All transcendental meromorphic solutions of the autonomous Schwarzian differential equation

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Theorem 1 (Malmquist[1], 1912)

If the differential equation

$$y' = R(z, y), \tag{1}$$

where R is a rational function in two variables, admits a non-rational meromorphic solution, then R is a polynomial and $\deg_y(R) \leq 2$.

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[2] Yosida, Japan. J. Math., 1933

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- A much simpler proof was given by Yosida[2] in 1933 using Nevanlinna theory.

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[2] Yosida, Japan. J. Math., 1933

Theorem 2 (Malmquist-Yosida)

Let $R(z, y)$ be rational in two variables. If the differential equation

$$(y')^n = R(z, y) \tag{2}$$

admits a non-rational meromorphic solution, then (2) reduces into

$$(y')^n = \sum_{i=0}^{2n} \alpha_i(z) y^i,$$

where at least one of the coefficients $\alpha_i(z)$ does not vanish.

Theorem 3 (Steinmetz[3], 1978)

Let $R(z,y)$ be rational in both of its arguments. If (2) *admits a transcendental meromorphic solution, then after a suitable Möbius transformation* $y=(\alpha v+\beta)/(\gamma v+\delta)$, (2) *reduces into one of the following types*

$$
\begin{aligned}
v' &= a(z)+b(z)v+c(z)v^2\\
(v')^2 &= a(z)(v-b(z))^2(v-\tau_1)(v-\tau_2)\\
(v')^2 &= a(z)(v-\tau_1)(v-\tau_2)(v-\tau_3)(v-\tau_4)\\
(v')^3 &= a(z)(v-\tau_1)^2(v-\tau_2)^2(v-\tau_3)^2\\
(v')^4 &= a(z)(v-\tau_1)^2(v-\tau_2)^3(v-\tau_3)^3\\
(v')^6 &= a(z)(v-\tau_1)^3(v-\tau_2)^4(v-\tau_3)^5
\end{aligned}
$$

where $\tau_1,\tau_2,\tau_3,\tau_4$ are complex constants, and the coefficients $a(z),b(z),c(z)$ are rational functions. Moreover, $a(z)\not\equiv 0$ in the lase five types.

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- Bank and Kaufman[4] obtained a precise growth estimate on meromorphic solutions of (2).

[3] Steinmetz, Dissertation, Karlsruhe Univ., 1978

[4] Bank and Kaufman, Acta Math., 1980

Conjecture

If the equation

$$f'' = R\left(z, f, f'\right), \tag{3}$$

where $R\left(z, f, f'\right)$ is rational in three variables has a non-rational meromorphic solution, then it reduces to (after a Möbius transformation)

$$f'' = L(z, f)\left(f'\right)^2 + M(z, f)f' + N(z, f), \tag{4}$$

where $L(z, f), M(z, f), N(z, f)$ are rational in two variables.

[5]Liao, Su, Yang, J. Differential Equations, 2003.

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Theorem 4 (Liao, Su, Yang[5], 2003)

The conjecture above is true if (3) possesses a meromorphic solution f of infinite order.

[5] Liao, Su, Yang, J. Differential Equations, 2003.

Difference equations (first-order)

$$f(z+1)^n = R(z,f) \tag{5}$$

- $n = 1$, if f is of finite order, then $\deg_f R = 1$. (Yanagihara[6])
- $n \in \mathbb{N}$, if $R(z,f) = R(f)$ and f is of finite order, then

  $$f(z+1) = Af + B \quad or \quad f(z+1)^2 = 1 - f^2. \quad \text{(Yanagihara}^7\text{)}$$

- $\deg_f R = n$, (5) can be reduced to one of 12 canonical equations. (Korhonen, Zhang[8])
- $\deg_f R \neq n$, (5) can be reduced to one of 16 canonical equations. (Korhonen, Zhang[9])

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[6] Yanagihara, Funkcial. Ekvac., 1980
[7] Yanagihara, Pitman Res. Notes Math. Ser., 1989
[8] Korhonen, Zhang, Constr. Approx., 2020
[9] Korhonen, Zhang, Constr. Approx., 2023

$$f(z+1)+f(z-1)=R(z,f(z)) \qquad (6)$$

Let $f(z)$ be an admissible meromorphic solution of (6) of finite order, where $R(z,f)$ is rational in f.

- $\deg_f R = 2$ (Ablowitz, Halburd, Herbst[10])

[10] Ablowitz, Halburd, Herbst, Nonlinearity, 2000

[11] Halburd, Korhonen, Proc. Lond. Math. Soc., 2007

Difference equations (second order)

$$f(z+1) + f(z-1) = R(z, f(z)) \tag{6}$$

Let $f(z)$ be an admissible meromorphic solution of (6) of finite order, where $R(z, f)$ is rational in f.

- $\deg_f R = 2$ (Ablowitz, Halburd, Herbst[10])
- (Halburd, Korhonen[11]) Either f satisfies a difference Riccati equation or equation (6) can be transformed by a linear change in f to one of the following 8 equations:

[10] Ablowitz, Halburd, Herbst, Nonlinearity, 2000

[11] Halburd, Korhonen, Proc. Lond. Math. Soc., 2007

Difference equations (second order)

$$f(z+1)+f(z)+f(z-1)=\frac{a_1z+a_2}{f}+b_1,$$
$$f(z+1)-f(z)+f(z-1)=\frac{a_1z+a_2}{f}+(-1)^z b_1,$$
$$f(z+1)+f(z-1)=\frac{a_1z+a_3}{f}+a_2,$$
$$f(z+1)+f(z-1)=\frac{a_1z+b_1}{f}+\frac{a_2}{f^2},$$
$$f(z+1)+f(z-1)=\frac{(a_1z+b_1)f+a_2}{(-1)^{-z}-f^2},$$
$$f(z+1)+f(z-1)=\frac{(a_1z+b_1)f+a_2}{1-f^2},$$
$$f(z+1)f(z)+f(z)f(z-1)=p,$$
$$f(z+1)f(z)+f(z)f(z-1)=pf+q,$$

where $p,q\in S(f)$, and $a_k,b_k\in S(f)$ are arbitrary finite-order periodic functions with period k.

- The Schwarzian differential equation is defined by

$$S(f,z)^p = R(z,f) = \frac{P(z,f)}{Q(z,f)}, \tag{7}$$

where p is a positive integer, and $R(z,f)$ is an irreducible rational function in f with meromorphic coefficients.

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- Let f be a meromorphic function. The Schwarzian derivative of f is

$$S_f(z) = S(f,z) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{8}$$

Theorem 5 (Ishizaki, 1997)

Suppose that the non-autonomous Schwarzian differential equation with $p = 1$ admits a transcendental meromorphic solution f such that all meromorphic coefficients of $R(z, f)$ are small with respect to f. Then

- *f satisfies a Riccati differential equation with small meromorphic coefficients; or*

Theorem 5 (Ishizaki, 1997)

Suppose that the non-autonomous Schwarzian differential equation with $p = 1$ *admits a transcendental meromorphic solution* f *such that all meromorphic coefficients of* $R(z, f)$ *are small with respect to* f*. Then*

- f *satisfies a Riccati differential equation with small meromorphic coefficients; or*
- f *satisfies a first order algebraic differential equation*

$$(f')^2 + B(z, f)f' + A(z, f) = 0 \tag{9}$$

  where $A(z, f), B(z, f)$ *are polynomials in* f *with small meromorphic coefficients; or*

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  where $A(z, f), B(z, f)$ *are polynomials in* f *with small meromorphic coefficients; or*
- *the Schwarzian differential equation reduces to one of the following two forms:*

  $$S(f, z) = \frac{P(z, f)}{(f + b(z))^2}$$
  $$S(f, z) = c(z)$$

  where $b(z), c(z)$ *are small meromorphic functions.*

Theorem A. (Ishizaki, 1991) Suppose that the autonomous Schwarzian differential equation (7) admits a transcendental meromorphic solution. Then for some Möbius transformation $u = (af + b)/(cf + d), ad - bc \neq 0$, (7) reduces into one of the following types

$$S(u,z) = c\frac{(u-\sigma_1)(u-\sigma_2)(u-\sigma_3)(u-\sigma_4)}{(u-\tau_1)(u-\tau_2)(u-\tau_3)(u-\tau_4)} \quad (10)$$

$$S(u,z)^3 = c\frac{(u-\sigma_1)^3(u-\sigma_2)^3}{(u-\tau_1)^3(u-\tau_2)^2(u-\tau_3)} \quad (11)$$

$$S(u,z)^3 = c\frac{(u-\sigma_1)^3(u-\sigma_2)^3}{(u-\tau_1)^2(u-\tau_2)^2(u-\tau_3)^2} \quad (12)$$

$$S(u,z)^2 = c\frac{(u-\sigma_1)^2(u-\sigma_2)^2}{(u-\tau_1)^2(u-\tau_2)(u-\tau_3)} \quad (13)$$

$$S(u,z) = c\frac{(u-\sigma_1)(u-\sigma_2)}{(u-\tau_1)(u-\tau_2)} \quad (14)$$

$$S(u,z) = c \quad (15)$$

where $c \in \mathbb{C}, \tau_j$ are distinct constants, and σ_j are constants, not necessarily distinct, $j = 1, \ldots, 4$.

Main result

Theorem 6 (Liao, W, Zhang, Zhao, 2023)

All transcendental meromorphic solutions of the autonomous Schwarzian differential equation can be constructed explicitly.

- Liao, W, Exact meromorphic solutions of Schwarzian differential equations, Math. Z., 300 (2022) 1657–1672.
- Liao, W, Zhang, Zhao, All meromorphic solutions of the autonomous Schwarzian differential equations, 2023, submitted.

Fifth canonical type

Theorem 7 (Liao, W, Zhang, Zhao, 2023)

Any transcendental meromorphic solution of the Schwarzian differential equation

$$S(u,z) = c\frac{(u-\sigma_1)(u-\sigma_2)}{(u-\tau_1)(u-\tau_2)}, \tag{16}$$

where $c, \tau_j, \sigma_j \in \mathbb{C}$ and $\tau_1 \neq \tau_2$, must have at least a Picard exceptional value on $\overline{\mathbb{C}}$.

All transcendental meromorphic solutions

Equations	Transcendental meromorphic solutions	Parameter values
$S(u,z)=c\frac{u^4+\beta u^2+\tau^2}{(u^2-1)(u^2-\tau^2)}$ with $\tau\in\mathbb{C}\backslash\{0,\pm1\}$ and $\beta=\frac{\tau^4-10\tau^2+1}{2(\tau^2+1)}$	$1-\frac{b}{\wp-d}$	$b=-\frac{c(\tau^2-1)}{2(\tau^2+1)}$ $d=\frac{c(\tau^2-5)}{12(\tau^2+1)}$ $g_2=\frac{c^2(\tau^4+14\tau^2+1)}{12(\tau^2+1)^2}$, $g_3=-\frac{c^3(\tau^4-34\tau^2+1)}{216(\tau^2+1)^2}$
$S(u,z)^3=c\frac{(u^2+5)^3}{(u-4)^3(u-3)^2u}$	$-\frac{3c}{c-74088\wp^3}$	$g_2=0$ $g_3=c/10584$
$S(u,z)^3=c\frac{(u^2+1/3)^3}{(u^3-u)^2}$	$\frac{9(9\wp+L^2)\wp'}{2L(81\wp^2-9L^2\wp+L^4)}$	$g_2=0, g_3=c/432$ $L^6=-27c/64$
$S(u,z)^2=c\frac{(u^2+1/4)^2}{u^2(u^2-1)}$	$-\frac{1}{2L}\frac{(8\wp+L^2)^2\wp'}{\wp(64\wp^2+L^4)}$	$g_2=-c/36, g_3=0$ $L^4=4c/9$
$S(u,z)=c\frac{u^2+2}{u^2-1}$	$\sin(\alpha z)$	$\alpha^2=2c$
$S(u,z)=c$	$\gamma(e^{\alpha z})$	$\alpha^2=-2c$

Remark 1

- **The conclusion of Theorem A does not hold for rational solutions** *of the autonomous Schwarzian differential equation. For instance, the function*

$$f(z) = -\frac{3}{2(z+a)^2},$$

where a is an arbitrary constant, satisfies the equation

$$S(u,z) = u$$

but it cannot be transformed into any type of (10)-(15) *via Möbius transformations.*

Theorem 8 (Liao, W, Zhang, Zhao, 2023)

If the autonomous Schwarzian differential equation has a non-constant rational solution, then it can be transformed via Möbius transformations into one of the following forms:

Form 1.

$$S_g^t = c^{t/k} g^2, \tag{17}$$

where $t \geq 2$ *is an integer, and all rational solutions are given by*

$$g(z) = c'(z - z_0)^{-t},$$

where c' *is a constant such that* $(1 - t^2)^t = 2^t c^{t/k} c'^2$;

Form 2.

$$S_f^3 = c^{3/k} \frac{(f - \sigma_1)^3}{(f - \tau_1)^2 (f - \tau_2)}, \tag{18}$$

where $\tau_1 \neq \tau_2$.

Theorem 9 (Briot, Bouquet[12], 1856)

Every meromorphic solution of the Briot-Bouquet (BB) equation

$$P(f, f') = 0, \tag{19}$$

where P is a polynomial in two variables, belongs to the class W.

- *Here, class W consists of elliptic functions and their degenerations, i.e., functions of the form $R(z)$ or $R(e^{az}), a \in \mathbb{C}$, where R is a rational function.*

[12] Briot, Bouquet, Intégration des équations différentielles au moyen de fonctions elliptiques, J. École Polytechnique, 1856

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- *Here, class W consists of elliptic functions and their degenerations, i.e., functions of the form $R(z)$ or $R(e^{az})$, $a \in \mathbb{C}$, where R is a rational function.*
- *W is chosen like* **Weierstrass** *as he proved that these are the only meromorphic functions that satisfy an* **algebraic addition theorem**

$$Q(y(z+\zeta), y(z), y(\zeta)) = 0, \quad \textit{where } Q \neq 0 \textit{ is a polynomial.}$$

[12] Briot, Bouquet, Intégration des équations différentielles au moyen de fonctions elliptiques, J. École Polytechnique, 1856

Let y be a meromorphic solution of the **higher order BB equations**

$$P\left(y^{(k)}, y\right) = 0, \quad k \geq 2, \tag{20}$$

then

- $k = 2$: $y \in W$ (Picard[13], 1880, Bank and Kaufman[14], 1981).

[13] Picard, C. R. Acad. Sci. Paris, 1880
[14] Bank, Kaufman, Math. Z., 1981
[15] Eremenko, Teor. Funktsii, Funk. Anal. i Prilozh., 1982
[16] Eremenko, TMath. Proc. Camb. Phil. Soc., 2009

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- $k \geq 3$: the conclusion is false in general.

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then

- $k = 2$: $y \in W$ (Picard[13], 1880, Bank and Kaufman[14], 1981).
- $k \geq 3$: the conclusion is false in general.
  - k is odd: non-entire $y \in W$ (Eremenko[15], 1982).
  - k is even: non-entire $y \in W$ (Eremenko, Liao, Ng[16], 2009)

[13] Picard, C. R. Acad. Sci. Paris, 1880

[14] Bank, Kaufman, Math. Z., 1981

[15] Eremenko, Teor. Funktsii, Funk. Anal. i Prilozh., 1982

[16] Eremenko, TMath. Proc. Camb. Phil. Soc., 2009

Theorem 10 (Eremenko, 2006)

All meromorphic solutions of the ODE

$$aw''' + bw'' + cw + w^2/2 + A = 0, \quad a, b, c, A \in \mathbb{C} \tag{21}$$

which describes the traveling wave reduction of the Kuramoto-Sivashinsky equation, belong to the class W.

- Eremenko, Meromorphic traveling wave solutions of the Kuramoto-Sivashinsky equation, J. Math. Phys. Anal. Geom., 2 (2006) 3 278–286.

Theorem 11 (Eremenko, 2006)

If an autonomous algebraic ODE

$$\sum_{\lambda\in I} a_{i_0,i_1,\dots,i_n} w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n} = 0, \tag{22}$$

where I *consists of finite multi-indices of the form* $\lambda = (i_0, i_1 \cdots, i_n)$, $i_k \in \mathbb{N}$, *satisfies*

Theorem 11 (Eremenko, 2006)

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where I consists of finite multi-indices of the form $\lambda = (i_0, i_1 \cdots, i_n)$, $i_k \in \mathbb{N}$, satisfies

i) *there is only one top degree term (the degree of each term in (22) is defined as $|\lambda| = i_0 + i_1 + \cdots + i_n$),*

ii) **(Finiteness property)** *there are finitely many choices of Laurent series expansion around the pole z_0 of w [Fuchs indices (= zeros of the indicial equation $Q = 0$, where Q is a polynomial of degree n) cannot be nonnegative integers],*

then all its meromorphic solutions belong to the class W.

- This method has been applied to many nonintegrable differential equations, such as the Kuramoto-Sivashinsky equation[17], the real cubic Swift-Hohenberg equation[18], the complex cubic-quintic Ginzburg-Landau equation[19][20], etc.

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[17] Eremenko, J. Math. Phys. Anal. Geom., 2006
[18] Conte, Ng, Wong, Stud. Appl. Math., 2012
[19] Conte, Musette, Ng, W, Phys. Rev. E, 2022
[20] Conte, Musette, Ng, W, Phys. Lett. A, 2023

Loewy Factorizable Algebraic ODEs

- This method has been applied to many nonintegrable differential equations, such as the Kuramoto-Sivashinsky equation[17], the real cubic Swift-Hohenberg equation[18], the complex cubic-quintic Ginzburg-Landau equation[19][20], etc.
- We propose to study the Loewy Factorizable Algebraic ODEs

$$[D - f_n(u)] \cdots [D - f_2(u)][D - f_1(u)](u - \alpha) = 0, \quad (23)$$

where $n \in \mathbb{N}, u = u(z), D = \dfrac{d}{dz}, f_i(u) = a_i u + b_i$ and $\alpha, a_i, b_i \in \mathbb{C}, i = 1, 2, \ldots, n$.

[17] Eremenko, J. Math. Phys. Anal. Geom., 2006
[18] Conte, Ng, Wong, Stud. Appl. Math., 2012
[19] Conte, Musette, Ng, W, Phys. Rev. E, 2022
[20] Conte, Musette, Ng, W, Phys. Lett. A, 2023

Loewy Factorizable Algebraic ODEs (general case)

Theorem 12 (Ng, W, 2019)

For all $n \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \backslash \Omega$, where Ω is the union of at most countably many hypersurfaces in $\mathbb{C}^n$, all meromorphic solutions (if they exist) of the ODE

$$[D - (a_n u + b_n)] \cdots [D - (a_1 u + b_1)](u - \alpha) = 0, \quad (24)$$

where $D = \dfrac{d}{dz}$, belong to class W.

- Ng, W, Nonlinear Loewy factorizable algebraic ODEs and Hayman's conjecture, Israel J. Math., 229 (2019) 1–38.

Theorem 13 (Ng, W, 2019)

Consider the ordinary differential equation

$$[D - f_2(u)][D - f_1(u)](u - \alpha) = 0, \tag{25}$$

where $u = u(z), D = \dfrac{d}{dz}, \alpha \in \mathbb{C}$ *and* $f_i(u) = a_i u + b_i, a_i, b_i \in \mathbb{C}, i = 1, 2.$
If either $a_1 a_2 = 0$ *or* $2 - \dfrac{4a_1}{a_2} \notin \mathbb{N}\backslash\{1, 2, 3, 4, 6\}$, *then all nontrivial meromorphic solutions of* (25) *can be constructed explicitly.*

Remark 2

There do exist meromorphic solutions of (25) outside the class W for certain choices of parameters, such as

$$u_1(z) = -\frac{q_i - q_k}{2} e^{-\frac{q_i - q_k}{\lambda} z} \frac{\wp'(e^{-\frac{q_i - q_k}{\lambda} z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_i - q_k}{\lambda} z} - \zeta_0; g_2, 0)} + q_k, \; g_2 \in \mathbb{C},$$

$$u_2(z) = \frac{\alpha a_1 - b_1}{2a_1} - \sqrt{\frac{\beta}{a_1}} \frac{e^{\frac{b_2 z}{2}} (c_1 J'_\nu(\zeta) + c_2 Y'_\nu(\zeta))}{(c_1 J_\nu(\zeta) + c_2 Y_\nu(\zeta))},$$

$$\zeta = \frac{2\sqrt{a_1 \beta}}{b_2} e^{\frac{b_2 z}{2}},$$

$$u_3(z) = \alpha - \frac{\sqrt{2} b_1 c_0 e^{b_1 z} \tanh\left(\frac{1}{2}\left(\sqrt{2} c_0 e^{b_1 z} + c_1\right)\right)}{a_2}.$$

Thank you!