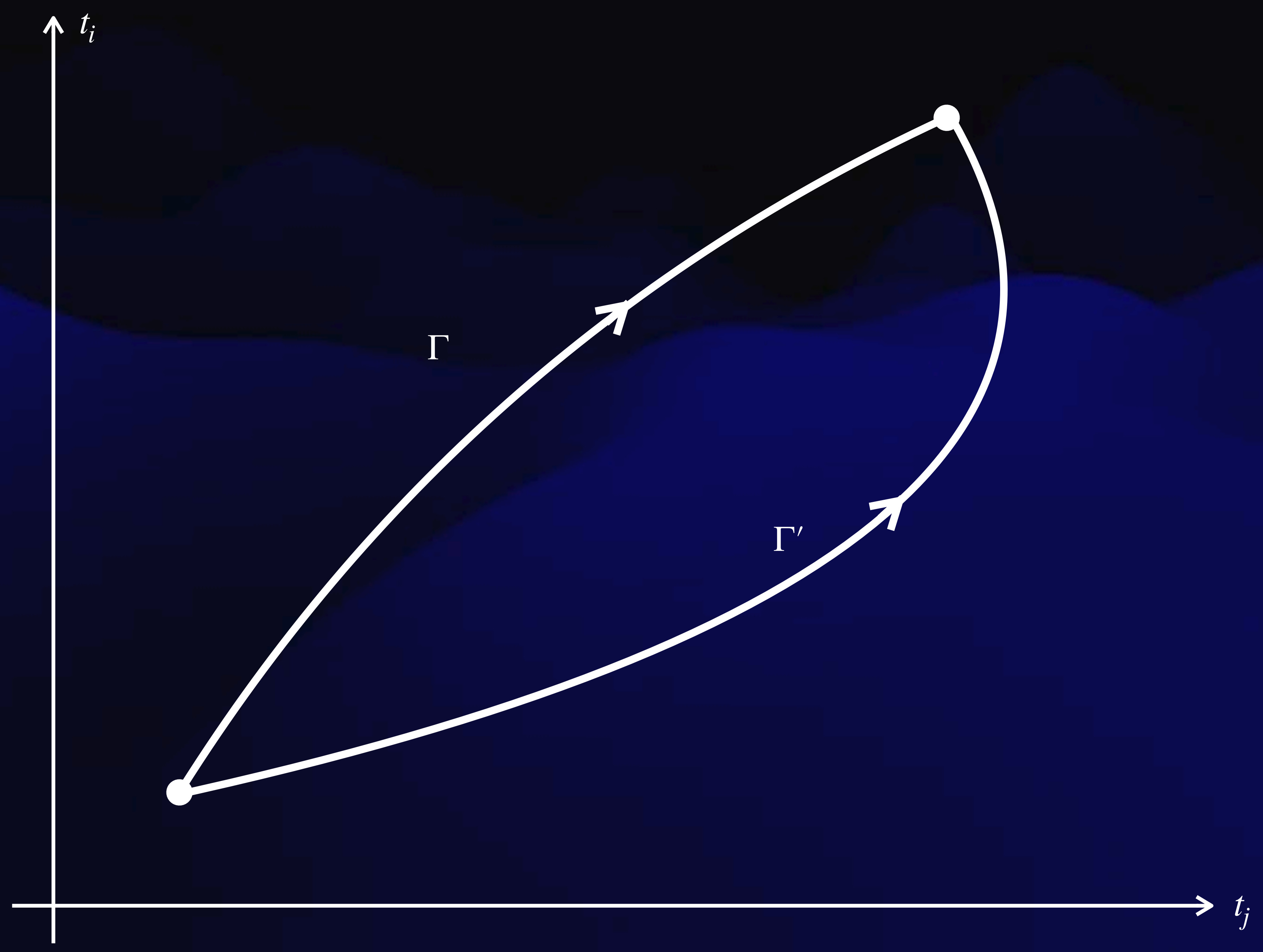
and

$$d(dS) = d^2 S = \sum_{i,j=1}^{N} \omega_{ij} dt_i \wedge dt_j = 0, \quad \text{where} \quad \omega_{ij} = \frac{dL_j}{dt_i} - \frac{dL_i}{dt_j} = 0$$

(Lagrangian 1-form closure relation).

Lagrangian 1-from and the closure relation

The condition $\frac{dL_j}{dt_i} - \frac{dL_i}{dt_j} = 0$ indicates that the value of the action does not depends on paths (sharing the same end points) on the space of independent variables.



Calogero-Moser systems

They are describing interacting system of N particles on a line (or circle). The Hamiltonians are given by

$$H_1(p,q) = \sum_{k=1}^{N} p_k$$

$$H_2(p,q) = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{l,k=1,k\neq l}^{N} \frac{1}{(q_k - q_j)^2}$$

$$H_3(p,q) = \frac{1}{3}\sum_{k=1}^{N} p_k^3 + \sum_{k,l=1,k\neq l}^{N} \frac{p_l}{(q_k - q_l)^2}$$

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Calogero-Moser systems

The Lagrangians are given by

There is no $L_1(\dot{q}, q)$

$$L_2(\dot{q}, q) = \frac{1}{2}\sum_{k=1}^{N} \dot{q}_k^2 - \sum_{l,k=1,k\neq l}^{N} \frac{1}{(q_k - q_j)^2}$$

$$L_3(\dot{q}, \bar{q}, q) = \sum_{k=1}^{N} \left(\dot{q}_i\bar{q}_i + \frac{1}{4}\dot{q}^3 \right) - \sum_{k,l=1,k\neq l}^{N} \frac{\dot{q}_k}{(q_k - q_l)^2}$$

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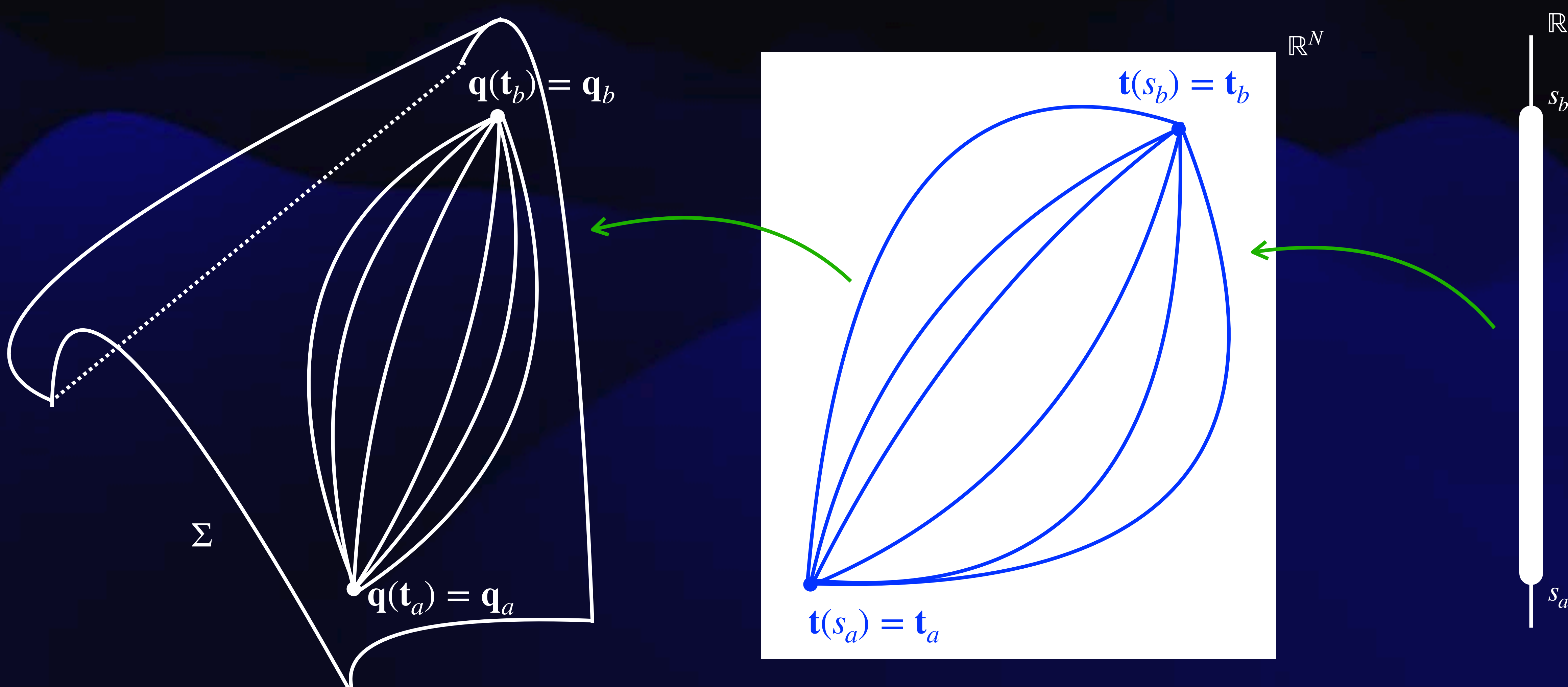
where $\dot{q} = \dfrac{dq}{dt_2}$ and $\bar{q} = \dfrac{dq}{dt_3}$.

Nijhoff's proposal

In 2013, Frank Nijhoff proposed a new form of the propagator in "Discrete Integrable Systems Fellow-up Meeting", Isaac Newton Institute, Cambridge.

$$K(\mathbf{q_b}, \mathbf{t_b}, s_b; \mathbf{q_a}, \mathbf{t}_a, s_a) = \int_{\mathbf{t}(s_a)=\mathbf{t}_a}^{\mathbf{t}(s_b)=\mathbf{t}_b} \mathscr{D}[\mathbf{t}(s)] \int_{\mathbf{x}(\mathbf{t}_a)=\mathbf{x}_a}^{\mathbf{x}(\mathbf{t}_b)=\mathbf{x}_b} \mathscr{D}_\Gamma[\mathbf{x}(\mathbf{t})] e^{\frac{i}{\hbar} S[\mathbf{x}(\mathbf{t});\Gamma]}$$



Dependent and independent variables are treated on the same equal footing!

Rovelli's idea, 2011

According to a view point of quantum gravity, one can work on background-independent theory: $q(t) \to q(\tau) = q(t(\tau))$. Then we have now $x = (q, t)$.

$$S[q] = \int dt \mathscr{L}(q, \dot{q}) \to \int \tau \dot{t} \mathscr{L}(q, \dot{q}/\dot{t}) = \int \tau(x, \dot{x}) \equiv S[x].$$

For example, a single particle Lagrangian $\mathscr{L} = \dfrac{m\dot{q}^2}{2} - V(q)$ will be rewritten as

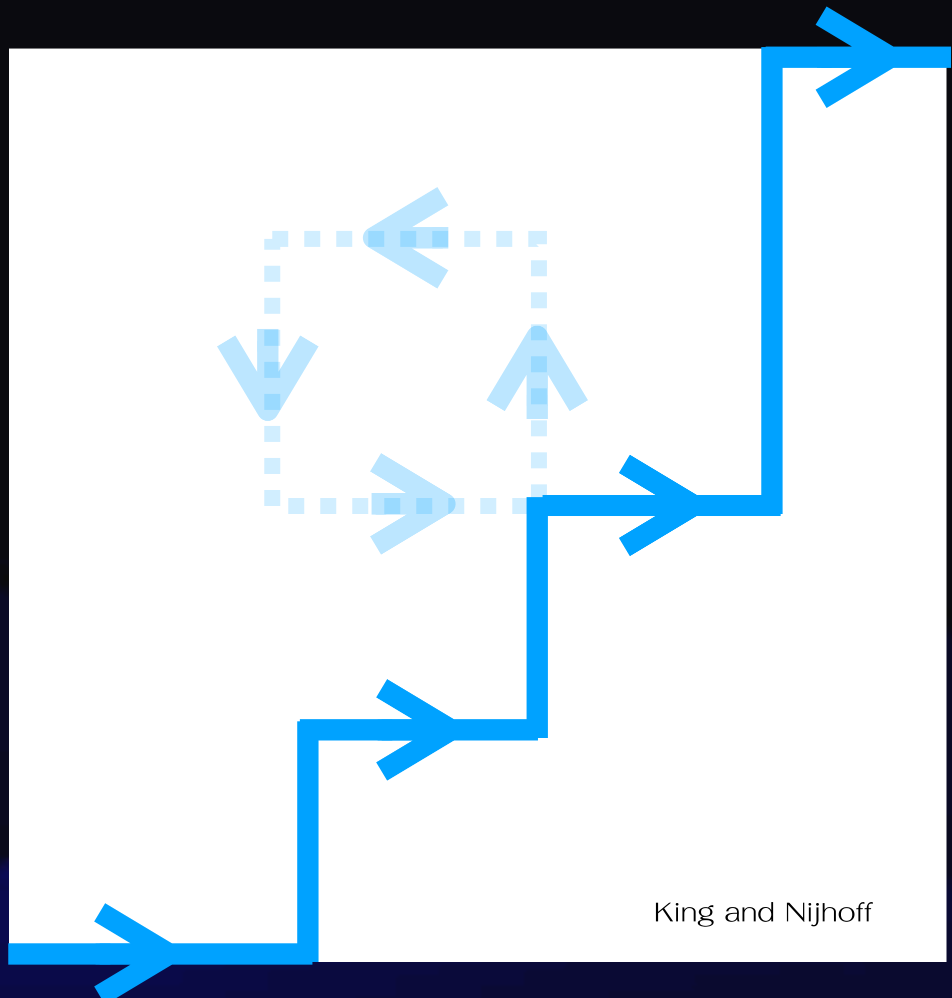
$$\mathscr{L} = \frac{m\dot{q}^2}{2\dot{t}} - \dot{t}V(q).$$

The propagator reads

$$K(x'', \tau'', x', \tau') = \int \mathscr{D}[x(\tau)] e^{\int_{\tau'}^{\tau''} d\tau \mathscr{L}} = \int \mathscr{D}[t(\tau)] \int \mathscr{D}[q(\tau)] e^{\int_{\tau'}^{\tau''} d\tau \mathscr{L}}$$

Carlo Rovelli, 2011, "On the structure of a background independent quantum theory: Hamilton function, transition amplitudes, classical limit and continuous limit",. arXiv: 1108.0832.

King and Nijhoff discrete propagator (quadratic case) 2019



Given discrete KdV equation and then they imposed the periodic condition resulting two discrete-time harmonic oscillators.

$$L_1 = \frac{1}{2}\left(\frac{\partial \mathbf{q}}{\partial t_1}\right)^2 - \frac{\omega_1^2 \mathbf{q}^2}{2} \quad \text{and} \quad L_2 = \frac{1}{2}\left(\frac{\partial \mathbf{q}}{\partial t_2}\right)^2 - \frac{\omega_2^2 \mathbf{q}^2}{2}$$

The discrete propagator processes a path-independent feature.

$$K_{\lrcorner} = K_{\ulcorner}$$

Multi-time propagator

Definition: Let $\mathscr{L} = \sum_{j=1}^{N} L_j dt_j$ be a Lagrangian 1-form , where $L_j = L_j\left(\mathbf{q}, \left\{\frac{\partial \mathbf{q}}{\partial t_j}; j = 1,2,\ldots,N\right\}; \mathbf{t}\right)$. On the space of independent variables (time variables) parameterised by a variable s such that $\mathbf{t}(s)$, where $s' < s < s''$, the multi-time propagator is given by

$$K(\mathbf{q}(\mathbf{t}(s'')), s''; \mathbf{q}(\mathbf{t}(s')), s') = \int_{\mathbf{q}(\mathbf{t}(s'))}^{\mathbf{q}(\mathbf{t}(s''))} \mathbb{D}[\mathbf{q}(\mathbf{t}(s)); \Gamma \in \mathscr{B}] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\mathscr{B}\}} \mathscr{L}},$$

where $\mathscr{L} = ds \sum_{j=1}^{N} L_j dt_j/ds$ and $\int \mathbb{D}[\mathbf{q}(s); \Gamma \in \mathscr{B}]$ is the functional measure over all possible spatial-temporal paths. Here $\Gamma \in \mathscr{B}$, where $\mathscr{B}$ is a family of paths connecting the point $\mathbf{t}(s')$ with the point $\mathbf{t}(s'')$ on space of time variables.

Multi-time propagator

$$\int_{\mathbf{q}(\mathbf{t}(s'))}^{\mathbf{q}(\mathbf{t}(s''))} \mathbb{D}[\mathbf{q}(\mathbf{t}(s)); \Gamma \in \mathscr{B}] = \lim_{\substack{\mathcal{N} \to \infty \\ \epsilon_{1,2,\ldots,N} \to 0}} \Bigg(\sum_{\alpha^N_{\mathcal{N}-1} \geq \cdots \alpha^N_2 \geq \alpha^N_1 \geq 0}^{\mathcal{N}} \cdots \sum_{\alpha^3_{\mathcal{N}-1} \geq \cdots \alpha^3_2 \geq \alpha^3_1 \geq 0}^{\mathcal{N}} \sum_{\alpha^2_{\mathcal{N}-1} \geq \cdots \alpha^2_2 \geq \alpha^2_1 \geq 0}^{\mathcal{N}} \mathcal{N}_\Gamma \int_{\mathbf{q}(0,0,\ldots,0)}^{\mathbf{q}(\epsilon_1,0,\ldots,0)} \mathscr{D}[\mathbf{q}(t_1)]$$

$$\times \left(\prod_{j=1}^{\mathcal{N}-1} \int_{-\infty}^{\infty} d^N q(j\epsilon_1, \alpha^2_{j-1}\epsilon_2, \ldots, \alpha^N_{j-1}\epsilon_N) \int_{\mathbf{q}(j\epsilon_1,\alpha^2_{j-1}\epsilon_2,\ldots\alpha^N_{j-1}\epsilon_N)}^{\mathbf{q}(j\epsilon_1,\alpha^2_j\epsilon_2,\ldots\alpha^N_j\epsilon_N)} \mathscr{P}\left[\mathscr{D}[\mathbf{q}(t_2)], \mathscr{D}[\mathbf{q}(t_3)], \cdots, \mathscr{D}[\mathbf{q}(t_N)]\right]\right.$$

$$\left. \times \int_{-\infty}^{\infty} d^N q(j\epsilon_1, \alpha^2_j\epsilon_2, \ldots, \alpha^N_j\epsilon_N) \int_{\mathbf{q}(j\epsilon_1,\alpha^2_j\epsilon_2,\ldots\alpha^N_j\epsilon_N)}^{\mathbf{q}((j+1)\epsilon_1,\alpha^2_j\epsilon_2,\ldots\alpha^N_j\epsilon_N)} \mathscr{D}[\mathbf{q}(t_1)] \right)$$

$$\times \int_{-\infty}^{\infty} d^N q(\mathcal{N}\epsilon_1, \alpha^2_{\mathcal{N}-1}\epsilon_2, \ldots, \alpha^N_{\mathcal{N}-1}\epsilon_N) \int_{\mathbf{q}(\mathcal{N}\epsilon_1,\alpha^2_{\mathcal{N}-1}\epsilon_2,\ldots,\alpha^N_{\mathcal{N}-1}\epsilon_N)}^{\mathbf{q}(\mathcal{N}\epsilon_1,\mathcal{N}\epsilon_2,\ldots,\mathcal{N}\epsilon_N)} \mathscr{P}\left[\mathscr{D}[\mathbf{q}(t_2)], \mathscr{D}[\mathbf{q}(t_3)], \cdots, \mathscr{D}[\mathbf{q}(t_N)]\right]$$

$$+ (\text{all symmetric terms}) \Bigg),$$

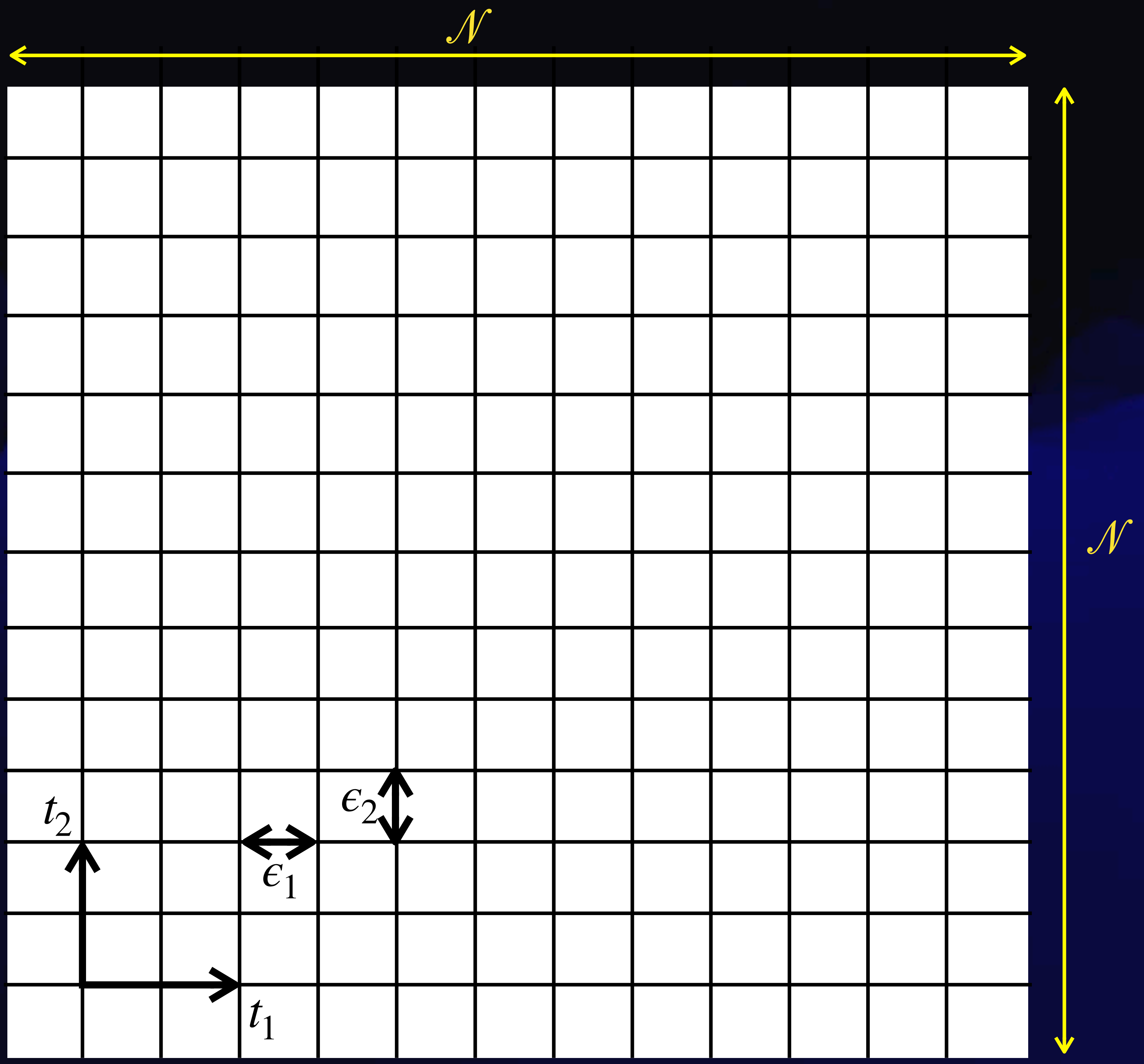
where

$$\int_{\mathbf{q}(j\epsilon_1,\alpha^2_{j-1}\epsilon_2,\ldots,\alpha^N_{j-1}\epsilon_N)}^{\mathbf{q}(j\epsilon_1,\alpha^2_j\epsilon_2,\ldots,\alpha^N_j\epsilon_N)} \mathscr{P}\left[\mathscr{D}[\mathbf{q}(t_2)], \mathscr{D}[\mathbf{q}(t_3)], \cdots, \mathscr{D}[\mathbf{q}(t_N)]\right]$$

$$= \frac{1}{\mathscr{P}(N-1, r_j)} \big(\text{Summation of all possible permutations}\big) \quad \text{and } r_j = \sum_{i=1}^{N} \delta_{\alpha^i_{j-1},\alpha^i_j}.$$

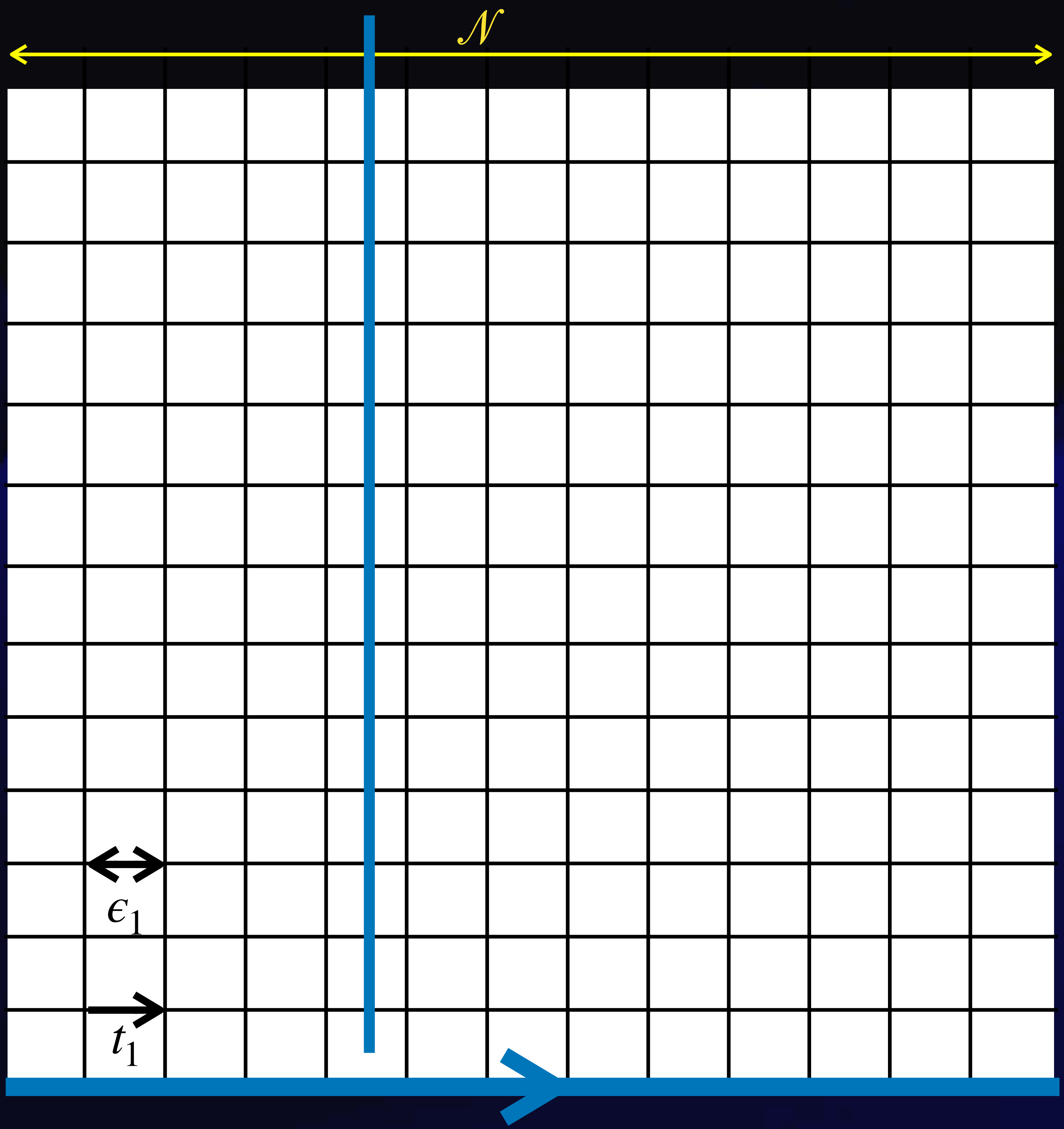
Multi-time propagator (2D)

To get a favour on constructing the multi-time propagator, we shall illustrate the case in the 2D space

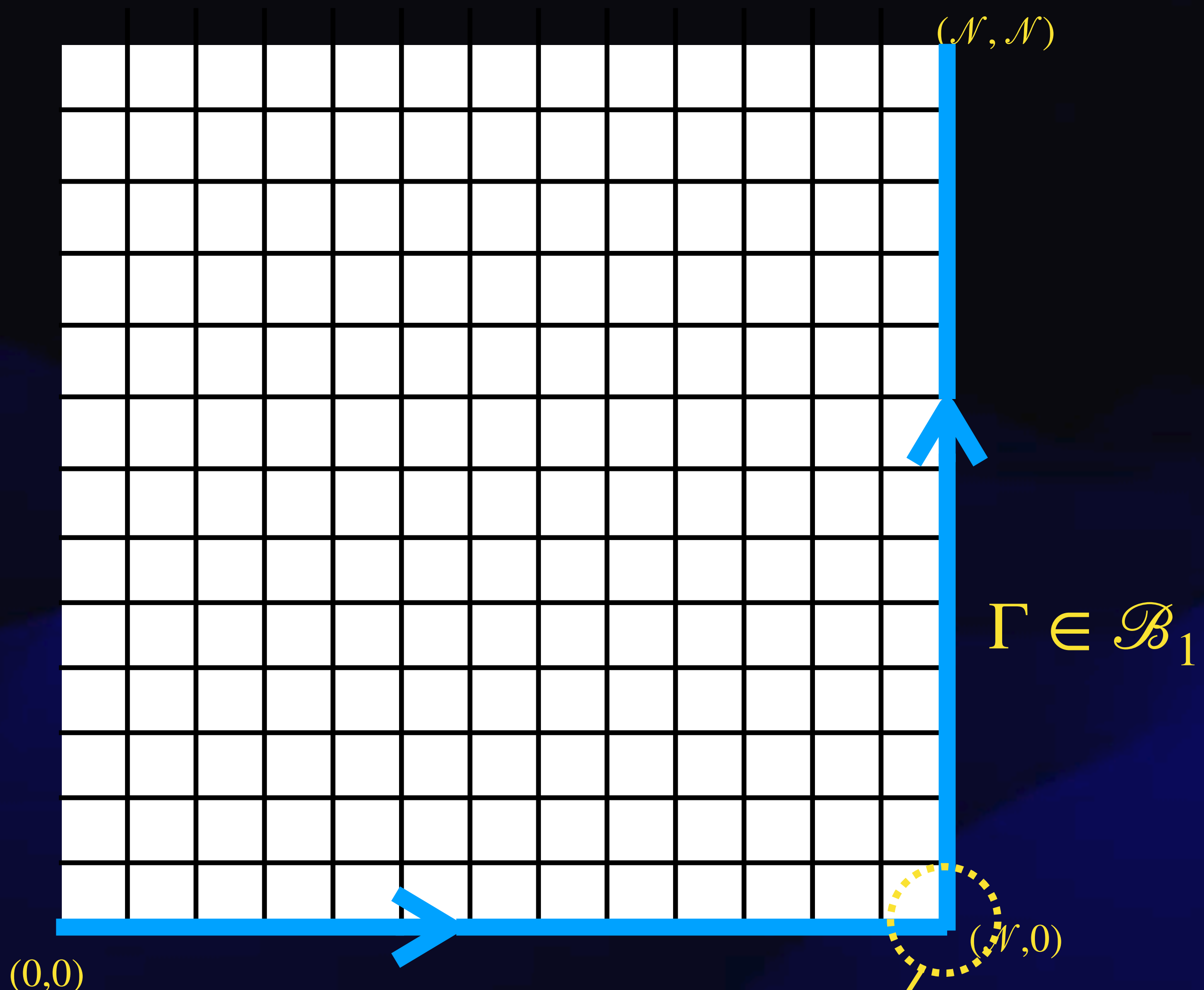


Multi-time propagator (2D)

$$K^{(1)} = \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\epsilon_1 \mathcal{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] e^{\frac{i}{\hbar}\int_{(0,0)}^{(\epsilon_1 \mathcal{N},0)} L_1(t_1,0) dt_1}$$



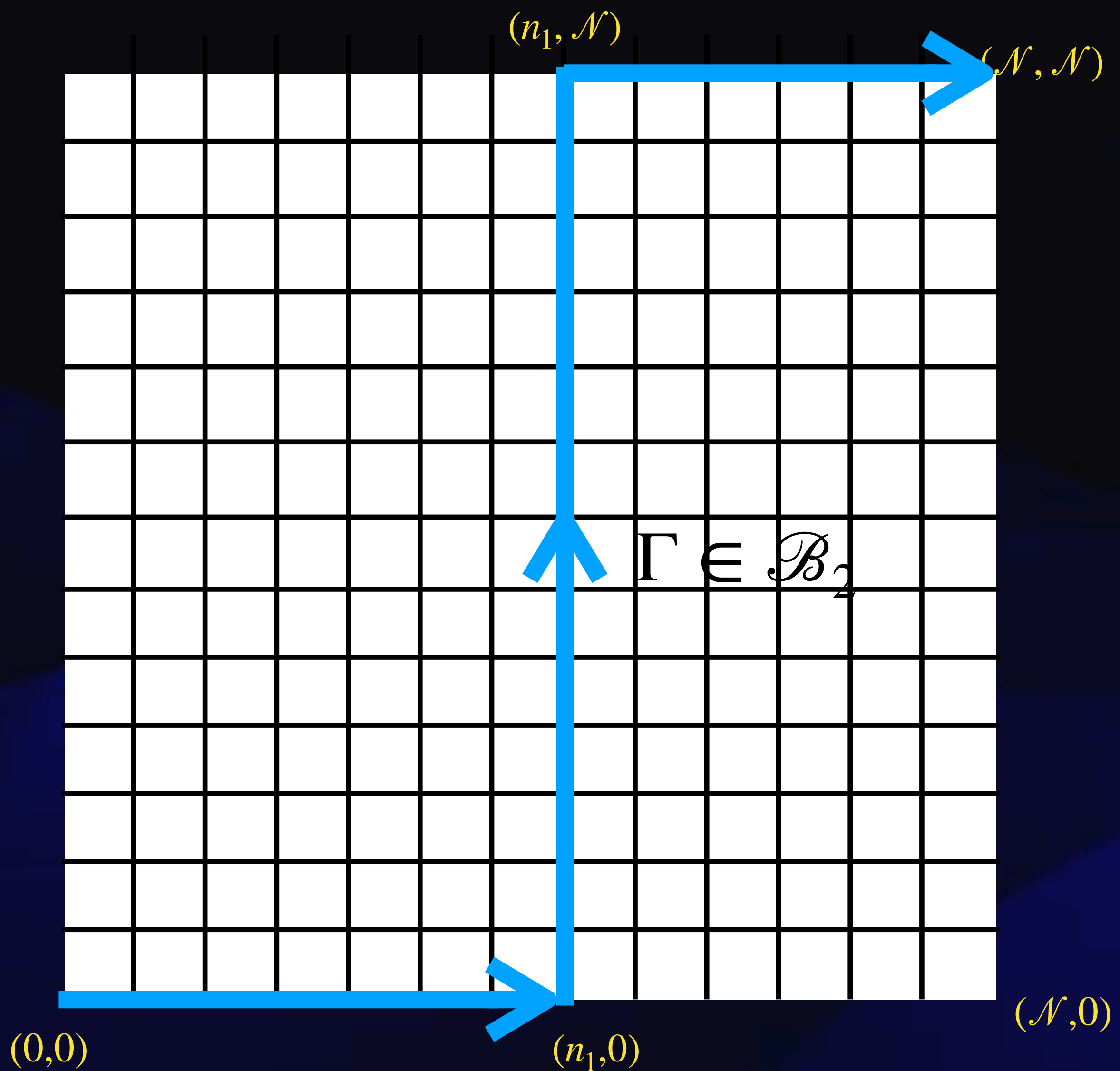
Multi-time propagator (2D)



$$K^{(1)} = \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\epsilon_1\mathcal{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(\epsilon_1\mathcal{N},0) \int_{\mathbf{q}(\epsilon_1\mathcal{N},0)}^{\mathbf{q}(\epsilon_1\mathcal{N},\epsilon_2\mathcal{N})} \mathscr{D}[\mathbf{q}(\epsilon_1\mathcal{N},t_2)] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(\epsilon_1\mathcal{N},0)} L_1(t_1,0)dt_1 + \int_{(\epsilon_1\mathcal{N},0)}^{(\epsilon_1\mathcal{N},\epsilon_2\mathcal{N})} L_2(\epsilon_1\mathcal{N},t_2)dt_2\right)}$$

$$K^{(1)} = \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\epsilon_1\mathcal{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(\epsilon_1\mathcal{N},0) \int_{\mathbf{q}(\epsilon_1\mathcal{N},0)}^{\mathbf{q}(\epsilon_1\mathcal{N},\epsilon_2\mathcal{N})} \mathscr{D}[\mathbf{q}(\epsilon_1\mathcal{N},t_2)] e^{\frac{i}{\hbar}\int_{\Gamma\in\mathscr{B}_1}\mathscr{L}}$$

Multi-time propagator (2D)



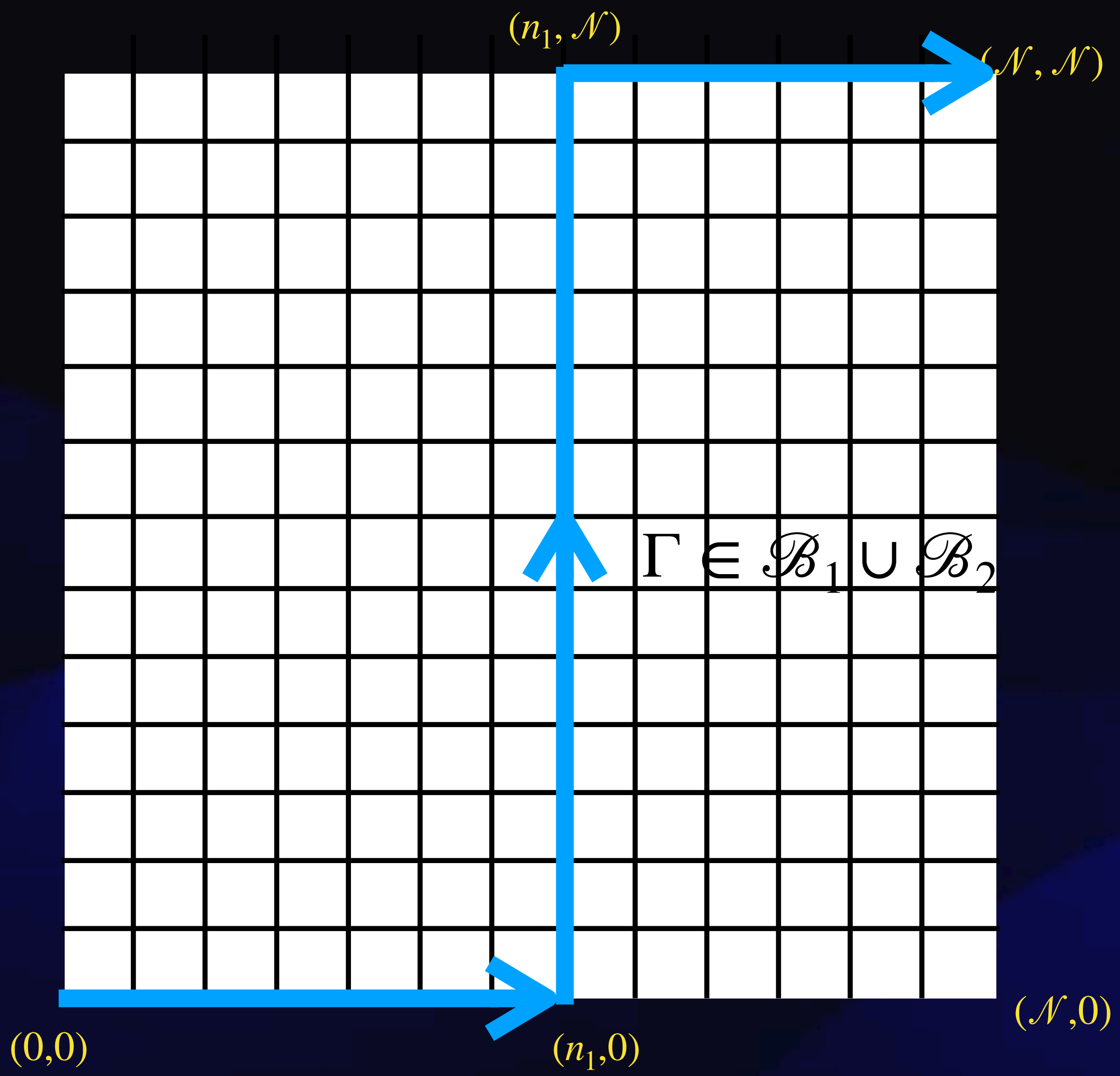
$K^{(1)}$

$$K^{(2)} = \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\mathcal{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(\mathcal{N},0) \int_{\mathbf{q}(\mathcal{N},0)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(\mathcal{N},t_2)] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(\mathcal{N},0)} L_1(t_1,0)dt_1 + \int_{(\mathcal{N},0)}^{(\mathcal{N},\mathcal{N})} L_2(\mathcal{N},t_2)dt_2 \right)}$$

$$+ \sum_{n_1=1}^{\mathcal{N}-1} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,\mathcal{N})} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,\mathcal{N})$$

$$\times \int_{\mathbf{q}(n_1,\mathcal{N})}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(t_1,\mathcal{N})] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(n_1,0)} L_1(t_1,0)dt_1 + \int_{(n_1,0)}^{(n_1,\mathcal{N})} L_2(n_1,t_2)dt_2 + \int_{(n_1,\mathcal{N})}^{(\mathcal{N},\mathcal{N})} L_1(t_1,\mathcal{N})dt_1 \right)}$$

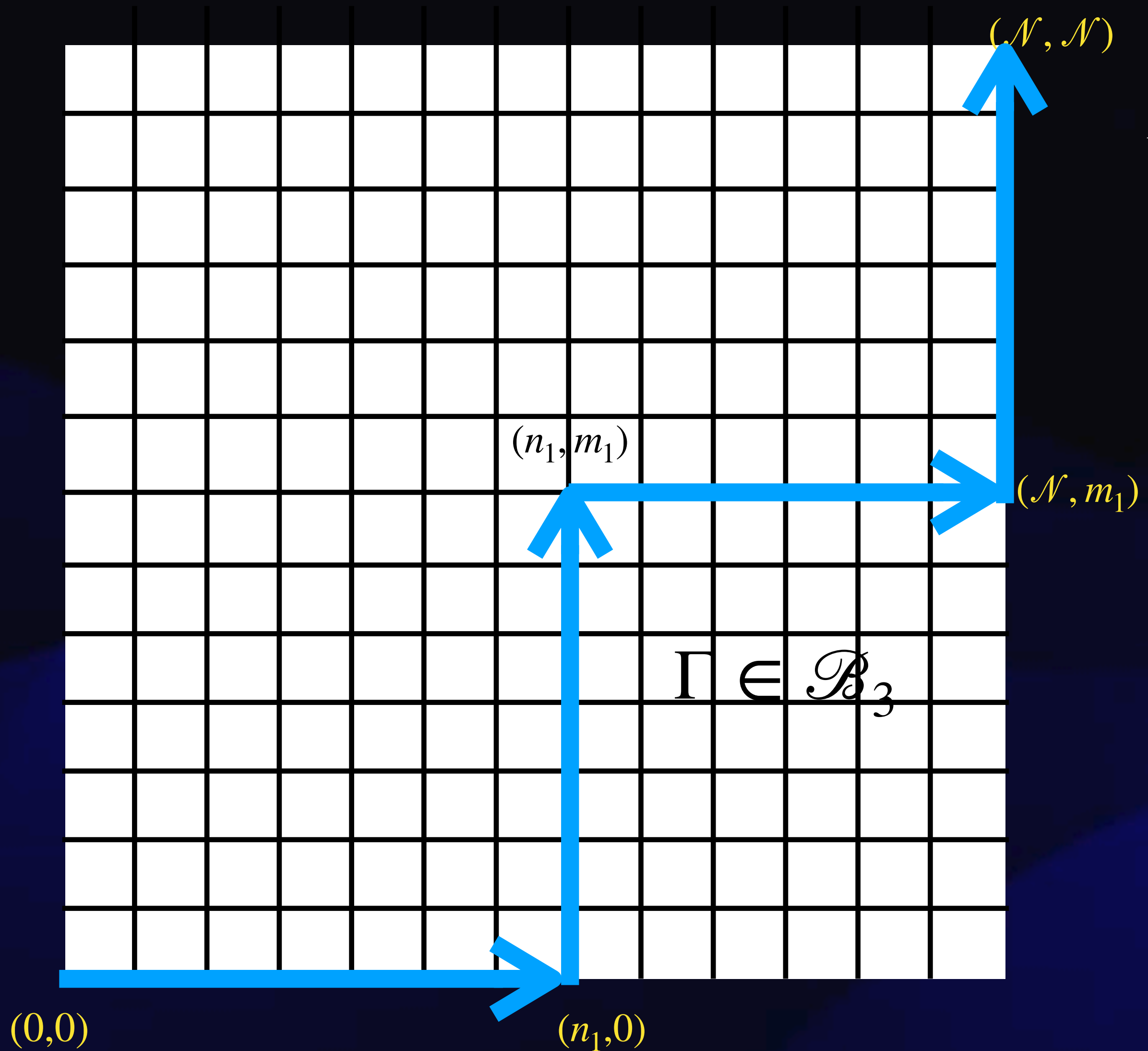
Multi-time propagator (2D)



$$K^{(2)} = \sum_{n_1=1}^{\mathcal{N}} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathcal{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,\mathcal{N})} \mathcal{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,\mathcal{N})$$

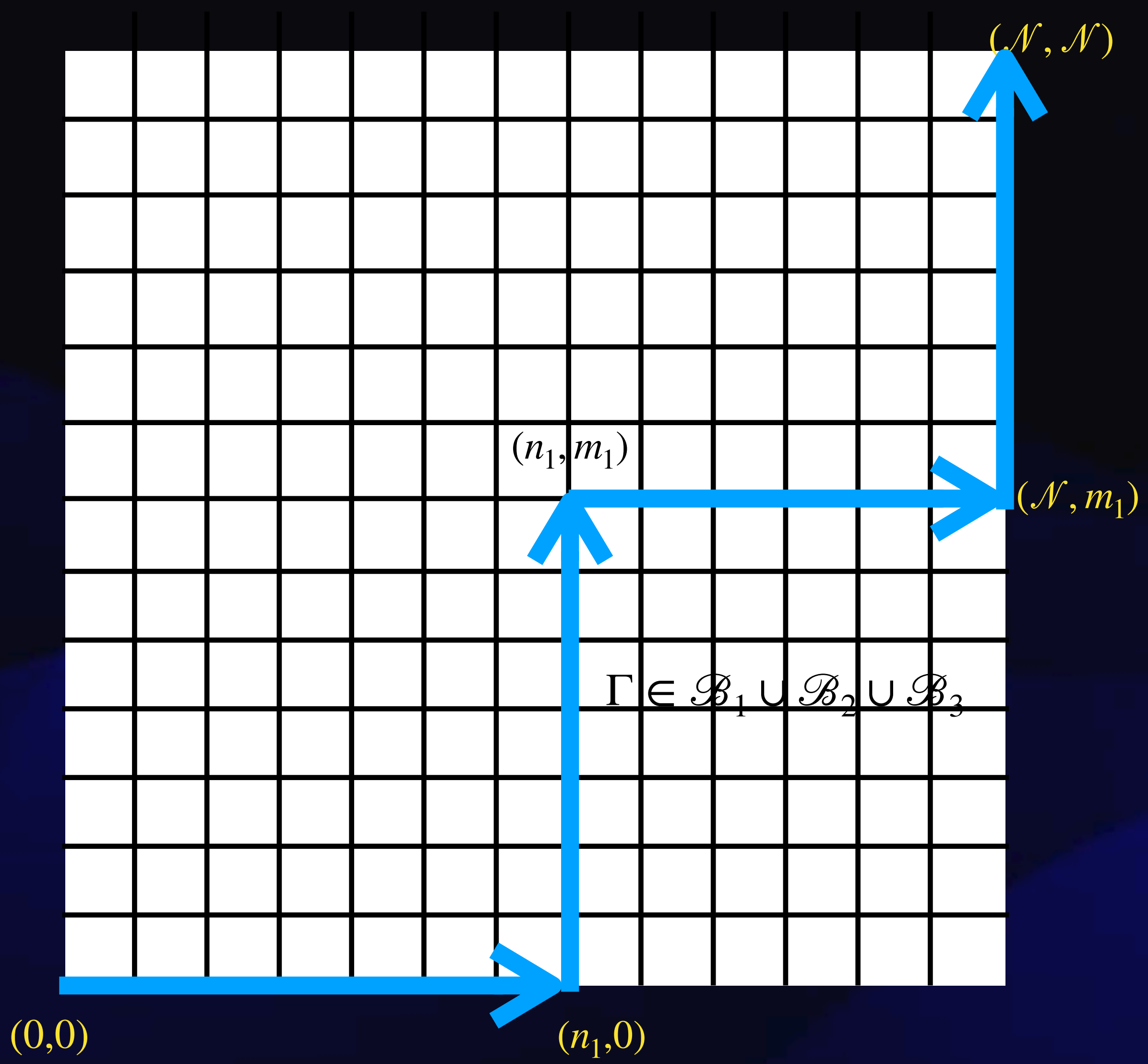
$$\times \int_{\mathbf{q}(n_1,\mathcal{N})}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathcal{D}[\mathbf{q}(t_1,\mathcal{N})] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\mathcal{B}_1\cup\mathcal{B}_2\}} \mathcal{L}}$$

Multi-time propagator (2D)



$$
\begin{aligned}
K^{(3)} = & \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\mathscr{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(\mathscr{N},0) \int_{\mathbf{q}(\mathscr{N},0)}^{\mathbf{q}(\mathscr{N},\mathscr{N})} \mathscr{D}[\mathbf{q}(\mathscr{N},t_2)] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(\mathscr{N},0)} L_1 dt_1 + \int_{(\mathscr{N},0)}^{(\mathscr{N},\mathscr{N})} L_2 dt_2 \right)} \\
& + \sum_{n_1=1}^{\mathscr{N}-1} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,N)} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,\mathscr{N}) \\
& \times \int_{\mathbf{q}(n_1,\mathscr{N})}^{\mathbf{q}(\mathscr{N},\mathscr{N})} \mathscr{D}[\mathbf{q}(t_1,\mathscr{N})] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(n_1,0)} L_1 dt_1 + \int_{(n_1,0)}^{(n_1,N)} L_2 dt_2 + \int_{(n_1,\mathscr{N})}^{(\mathscr{N},\mathscr{N})} L_1 dt_1 \right)} \\
& + \sum_{m_1=1}^{\mathscr{N}-1} \sum_{n_1=1}^{\mathscr{N}-1} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,m_1)} \mathscr{D}[\mathbf{q}(n_1,t_2)] \\
& \times \int_{-\infty}^{\infty} d^N q(n_1,m_1) \int_{\mathbf{q}(n_1,m_1)}^{\mathbf{q}(\mathscr{N},m_1)} \mathscr{D}[\mathbf{q}(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(\mathscr{N},m_1) \\
& \times \int_{\mathbf{q}(\mathscr{N},m_1)}^{\mathbf{q}(\mathscr{N},\mathscr{N})} \mathscr{D}[\mathbf{q}(\mathscr{N},t_2)] e^{\frac{i}{\hbar}\left(\int_{(0,0)}^{(n_1,0)} L_1 dt_1 + \int_{(n_1,0)}^{(n_1,m_1)} L_2 dt_2 + \int_{(n_1,m_1)}^{(\mathscr{N},m_1)} L_1 dt_1 + \int_{(\mathscr{N},m_1)}^{(\mathscr{N},\mathscr{N})} L_2 dt_2 \right)}
\end{aligned}
$$

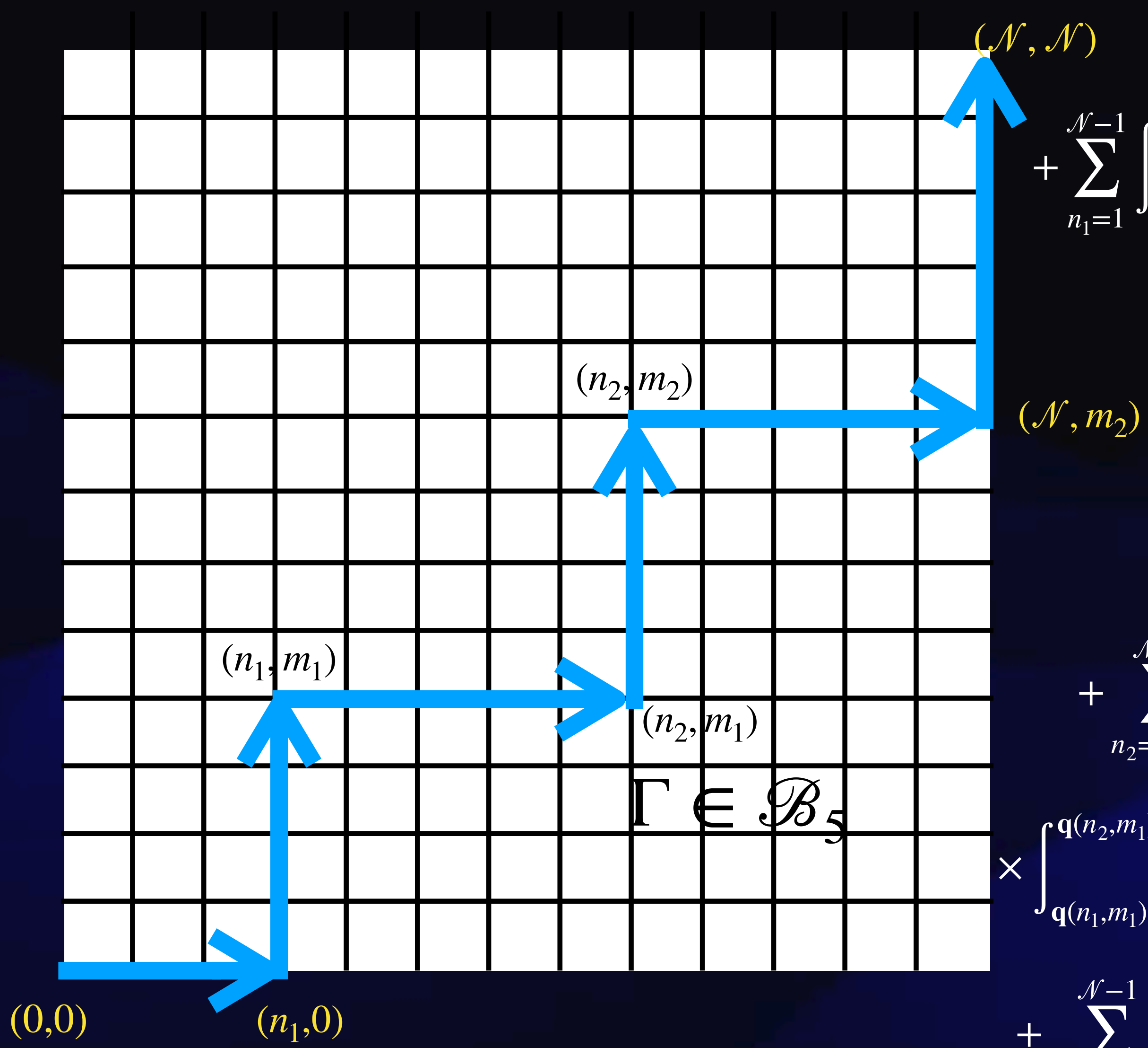
Multi-time propagator (2D)



$$K^{(3)} = \sum_{n_1=1}^{\mathcal{N}-1} \sum_{m_1=0}^{\mathcal{N}} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathcal{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,m_1)} \mathcal{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1)$$

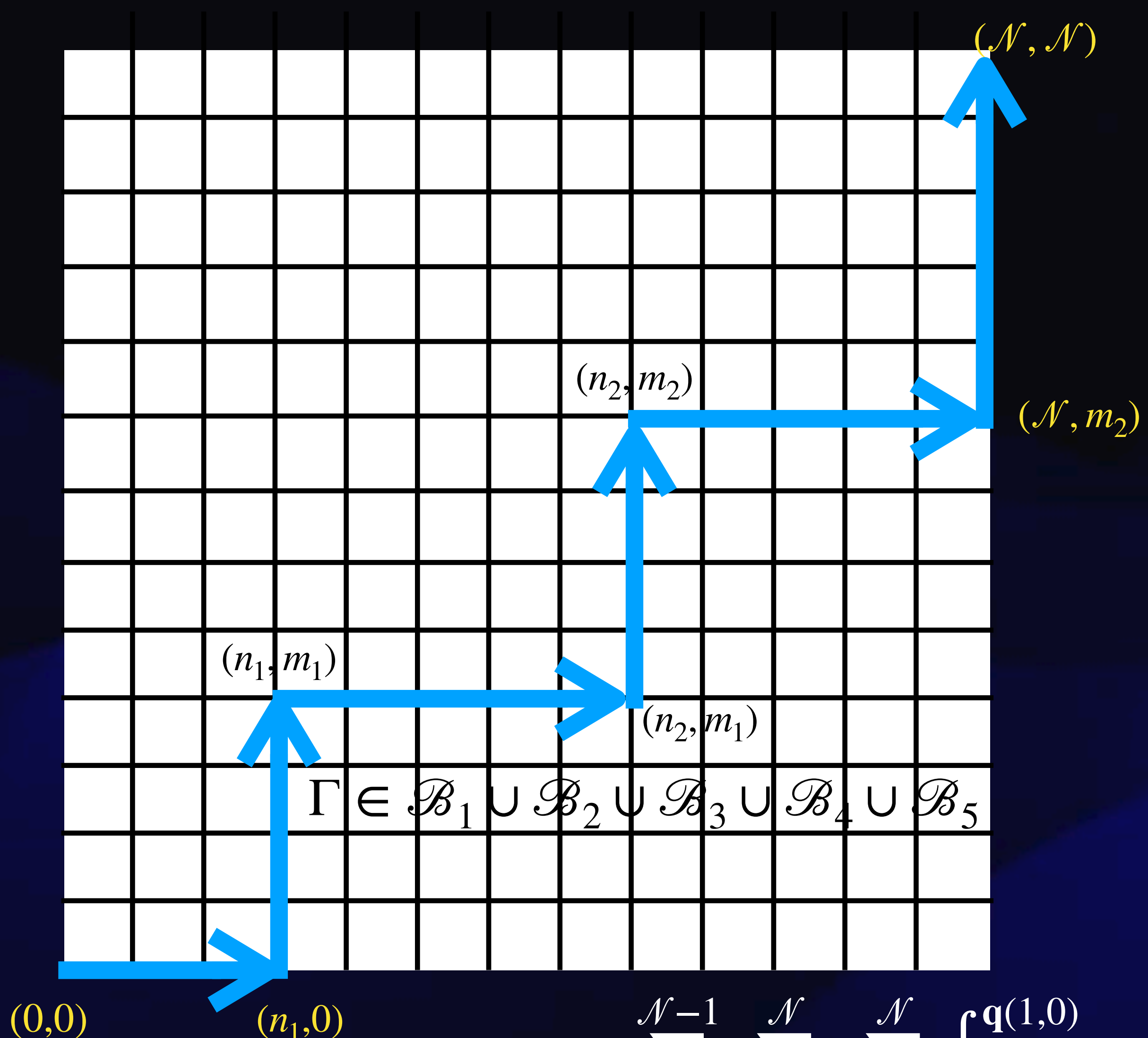
$$\times \int_{\mathbf{q}(n_1,m_1)}^{\mathbf{q}(\mathcal{N},m_1)} \mathcal{D}[\mathbf{q}(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(\mathcal{N},m_1) \int_{\mathbf{q}(\mathcal{N},m_1)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathcal{D}[\mathbf{q}(\mathcal{N},t_2)] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\}}\mathcal{L}}$$

Multi-time propagator (2D)



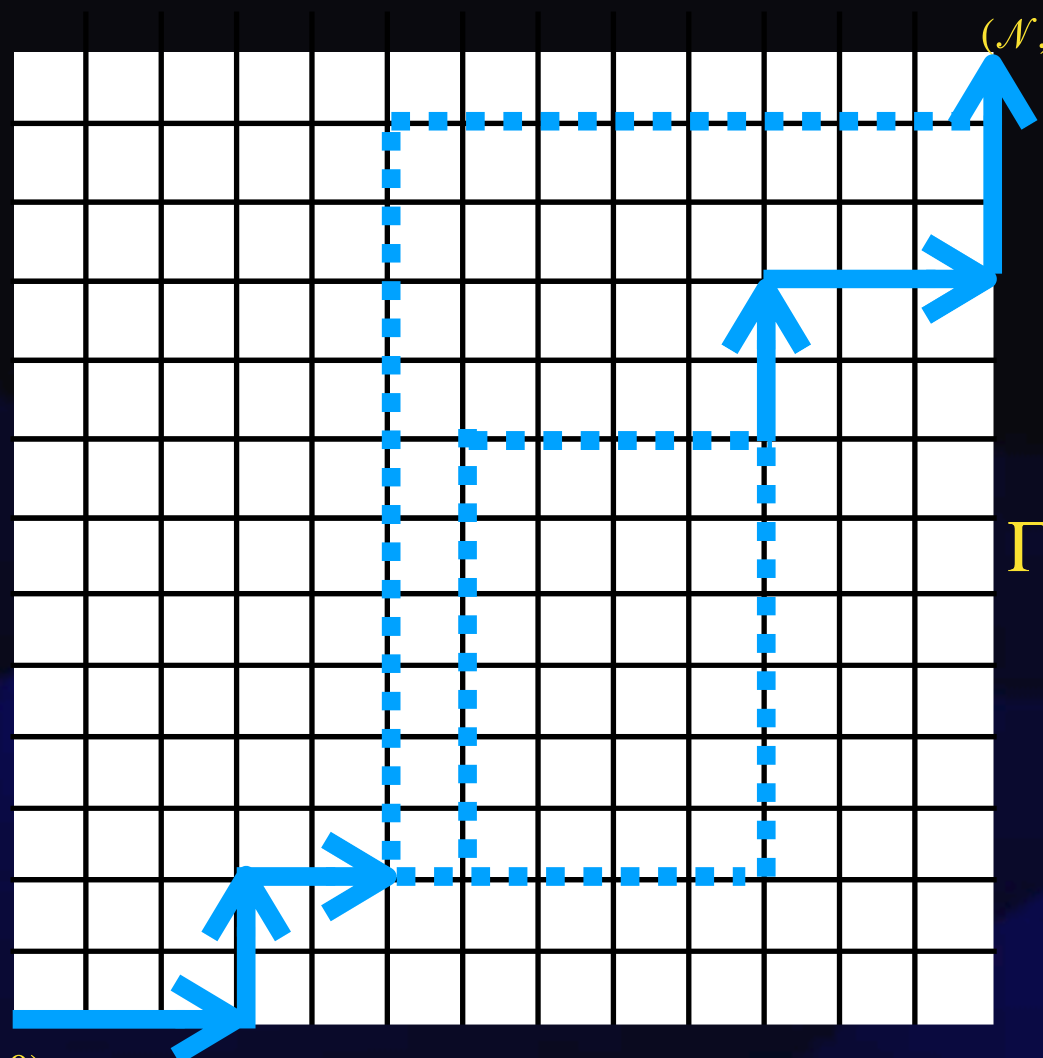
$$
\begin{aligned}
K^{(5)} = \Bigg(& \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\mathcal{N},0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(\mathcal{N},0) \int_{\mathbf{q}(\mathcal{N},0)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(\mathcal{N},t_2)] \\
& + \sum_{n_1=1}^{\mathcal{N}-1} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,\mathcal{N})} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,\mathcal{N}) \int_{\mathbf{q}(n_1,\mathcal{N})}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(t_1,\mathcal{N})] \\
& + \sum_{m_1=1}^{\mathcal{N}-1} \sum_{n_1=1}^{\mathcal{N}-1} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,m_1)} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \\
& \times \int_{\mathbf{q}(n_1,m_1)}^{\mathbf{q}(\mathcal{N},m_1)} \mathscr{D}[\mathbf{q}(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(\mathcal{N},m_1) \int_{\mathbf{q}(\mathcal{N},m_1)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(\mathcal{N},t_2)] \\
& + \sum_{n_2=n_1+1}^{\mathcal{N}-1} \sum_{m_1=1}^{\mathcal{N}-1} \sum_{n_1=1}^{\mathcal{N}-2} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,m_1)} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \\
& \times \int_{\mathbf{q}(n_1,m_1)}^{\mathbf{q}(n_2,m_1)} \mathscr{D}[\mathbf{q}(t_1,m_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_1) \int_{\mathbf{q}(n_2,m_1)}^{\mathbf{q}(n_2,\mathcal{N})} \mathscr{D}[\mathbf{q}(n_2,t_2)] \int_{-\infty}^{\infty} d^N q(n_2,\mathcal{N}) \int_{\mathbf{q}(n_2,\mathcal{N})}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(t_1,\mathcal{N})] \\
& + \sum_{m_2=m_1+1}^{\mathcal{N}-1} \sum_{n_2=n_1+1}^{\mathcal{N}-1} \sum_{m_1=1}^{\mathcal{N}-2} \sum_{n_1=1}^{\mathcal{N}-2} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(n_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{\mathbf{q}(n_1,0)}^{\mathbf{q}(n_1,m_1)} \mathscr{D}[\mathbf{q}(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \\
& \times \int_{\mathbf{q}(n_1,m_1)}^{\mathbf{q}(n_2,m_1)} \mathscr{D}[\mathbf{q}(t_1,m_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_1) \int_{\mathbf{q}(n_2,m_1)}^{\mathbf{q}(n_2,m_2)} \mathscr{D}[\mathbf{q}(n_2,t_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_2) \int_{\mathbf{q}(n_2,m_2)}^{\mathbf{q}(\mathcal{N},m_2)} \mathscr{D}[\mathbf{q}(t_1,m_2)] \\
& \times \int_{-\infty}^{\infty} d^N q(\mathcal{N},m_2) \int_{\mathbf{q}(\mathcal{N},m_2)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(\mathcal{N},t_2)] \Bigg) e^{\frac{i}{\hbar} \int_{\{\Gamma:\Gamma\in\bigcup_{l=1}^{5}\mathscr{B}_l\}} \mathscr{L}}
\end{aligned}
$$

Multi-time propagator (2D)



$$
\begin{aligned}
K^{(5)} = \sum_{n_2=2}^{\mathcal{N}-1} \sum_{m_2=m_1}^{\mathcal{N}} \sum_{m_1=0}^{\mathcal{N}} &\int_{\mathbf{q}(0,0)}^{\mathbf{q}(1,0)} \mathcal{D}[\mathbf{q}(t_1,0)] \int_{-\infty}^{\infty} d^N q(1,0) \int_{\mathbf{q}(1,0)}^{\mathbf{q}(1,m_1)} \mathcal{D}[\mathbf{q}(1,t_2)] \int_{-\infty}^{\infty} d^N q(1,m_1) \\
\times &\int_{\mathbf{q}(1,m_1)}^{\mathbf{q}(n_2,m_1)} \mathcal{D}[\mathbf{q}(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(n_2,m_1) \int_{\mathbf{q}(n_2,m_1)}^{\mathbf{q}(n_2,m_2)} \mathcal{D}[\mathbf{q}(n_2,t_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_2) \\
\times &\int_{\mathbf{q}(n_2,m_2)}^{\mathbf{q}(\mathcal{N},m_2)} \mathcal{D}[\mathbf{q}(t_1,m_2)] \int_{-\infty}^{\infty} d^N q(\mathcal{N},m_2) \int_{\mathbf{q}(\mathcal{N},m_2)}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathcal{D}[\mathbf{q}(\mathcal{N},t_2)] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\bigcup_{l=1}^{5}\mathcal{B}_l\}}\mathcal{L}}
\end{aligned}
$$

Multi-time propagator (2D)



$$K^{(\mathrm{All})} = \sum_{m_{\mathcal{N}-1}\geq\cdots\geq m_2\geq m_1\geq 0}^{\mathcal{N}} \mathcal{N}_{m_I} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \left(\prod_{i=1}^{\mathcal{N}-1} \int_{-\infty}^{\infty} d^N q(i,m_{i-1}) \int_{\mathbf{q}(i,m_{i-1})}^{\mathbf{q}(i,m_i)} \mathscr{D}[\mathbf{q}(i,t_2)] \int_{-\infty}^{\infty} d^N q(i,m_i) \right.$$

$$\left. \times \int_{\mathbf{q}(i,m_i)}^{\mathbf{q}(i+1,m_i)} \mathscr{D}[\mathbf{q}(t_1,m_{i+1})] \right) \int_{-\infty}^{\infty} d^N q(\mathcal{N},m_{\mathcal{N}-1}) \int_{\mathbf{q}(\mathcal{N},m_{\mathcal{N}-1})}^{\mathbf{q}(\mathcal{N},\mathcal{N})} \mathscr{D}[\mathbf{q}(\mathcal{N},t_2)] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\cup_{l=1}^{2\mathcal{N}-1}\mathscr{B}_l\}} \mathscr{L}}$$

where $m_I = m_1 m_2 \cdots m_{\mathcal{N}-1}$.

Multi-time propagator (2D)

Employing the symmetry of the lattice, we finally obtain

$$K(\mathbf{q}(\mathbf{t}(s'')), s''; \mathbf{q}(\mathbf{t}(s')), s') = \int_{\mathbf{q}(\mathbf{t}(s'))}^{\mathbf{q}(\mathbf{t}(s''))} \mathbb{D}[\mathbf{q}(\mathbf{t}(s)); \Gamma \in \mathscr{B}] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\mathscr{B}\}} \mathscr{L}}$$

where

$$\begin{aligned}
\int_{\mathbf{q}(\mathbf{t}(s'))}^{\mathbf{q}(\mathbf{t}(s''))} \mathbb{D}[\mathbf{q}(\mathbf{t}(s)); \Gamma \in \mathscr{B}] = \lim_{\substack{\mathcal{N}\to\infty \\ \epsilon_{1,2}\to 0}} \Bigg\{ & \sum_{m_{\mathcal{N}-1}\geq\cdots\geq m_2\geq m_1\geq 0}^{\mathcal{N}} \mathcal{N}_{m_I} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(\epsilon_1,0)} \mathscr{D}[\mathbf{q}(t_1,0)] \Bigg(\prod_{i=1}^{\mathcal{N}-1} \int_{-\infty}^{\infty} d^N q(i\epsilon_1, m_{i-1}\epsilon_2) \\
& \times \int_{\mathbf{q}(i\epsilon_1,m_{i-1}\epsilon_2)}^{\mathbf{q}(i\epsilon_1,m_i\epsilon_2)} \mathscr{D}[\mathbf{q}(i\epsilon_1,t_2)] \int_{-\infty}^{\infty} d^N q(i\epsilon_1, m_i\epsilon_2) \int_{\mathbf{q}(i\epsilon_1,m_i\epsilon_2)}^{\mathbf{q}((i+1)\epsilon_1,m_i\epsilon_2)} \mathscr{D}[\mathbf{q}(t_1, m_i\epsilon_2)] \Bigg) \\
& \times \int_{\mathbf{q}(i\epsilon_1,m_{i-1}\epsilon_2)}^{\mathbf{q}(i\epsilon_1,m_i\epsilon_2)} \mathscr{D}[\mathbf{q}(i\epsilon_1,t_2)] \int_{-\infty}^{\infty} d^N q(i\epsilon_1, m_i\epsilon_2) \int_{\mathbf{q}(i\epsilon_1,m_i\epsilon_2)}^{\mathbf{q}((i+1)\epsilon_1,m_i\epsilon_2)} \mathscr{D}[\mathbf{q}(t_1, m_i\epsilon_2)] \Bigg) \\
& + \sum_{n_{\mathcal{N}-1}\geq\cdots\geq n_2\geq n_1\geq 0}^{\mathcal{N}} \mathcal{N}_{n_I} \int_{\mathbf{q}(0,0)}^{\mathbf{q}(0,\epsilon_2)} \mathscr{D}[\mathbf{q}(0,t_2)] \Bigg(\prod_{i=1}^{\mathcal{N}-1} \int_{-\infty}^{\infty} d^N q(n_{i-1}\epsilon_1, i\epsilon_2) \\
& \times \int_{\mathbf{q}(n_{i-1}\epsilon_1,i\epsilon_2)}^{\mathbf{q}(n_i\epsilon_1,i\epsilon_2)} \mathscr{D}[\mathbf{q}(t_1,i\epsilon_2)] \int_{-\infty}^{\infty} d^N q(n_i\epsilon_1, i\epsilon_2) \int_{\mathbf{q}(n_i\epsilon_1,i\epsilon_2)}^{\mathbf{q}(n_i\epsilon_1,(i+1)\epsilon_2)} \mathscr{D}[\mathbf{q}(n_i\epsilon_1, t_2)] \Bigg) \\
& \times \int_{-\infty}^{\infty} d^N q(n_{\mathcal{N}-1}\epsilon_1, \mathcal{N}\epsilon_2) \int_{\mathbf{q}(n_{\mathcal{N}-1}\epsilon_1,\mathcal{N}\epsilon_2)}^{\mathbf{q}(\mathcal{N}\epsilon_1,\mathcal{N}\epsilon_2)} \mathscr{D}[\mathbf{q}(t_1, \mathcal{N}\epsilon_2)] \Bigg\}
\end{aligned}$$

Multi-time propagator - semiclassical approximation

Effectively, one can put the propagator as a function of a single time s

$$K(\mathbf{q}(s''), s''; \mathbf{q}(s'), s') = \int_{\mathbf{q}(s')}^{\mathbf{q}(s'')} \mathbb{D}[\mathbf{q}(s); \Gamma \in \mathscr{B}] e^{\frac{i}{\hbar}\int_{\{\Gamma:\Gamma\in\mathscr{B}\}} \mathscr{L}}$$

Therefore, we have

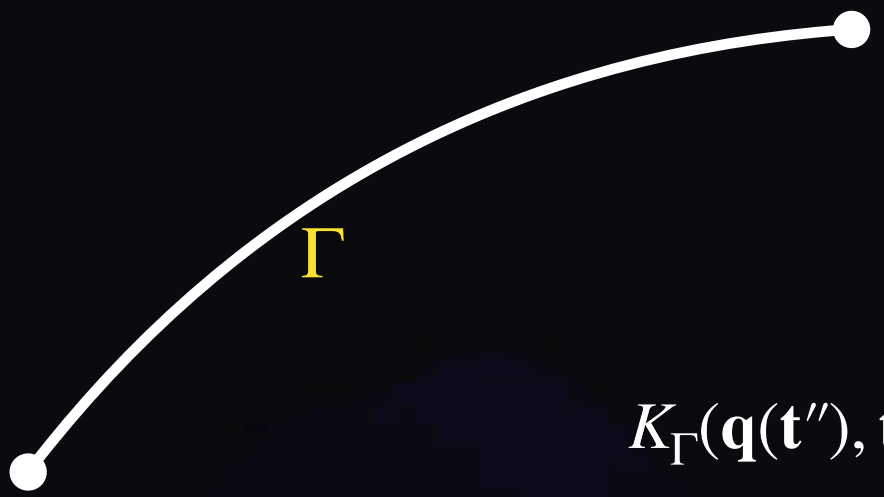
$$K(\mathbf{q}'', s''; \mathbf{q}', s') = e^{\frac{i}{\hbar}S[\mathbf{q}_c(s)]} \mathscr{Q}(\mathbf{q}'', s'', \mathbf{q}', s')\left[1 + \mathscr{O}(\hbar)\right]$$

where $\mathbf{q}_c$ is a classical solution and

$$\mathscr{Q}(\mathbf{q}'', s'', \mathbf{q}', s') = \int_{\mathbf{y}(s')=0}^{\mathbf{y}(s'')=0} \mathbb{D}[\mathbf{y}(s); \Gamma] e^{\frac{i}{2\hbar}\int_{s'}^{s''} d\tau \int_{s'}^{s''} d\sigma \left(\mathbf{y}(\tau)\frac{\delta^2 S[\mathbf{q}_c(s)]}{\delta \mathbf{q}(\tau)\delta \mathbf{q}(\sigma)}\mathbf{y}(\sigma)\right)}$$

is a smooth function of end points.

Multi-time propagator - semiclassical approximation



We now consider a path Γ connecting between end points $\mathbf{t}''$ and $\mathbf{t}'$ on space of time variables. Then what we have now is

$$K_\Gamma(\mathbf{q}(\mathbf{t}''),\mathbf{t}'';\mathbf{q}(\mathbf{t}'),\mathbf{t}') = e^{\frac{i}{\hbar}S_\Gamma[\mathbf{q}_c(\mathbf{t})]}\mathcal{Q}_\Gamma(\mathbf{q}'',\mathbf{q}',\mathbf{t}'',\mathbf{t}')\big[1+\mathcal{O}(\hbar)\big]$$

where

$$\mathcal{Q}_\Gamma(\mathbf{q}'',\mathbf{q}',\mathbf{t}'',\mathbf{t}') = \int_{\mathbf{y}(\mathbf{t}')=0}^{\mathbf{y}(\mathbf{t}'')=0}\mathcal{D}_\Gamma[\mathbf{y}(\mathbf{t})]e^{\frac{i}{2\hbar}\sum_{j=1}^{N}\int_\Gamma du_j\int_\Gamma dv_j\left(\mathbf{y}(\mathbf{u})\frac{\delta^2 S_{j,\Gamma}[\mathbf{q}_c(\mathbf{t})]}{\delta\mathbf{q}(\mathbf{u})\delta\mathbf{q}(\mathbf{v})}\mathbf{y}(\mathbf{v})\right)}$$

$$S_{j,\Gamma}[\mathbf{q}(\mathbf{t})] = \int_\Gamma L_j dt_j$$

The function $\mathcal{Q}_\Gamma$ can be reduced into

$$\mathcal{Q}_\Gamma(\mathbf{q}'',\mathbf{q}',\mathbf{t}'',\mathbf{t}') = \det\left(\frac{i}{2\pi\hbar}\frac{\partial^2 S_\Gamma[\mathbf{q}_c(\mathbf{t})]}{\partial\mathbf{q}(\mathbf{t}'')\partial\mathbf{q}(\mathbf{t}')}\right)^{\frac{1}{2}}$$

Integrability

Theorem: Let $\{L_1, L_2, \ldots, L_N\}$ be a set of Lagrangians satisfying the Lagrangian closure relation and $\mathscr{L} = \sum_{j=1}^{N} L_j dt_j$ be the Lagrangian 1-form, where $L_j = L_j\left(\mathbf{q}, \left\{ \frac{\partial \mathbf{q}}{\partial t_j}; j = 1,2,\ldots,N \right\}; \mathbf{t}\right)$. On the space of independent variables (time variables), the multi-time propagator for any $\Gamma \in \mathscr{B}$, where $\mathscr{B}$ is a family of paths connecting between $\mathbf{t}'$ and $\mathbf{t}''$, gives equally contribution leading to

$$\oint \mathscr{D}_{C=\partial\mathcal{S}}[\mathbf{q}(\mathbf{t})] e^{\frac{i}{\hbar}\oint_{C=\partial\mathcal{S}} \mathscr{L}} = \mathbb{1}$$

where $\mathcal{S}$ is an arbitrary surface bounded by a contractible loop C on the space of time variables, and therefore the multi-time quantum system is integrable.

Integrability

Proof: Recalling the multi-time propagator for path Γ

$$K_\Gamma(\mathbf{q}(\mathbf{t}''),\mathbf{t}'';\mathbf{q}(\mathbf{t}'),\mathbf{t}') = e^{\frac{i}{\hbar}S_\Gamma[\mathbf{q}_c(\mathbf{t})]}\mathcal{Q}_\Gamma(\mathbf{q}'',\mathbf{q}',\mathbf{t}'',\mathbf{t}')\big[1+\mathcal{O}(\hbar)\big]$$

$$\mathcal{Q}_\Gamma = \det\left(\frac{i}{2\pi\hbar}\frac{\partial^2 S_\Gamma[\mathbf{q}_c(\mathbf{t})]}{\partial\mathbf{q}(\mathbf{t}'')\partial\mathbf{q}(\mathbf{t}')}\right)^{\frac{1}{2}}$$

and for path Γ'

$$K_{\Gamma'}(\mathbf{q}(\mathbf{t}''),\mathbf{t}'';\mathbf{q}(\mathbf{t}'),\mathbf{t}') = e^{\frac{i}{\hbar}S_{\Gamma'}[\mathbf{q}_c(\mathbf{t})]}\mathcal{Q}_{\Gamma'}(\mathbf{q}'',\mathbf{q}',\mathbf{t}'',\mathbf{t}')\big[1+\mathcal{O}(\hbar)\big]$$

$$\mathcal{Q}_{\Gamma'} = \det\left(\frac{i}{2\pi\hbar}\frac{\partial^2 S_{\Gamma'}[\mathbf{q}_c(\mathbf{t})]}{\partial\mathbf{q}(\mathbf{t}'')\partial\mathbf{q}(\mathbf{t}')}\right)^{\frac{1}{2}}$$

The closure relation for the classical Lagrangian 1-forms $\mathscr{L}_c$ provides

$$S_\Gamma[\mathbf{q}_c(\mathbf{t})] - S_{\Gamma'}[\mathbf{q}_c(\mathbf{t})] = \left(\int_\Gamma - \int_{\Gamma'}\right)\mathscr{L}_c = \oint_{C=\partial\mathcal{S}}\mathscr{L}_c = \iint_{\mathcal{S}}\sum_{k\geq 1}^{N}\sum_{l=1}^{N}\left(\frac{\partial L_l}{\partial t_k} - \frac{\partial L_k}{\partial t_l}\right)dt_k\wedge dt_l = 0$$

which is nothing but the path independent feature on independent variables space. Here $\mathcal{S}$ is an arbitrary surface bounded by a contractible loop C on the space of time variables. Therefore, $\mathcal{Q}_\Gamma = \mathcal{Q}_{\Gamma'}$ and consequently we have

$$K_\Gamma(\mathbf{q}(\mathbf{t}''),\mathbf{t}'';\mathbf{q}(\mathbf{t}'),\mathbf{t}') = K_{\Gamma'}(\mathbf{q}(\mathbf{t}''),\mathbf{t}'';\mathbf{q}(\mathbf{t}'),\mathbf{t}')$$

Integrability

For a contractible loop $C = \partial\mathcal{S}$ on space of time variable, the propagator can be captured as

$$K_{C=\partial\mathcal{S}} = \lim_{(\mathbf{t}'-\tilde{\mathbf{t}})\to 0} \int_{-\infty}^{\infty} d^N q'' \int_{-\infty}^{\infty} d^N q' \det\left(\left(\frac{i}{2\pi\hbar}\right)^2 \frac{\partial^2 S_\Gamma[\mathbf{q}_c(\mathbf{t})]}{\partial\mathbf{q}(\mathbf{t}'')\partial\mathbf{q}(\mathbf{t}')} \frac{\partial^2(-S_{\Gamma'}[\mathbf{q}_c(\mathbf{t})])}{\partial\mathbf{q}(\mathbf{t}'')\partial\mathbf{q}(\tilde{\mathbf{t}})}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\left(\int_{\mathbf{t}',\Gamma}^{\mathbf{t}''} - \int_{\tilde{\mathbf{t}},\Gamma'}^{\mathbf{t}''}\right)\mathcal{L}_c} .$$

Then we write

$$S_c[\mathbf{q}', \mathbf{q}''] =: \int_{\mathbf{t}'}^{\mathbf{t}''} \mathcal{L}_c$$

$$S_c[\tilde{\mathbf{q}}, \mathbf{q}''] =: \int_{\tilde{\mathbf{t}}}^{\mathbf{t}''} \mathcal{L}_c$$

Dropping out the subscripts Γ and Γ' because of path independent feature, we obtain

$$\lim_{(\mathbf{t}'-\tilde{\mathbf{t}})\to 0}\left(\int_{\mathbf{t}'}^{\mathbf{t}''} - \int_{\tilde{\mathbf{t}}}^{\mathbf{t}''}\right)\mathcal{L}_c = \lim_{(\mathbf{t}'-\tilde{\mathbf{t}})\to 0} \sum_{i=1}^{N} \frac{S_c[\mathbf{q}', \mathbf{q}''] - S_c[\tilde{\mathbf{q}}, \mathbf{q}'']}{\tilde{q}_i - q_i'}(\tilde{q}_i - q_i') = -\frac{\partial S_c}{\partial\tilde{\mathbf{q}}} \cdot (\tilde{\mathbf{q}} - \mathbf{q}') .$$

Therefore, we have

$$K_{C=\partial\mathcal{S}} = \left(\frac{1}{2\pi}\right)^N \int_{-\infty}^{\infty} d^N q' \int_{-\infty}^{\infty} d^N \left(\frac{1}{\hbar}\frac{\partial S_c}{\partial\tilde{q}}\right) e^{-\frac{i}{\hbar}\frac{\partial S_c}{\partial\tilde{\mathbf{q}}}\cdot(\tilde{\mathbf{q}}-\mathbf{q}')} = \int_{-\infty}^{\infty} d^N q' \delta^N(\tilde{q} - q') = \mathbb{1}$$

Example

To verify the idea, we shall explore a simplest example: two-time harmonic oscillators, see Kings and Nijhoff.

$$L_1 = \frac{1}{2}\left(\frac{\partial \mathbf{q}}{\partial t_1}\right)^2 - \frac{\omega_1^2 \mathbf{q}^2}{2} \quad \text{and} \quad L_2 = \frac{1}{2}\left(\frac{\partial \mathbf{q}}{\partial t_2}\right)^2 - \frac{\omega_2^2 \mathbf{q}^2}{2},$$

where $\mathbf{q}(t_1, t_2) = (q_1(t_1, t_2), q_2(t_2, t_2))$ and $\omega_{1,2}$ are constant.

The quantity that we need to focus on is

$$\frac{i}{2\hbar}\sum_{j=1}^{2}\int_\Gamma du_j \int_\Gamma dv_j \left(\mathbf{y}(\mathbf{u})\frac{\delta^2 S_{j,\Gamma}[\mathbf{q}_c(\mathbf{t})]}{\delta \mathbf{q}(\mathbf{u})\delta \mathbf{q}(\mathbf{v})}\mathbf{y}(\mathbf{v})\right) = \frac{i}{2\hbar}\left(\int_\Gamma dt_1 \left[\left(\frac{\partial \mathbf{y}}{\partial t_1}\right)^2 - \omega_1^2 \mathbf{y}^2\right] + dt_2 \left[\left(\frac{\partial \mathbf{y}}{\partial t_2}\right)^2 - \omega_2^2 \mathbf{y}^2\right]\right)$$

Therefore, we have

$$\mathcal{Q}_\Gamma = \int_{\mathbf{y}(0,0)=0}^{\mathbf{y}(T_1,T_2)=0} \mathcal{D}_\Gamma[\mathbf{y}(t_1, t_2)] e^{\frac{i}{2\hbar}\left(\int_{(0,0)}^{(T_1,0)} dt_1 \left[\left(\frac{\partial \mathbf{y}}{\partial t_1}\right)^2 - \omega_1^2 \mathbf{y}^2\right] + \int_{(T_1,0)}^{(T_1,T_2)} dt_2 \left[\left(\frac{\partial \mathbf{y}}{\partial t_2}\right)^2 - \omega_2^2 \mathbf{y}^2\right]\right)}.$$

Example

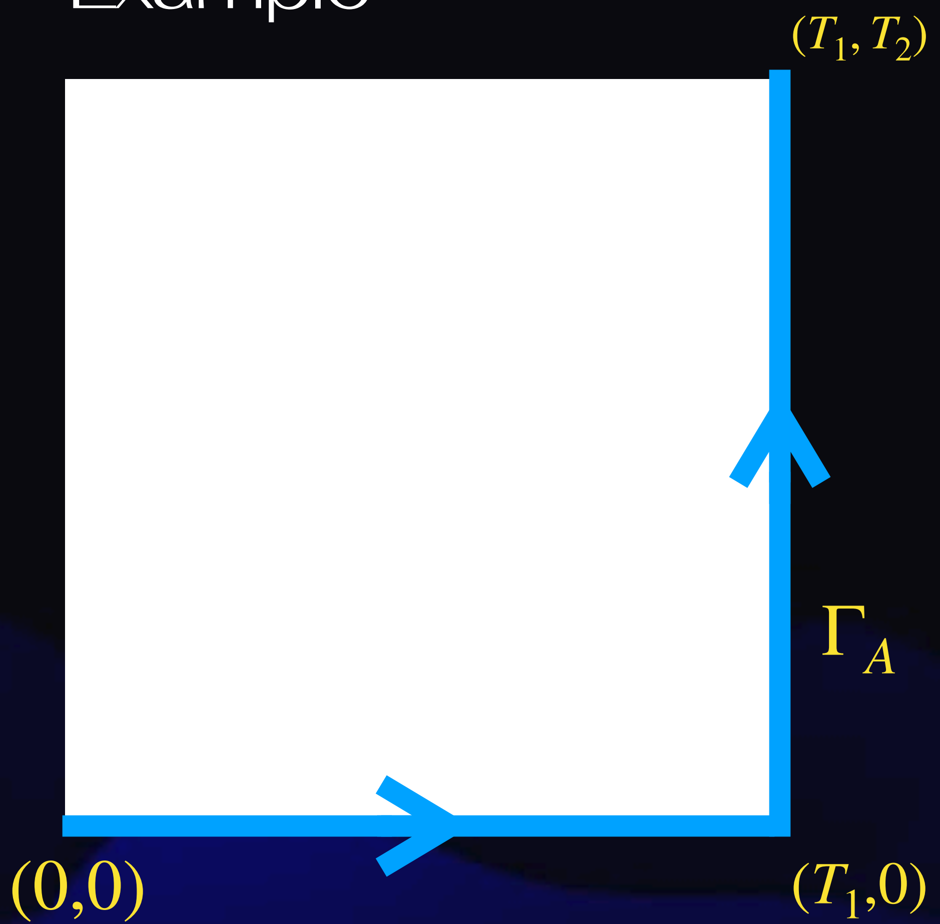
The whole exponent term of the equation Q_Γ can be written as

$$\int_{(0,0)}^{(T_1,0)} dt_1 \left[\left(\frac{\partial \mathbf{y}}{\partial t_1}\right)^2 - \omega_1^2 \mathbf{y}^2\right] + \int_{(T_1,0)}^{(T_1,T_2)} dt_2 \left[\left(\frac{\partial \mathbf{y}}{\partial t_2}\right)^2 - \omega_2^2 \mathbf{y}^2\right]$$

$$= \int_{(0,0)}^{(T_1,0)} dt_1 \mathbf{y} \left[-\left(\frac{\partial}{\partial t_1}\right)^2 - \omega_1^2\right] \mathbf{y} + \mathbf{y}(T_1,0)\frac{\partial \mathbf{y}(T_1,0)}{\partial t_1}$$

$$+ \int_{(T_1,0)}^{(T_1,T_2)} dt_2 \mathbf{y} \left[-\left(\frac{\partial}{\partial t_2}\right)^2 - \omega_2^2\right] \mathbf{y} - \mathbf{y}(T_1,0)\frac{\partial \mathbf{y}(T_1,0)}{\partial t_2} \ .$$

Example



For this particular path, we find that

$$\mathbf{y}(t_1,t_2) = \sum_n a_n \mathbf{y}_{n,A}(t_1,t_2) = \sum_n a_n \left(\sqrt{\frac{2}{T_1}} \sin\left(\frac{n\pi}{T_1}t_1\right) \cos\left(\frac{n\pi}{T_2}t_2\right) + \sqrt{\frac{2}{T_2}} \sin\left(\frac{n\pi}{T_2}t_2\right) \cos\left(\frac{n\pi}{T_1}t_1\right) \right)$$

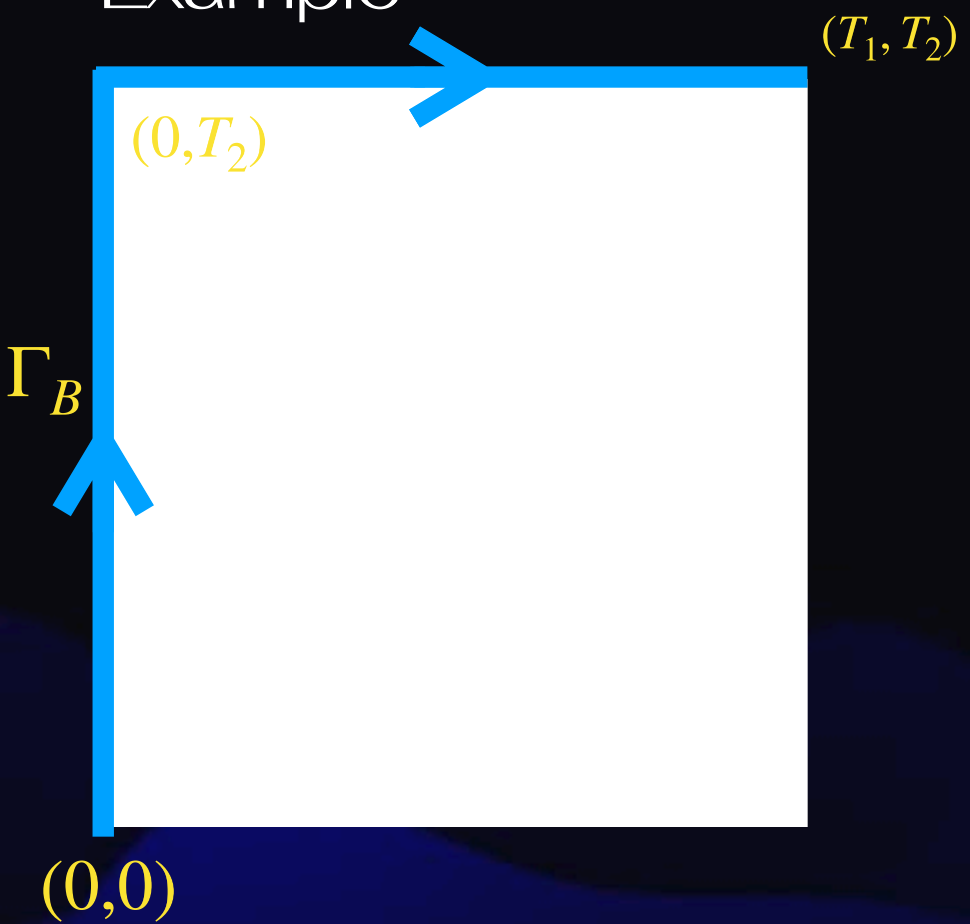
where $0 \leq t_{1,2} \leq T_{1,2}$ and it is not difficult to show that the orthonormality condition holds

$$\int_{(0,0)}^{(T_1,0)} dt_1 \mathbf{y}_{n,A} \mathbf{y}_{m,A} = \int_{(T_1,0)}^{(T_1,T_2)} dt_2 \mathbf{y}_{n,A} \mathbf{y}_{m,A} = \delta_{nm} \, .$$

Then, the Q_Γ function becomes

$$\mathcal{Q}_{\Gamma_A} = \int \mathcal{D}[a_n] e^{\frac{i}{2\hbar} \sum_n |a_n|^2 \left(-\omega_1^2 - \omega_2^2 + \left(\frac{n\pi}{T_1}\right)^2 + \left(\frac{n\pi}{T_2}\right)^2 \right)} .$$

Example



For this particular path, we can still use

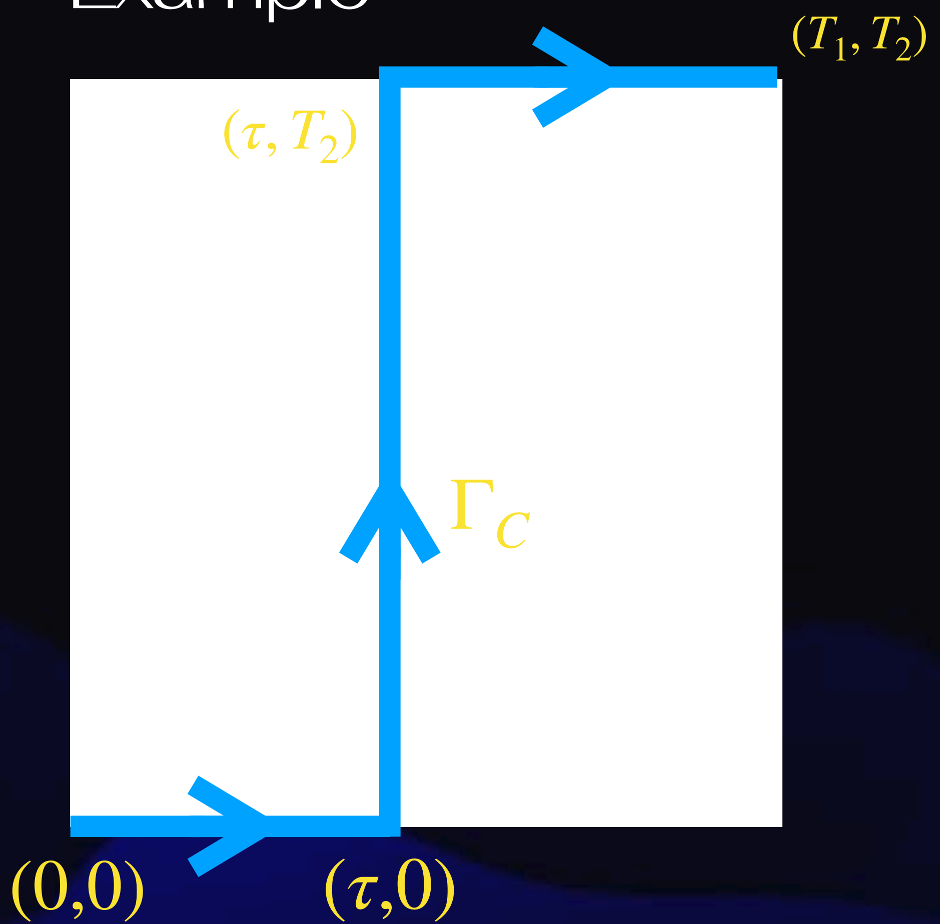
$$\mathbf{y}(t_1,t_2) = \sum_n a_n \mathbf{y}_{n,A}(t_1,t_2) = \sum_n a_n \left(\sqrt{\frac{2}{T_1}} \sin\left(\frac{n\pi}{T_1} t_1\right) \cos\left(\frac{n\pi}{T_2} t_2\right) + \sqrt{\frac{2}{T_2}} \sin\left(\frac{n\pi}{T_2} t_2\right) \cos\left(\frac{n\pi}{T_1} t_1\right) \right)$$

where $0 \le t_{1,2} \le T_{1,2}$.

Then, the Q_Γ function becomes

$$\mathcal{Q}_{\Gamma_B} = \int \mathcal{D}[a_n] e^{\frac{i}{2\hbar} \sum_n |a_n|^2 \left(-\omega_1^2 - \omega_2^2 + \left(\frac{n\pi}{T_1}\right)^2 + \left(\frac{n\pi}{T_2}\right)^2 \right)}.$$

Example



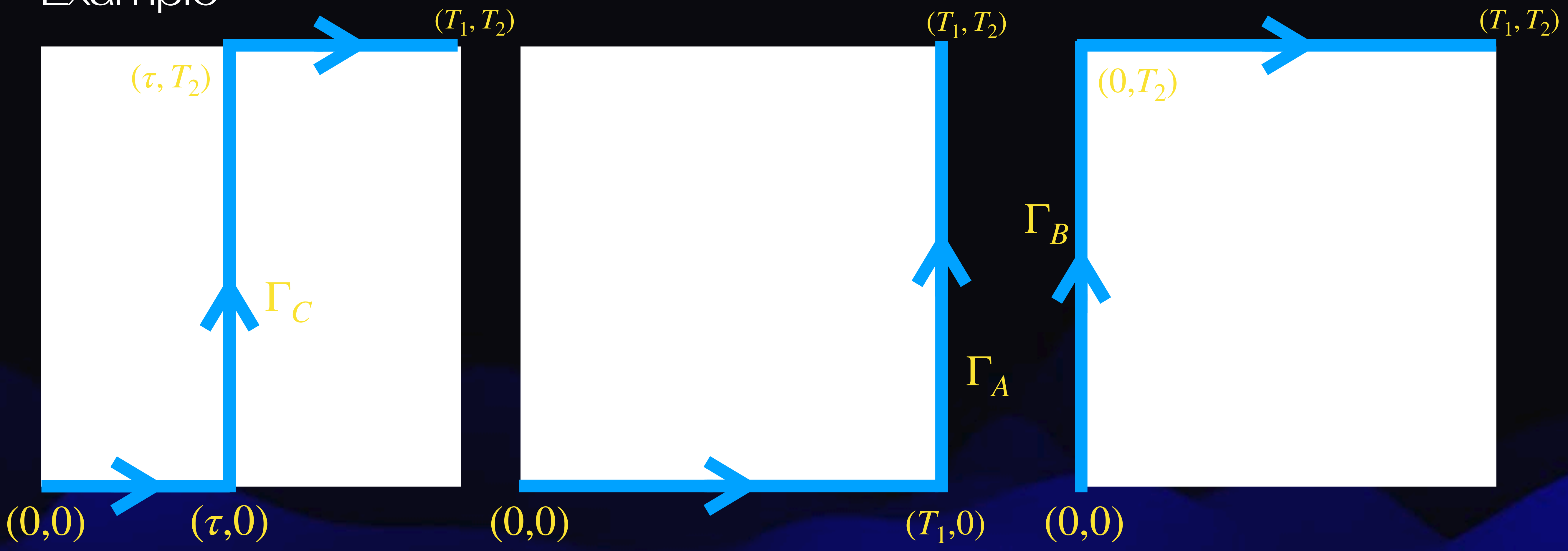
For this particular path, the fluctuation $\mathbf{y}$ is

$$\mathbf{y}(t_1,t_2) = \sum_n a_n \mathbf{y}_{n,C}(t_1,t_2) = \begin{cases} \sum_n a_n \sqrt{\frac{2}{T_1}} \sin\left(\frac{n\pi}{T_1} t_1\right) \cos\left(\frac{n\pi}{T_2} t_2\right) & ; \ (t_1 \leq \tau \text{ at } t_2 = 0) \cup (t_1 \geq \tau \text{ at } t_2 = T_2) \\ \sum_n a_n \sqrt{\frac{2}{T_1}} \cos\left(\frac{n\pi}{\tau} t_1\right) \sin\left(\frac{n\pi}{T_2} t_2\right) & ; \ (t_2 \in [0, T_2] \text{ at } t_1 = \tau) \end{cases}$$

Then, the Q_Γ function becomes

$$\mathcal{Q}_{\Gamma_C} = \int \mathcal{D}[a_n] e^{\frac{i}{2\hbar} \sum_n |a_n|^2 \left(-\omega_1^2 - \omega_2^2 + \left(\frac{n\pi}{T_1}\right)^2 + \left(\frac{n\pi}{T_2}\right)^2\right)}.$$

Example



Since $\mathcal{Q}_{\Gamma_A} = \mathcal{Q}_{\Gamma_B} = \mathcal{Q}_{\Gamma_C}$ and, with the closure relation, $S_A = S_B = S_C$, we have

$$K\left(\mathbf{q}(T_1, T_2), (T_1, T_2); \mathbf{q}(0,0), (0,0)\right) = \mathcal{Q}_A e^{\frac{i}{\hbar}S_A[\mathbf{q}_c]} = \mathcal{Q}_B e^{\frac{i}{\hbar}S_B[\mathbf{q}_c]} = \mathcal{Q}_C e^{\frac{i}{\hbar}S_C[\mathbf{q}_c]} \,,$$

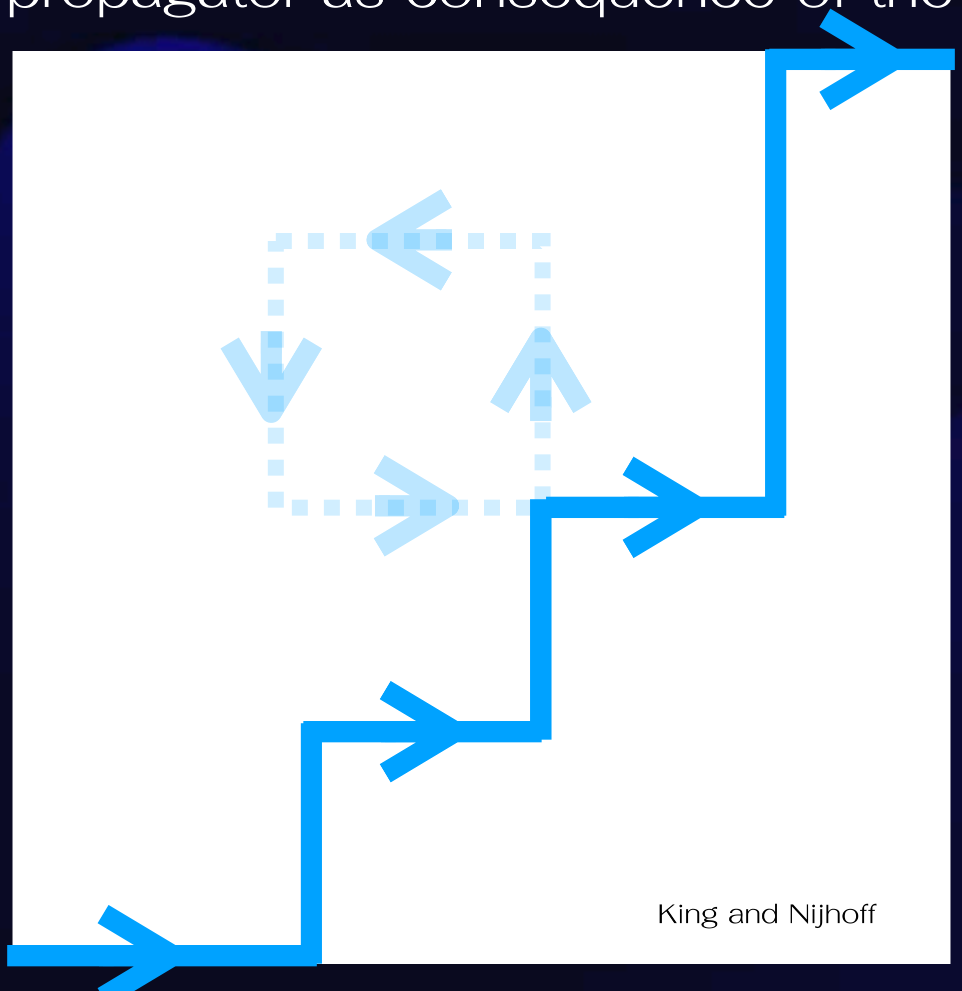
which is nothing but the path independent feature of the multi-time propagator in case of quadratic Lagrangian 1-forms.

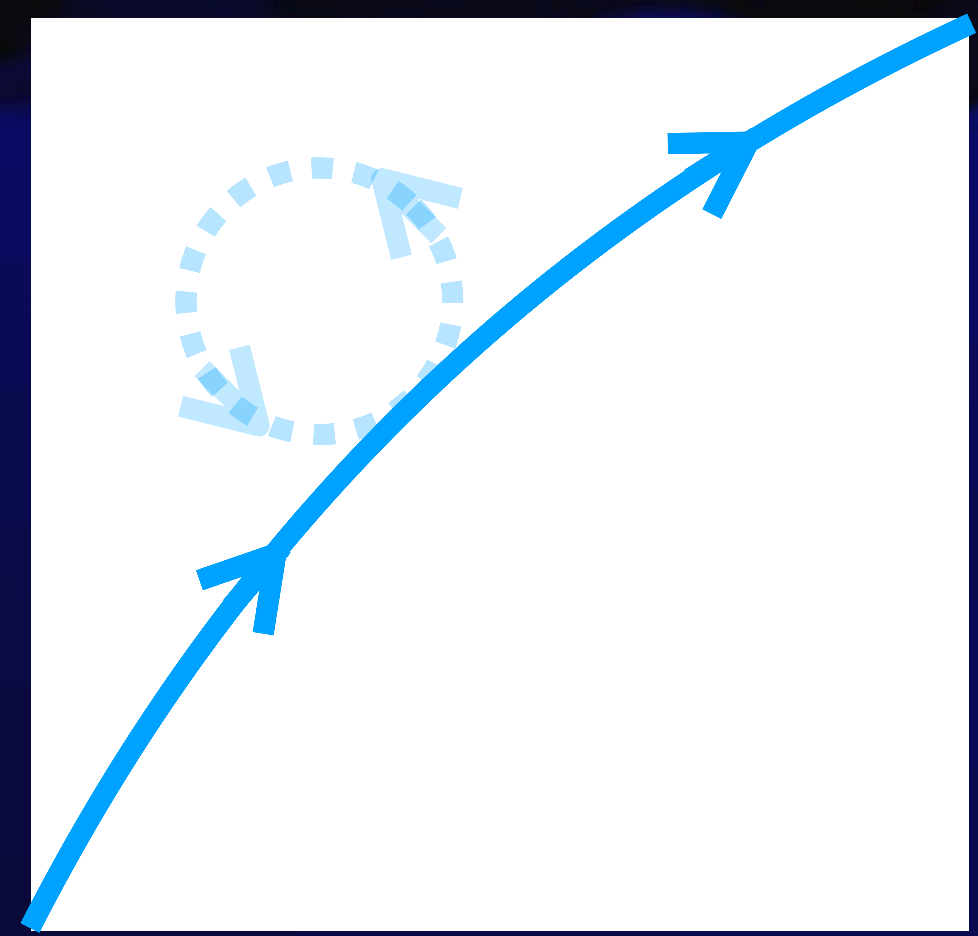
Summary

1) The propagator possessed the path independent feature on the space of time variables is the one that comes with a special set of Lagrangians satisfying the closure relation. This structure gives us an on top feature of the classical variational principle in the sense that this special set of Lagrangians plays a role of critical point resulting path independent propagator on the space of independent variables coined as the quantum variation, see King and Nijhoff.

Summary

2) The interesting point is that this multi-time propagator comes with a new feature on sum over all possible paths. One needs to take into account not only all possible paths on the space of dependent variables, but also on the space of independent variables(time variables). Of course, this idea is not new and it was first introduced by Nijhoff in 2013*. We point at this stage that what we come up for the formula of the continuous multi-time propagator in the 1-form case is not the same with Nijhoff's proposal. However, they do share the exactly the same interpretation. Moreover, loops will not contribute into the propagator as consequence of the theorem.





*This new perspective of treating the dependent and independent variables on the same equal footing was suggested in many places, see Atkinson and Rovelli, see further discussion in King's thesis.

Summary

3) Beyond the quadratic cases ?

Speculation: The propagator for each Lagrangian in the hierarchy can be expressed in the form [*]

$$K_j(\mathbf{q}(\ldots, t_j(s''), \ldots), s''; q(\ldots, t_j(s'), \ldots), s') = \int_{\mathbf{q}(\ldots,t_j(s''),\ldots)}^{\mathbf{q}(\ldots,t_j(s'),\ldots)} \mathscr{D}[\mathbf{q}(\ldots, t_j(s), \ldots)] e^{\frac{i}{\hbar} S[\mathbf{q}(\ldots,t_j(s'),\ldots)]}$$

where

$$S[\mathbf{q}(\ldots, t_j(s'), \ldots)] = \int_{t_j'}^{t_j''} dt_j L_j \ .$$

[*] Kazuyuki Jujit, 2011, “Beyond the Gaussian”, SIGMA, 7, 022.

Different from the quadratic ones

References

1) S. D. King and F. W. Nijhoff, Quantum variational principle and quantum multiform structure: The case of quadratic Lagrangians, 2019, Nuclear Physics B, 947:114686.
2) F.W. Nijhoff, Lagrangian multiform theory and variational principle for integrable systems, 2013, https://www.newton.ac.uk/files/seminar/20130710093010001-153650.pdf
3) J. Atkinson and S. B. Lobb and F. W. Nijhoff, An Integrable Multicomponent Quad-Equation and Its Lagrangian Formulation, 2012, Theoretical and Mathematical Physics, 173(3), pp. 1644-1653.
4) C. Rovelli, Discretising parametrised systems: the magic of Ditt-invariance, 2011, ARXIV.1107.2310.
5) S. D. King, Lagrangian Multiform Structures, Discrete Systems and Quantisation, 2017. PhD Thesis, University of Leeds.
6) Kongkoom T and Yoo-Kong S, Quantum integrability: Lagrangian 1-form case, 2023, Nuclear Physics B, 987: 11610.

Thank you!