Periodic Reductions of Discrete Lagrangian Multiforms

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Motivation

- A novel and general framework to carry out periodic reductions on discrete Lagrangian multiforms
- Reduce discrete 2-forms to discrete 1-forms
- In future work this framework will find novel 1-forms to generalise work on quantum multiforms and path integrals[1]
- This elevates periodic discrete integrable equations to the Lagrangian (multiform) level
- Note a connection has been made before between discrete 1-forms and 2-forms in the case of Toda-type systems[2]

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[1] S.D. King, F.W. Nijhoff, *Nucl. Phys.* **B947** (2019), 114686

[2] R. Boll, M. Petrera, Y.B. Suris, *J. Phys. A: Math. Theor.* **48** (2015), 085203

Discrete Quad Equations

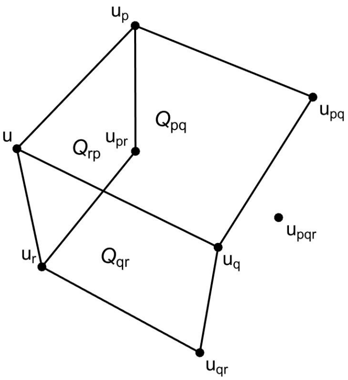
- I focus on (multicomponent) discrete quad equations, although the framework should generalise
- We consider a 4D square lattice
- Some discrete field $u(n_p, n_q, n_r, n_s)$ where $n_p, n_q, n_r, n_s \in \mathbb{Z}$
- p, q, r, s are the labels of complex lattice parameters associated with each direction
- Shift operator T_i

$$T_p u = u_p = u(n_p + 1, n_q, n_r, n_s)$$
$$T_q u = u_q = u(n_p, n_q + 1, n_r, n_s)$$

- Quad equations are partial difference equations $Q_{pq}(u, u_p, u_q, u_{pq}) = 0$
- Defined on the four corners of a square

Multidimensional Consistency (MDC)

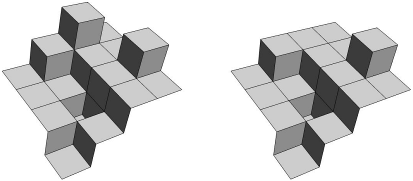
- Given Initial Data u, u_p, u_q, u_r
- One can then use three equations to calculate each of u_{pq}, u_{qr}, u_{pr}
- Then any method to calculate u_{pqr} must be consistent
- i.e. shifts commute $T_r T_p T_q = T_p T_q T_r = T_p T_q T_r$



$$Q_{pq}(u, u_p, u_q, u_{pq}) = 0$$
$$Q_{qr}(u, u_q, u_r, u_{qr}) = 0$$
$$Q_{rp}(u, u_r, u_p, u_{pr}) = 0$$
$$T_r Q_{pq} = Q_{pq}(u_r, u_{pr}, u_{qr}, u_{pqr}) = 0$$
$$T_p Q_{qr} = Q_{qr}(u_p, u_{pq}, u_{pr}, u_{pqr}) = 0$$
$$T_q Q_{rp} = Q_{rp}(u_q, u_{qr}, u_{pq}, u_{pqr}) = 0$$

The Multiform Structure of Discrete 2-Forms

- A quad surface $\sigma := \{\sigma_{ij}\}_{ij}$ in a N-dimensional lattice
- Each quad σ_{ij} is associated with a Lagrangian $\mathcal{L}_{ij}$ and a quad equation Q_{ij}



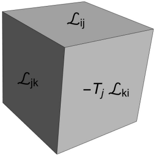
- An action summing discrete Lagrangians over a quad surface

$$A[u(n_p, n_q, n_r, n_s),\ \sigma] := \sum_{\sigma_{ij} \in \sigma} \mathcal{L}_{ij}$$

- Variational principle where action is minimised by u and unaffected by σ
- Encapsulates multidimensionally consistent (MDC) quad eqns

Definition for Discrete (Quad) 2-Form

- The 2-form structure can be viewed as behaviour of Lagrangians on cubes
- Let $\{i, j, k\} \in \{p, q, r, s\}$



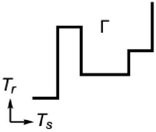
$$\mathcal{L}_{ij} : (u, u_i, u_j, u_{ij}; i, j) \mapsto \mathbb{C}$$
$$\text{Cube}_{ijk} := \mathcal{L}_{ij} - T_k\mathcal{L}_{ij} + \mathcal{L}_{jk} - T_i\mathcal{L}_{jk} + \mathcal{L}_{ki} - T_j\mathcal{L}_{ki}$$

- If u satisfies associated MDC Quad equations then cubic closure and Euler-Lagrange equations vanish

$$\begin{matrix} Q_{ij}(u, u_i, u_j, u_{ij}), Q_{jk}, Q_{ki} = 0 \\ T_k Q_{ij}, T_i Q_{jk}, T_j Q_{ki} = 0 \end{matrix} \implies \begin{matrix} \text{Cube}_{ijk} = 0 \\ \dfrac{\partial}{\partial\,\text{corner}}\text{Cube}_{ijk} = 0 \end{matrix}$$

Definition for Discrete 1-Form

- Ordinary difference equations $Q_j(u, u_j)$ with commuting flows
- Each edge Γ_j in the path has a corresponding Lagrangian $\mathcal{L}_j$
- The 1-form action acts on discrete paths Γ



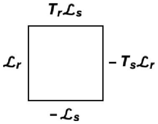
$$A[u(n_p, n_q, n_r, n_s),\ \Gamma] := \sum_{\Gamma_j \in \sigma} \mathcal{L}_j$$

- Path Independence about behaviour of Lagrangians on squares

$$\mathcal{L}_j : (u, u_j; j) \mapsto \mathbb{C}$$

$$\text{Square}_{rs} := \mathcal{L}_r - T_s\mathcal{L}_r - \mathcal{L}_s + T_r\mathcal{L}_s$$

$$\begin{aligned} Q_r(u, u_r; r), Q_s &= 0 \\ T_r Q_s, T_s Q_r &= 0 \end{aligned} \implies \begin{aligned} \text{Square}_{rs} &= 0 \\ \frac{\partial}{\partial\, \text{corner}} \text{Square}_{rs} &= 0 \end{aligned}$$



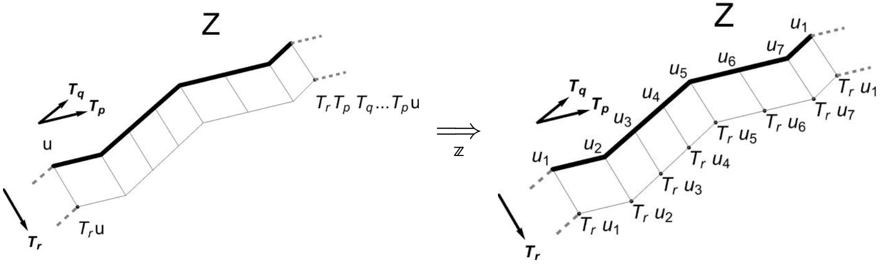
Periodic Reductions

- 2D periodic staircase Z with period N
- Along the staircase we have $T_1 \cdots T_N = \mathbb{1}$
- Example Staircase below

$$Z = \{u,\ T_p u,\ T_q T_p u,\ \cdots,\ (T_p T_q \cdots T_q)u\}$$

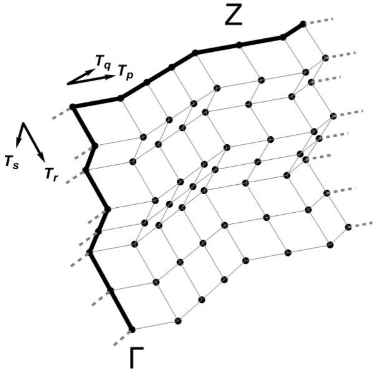
- Relabelling map $\mathbb{z}$ from (u) to staircase variables $(u_1, \cdots, u_N)$

$$\mathbb{z}\, Z = \{u_1, u_2, \cdots, u_N\} \quad \mathbb{z}\, T_r = T_r\, \mathbb{z} \quad \mathbb{z}\, T_s = T_s\, \mathbb{z}$$



Multiform Reduction Framework

- 4D quad surface $\sigma = \Gamma \, x \, Z$, represented in 3D
- Made from 2D Periodic Staircase Z
- Made from 2D 1-form path Γ



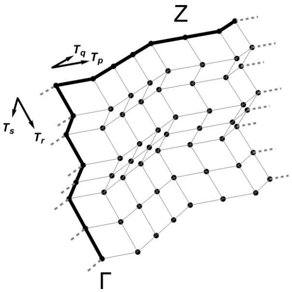
$$A_{\text{1-Form}}[u_1 \cdots u_N,\ \Gamma] =_{\mathbb{Z}} A_{\text{2-Form}}[u,\ \Gamma \times S]$$

Multiform Reduction Framework

- Edge in Γ associated with multicomponent 1-form Lagrangian
- Relabelled sums of N single-component 2-form Lagrangians

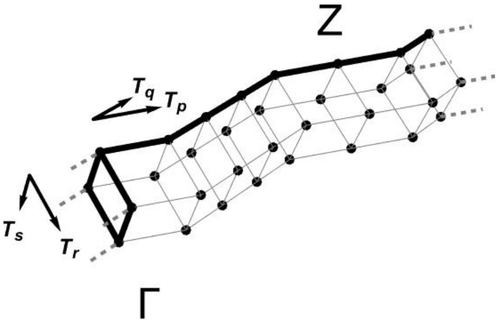
$$\mathcal{L}_r(u_1,\cdots,u_N,T_r u_1,\cdots,T_r u_N) := \mathbb{Z}(\underbrace{\mathcal{L}_{pr} + T_p\mathcal{L}_{qr} + T_q T_p\mathcal{L}_{qr} + \cdots}_{N \text{ Lagrangians}})$$

$$\mathcal{L}_s(u_1,\cdots,u_N,T_s u_1,\cdots,T_s u_N) := \mathbb{Z}(\mathcal{L}_{ps} + T_p\mathcal{L}_{qs} + T_q T_p\mathcal{L}_{qs} + \cdots)$$



Multiform Reduction Framework

- Square closure and Euler-Lagrange equations are relabelled sums of cubes
- End faces of cuboid equal and opposite sign



$$\text{Square}_{rs} = \mathbb{Z}\Big(\overbrace{\text{Cube}_{prs} + T_p\text{Cube}_{qrs} + T_q T_p\text{Cube}_{qrs} + \cdots}^{N \text{ cubes}}\Big)$$

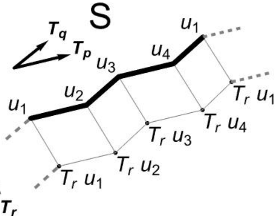
$$\frac{\partial}{\partial\,\text{corner}}\text{Square}_{rs} = \mathbb{Z}\,\frac{\partial}{\partial\,\text{corner}}\Big(\text{Cube}_{prs} + T_p\text{Cube}_{qrs} + T_q T_p\text{Cube}_{qrs} + \cdot\cdot$$

Start with pdKdV 2-Form

- Apply the periodic sequence to dpKdV 2-form

$$\mathcal{L}_{ij} = (T_i u - T_j u)u - (i^2 - j^2)\ln(T_i u - T_j u)$$

$$Q_{ij} = (u - u_{ij}) + \frac{j^2 - i^2}{u_i - u_j}$$



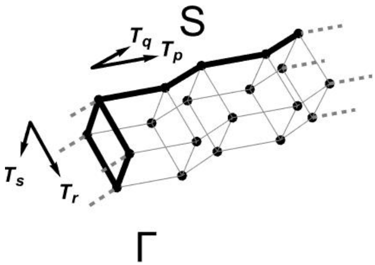
$$\begin{aligned}
\mathcal{L}_r := \; & (u_2 - T_r u_1)u_1 - (p^2 - r^2)\ln(u_2 - T_r u_1) && \mathbb{Z}\, \mathcal{L}_{pr} \\
& + (u_3 - T_r u_2)u_2 - (q^2 - r^2)\ln(u_3 - T_r u_2) && \mathbb{Z}\, T_p \mathcal{L}_{qr} \\
& + (u_4 - T_r u_3)u_3 - (p^2 - r^2)\ln(u_4 - T_r u_3) && \mathbb{Z}\, T_q T_p \mathcal{L}_{pr} \\
& + (u_1 - T_r u_4)u_4 - (q^2 - r^2)\ln(u_1 - T_r u_4) && \mathbb{Z}\, T_p T_q T_p \mathcal{L}_{qr}
\end{aligned}$$

Periodic Multiform Reduction of dpKdV

- The Lagrangians are larger but are multicomponent
- The single-component 2-form equations have similar form to the multicomponent 1-form equations

$$\frac{\partial}{\partial u_p}\mathrm{Cube}_{prs} = u_{pr} - u_{ps} + \frac{p^2 - r^2}{u_p - u_r} - \frac{p^2 - s^2}{u_p - u_s}$$

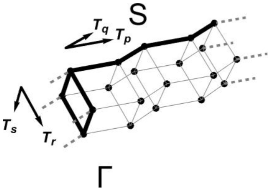
$$\frac{\partial}{\partial u_2}\mathrm{Square}_{rs} = T_r u_2 - T_s u_2 + \frac{p^2 - r^2}{u_2 - T_r u_1} - \frac{p^2 - s^2}{u_2 - T_s u_1}$$



Periodic Multiform Reduction of dpKdV

- Periodic reduction leads to 16 KdV-like equations on edges
- Implicit commuting maps $(u_1, u_2, u_3, u_4) \mapsto T_r(u_1, u_2, u_3, u_4)$ and $(u_1, u_2, u_3, u_4) \mapsto T_s(u_1, u_2, u_3, u_4)$

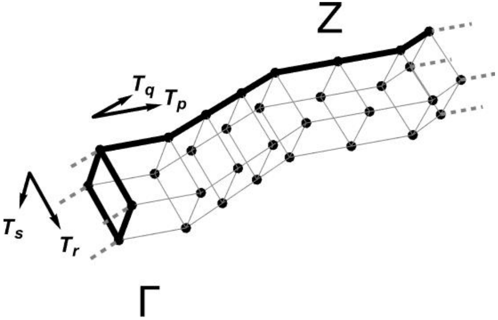
$$Q_r^{(1)}(u_1, u_2, T_r u_1, T_r u_2) := \mathbb{Z} Q_{pr}(u, u_p, u_r, u_{pr}; p, r) = 0$$
$$Q_r^{(2)}(u_2, u_3, T_r u_2, T_r u_3) := \mathbb{Z} T_p Q_{qr} = 0$$
$$Q_r^{(3)}(u_3, u_4, T_r u_3, T_r u_4) := \mathbb{Z} T_q T_p Q_{pr} = 0$$
$$Q_r^{(4)}(u_4, u_1, T_r u_4, T_r u_1) := \mathbb{Z} T_p T_q T_p Q_{qr} = 0$$
$$Q_s^{(1)}, Q_s^{(2)}, Q_s^{(3)}, Q_s^{(4)}, T_s Q_r^{(1)}, \cdots, T_r Q_s^{(4)} = 0$$



$$\Longrightarrow \quad \text{Square}_{rs} = 0, \qquad \frac{\partial}{\partial\,\text{corner}} \text{Square}_{rs} = 0$$

Multiform Reduction Framework

- Square closure and Euler-Lagrange equations are relabelled sums of cubes
- End faces of cuboid equal and opposite sign



$$\text{Square}_{rs} = \mathbb{Z}\,(\overbrace{\text{Cube}_{prs} + T_p\text{Cube}_{qrs} + T_q T_p\text{Cube}_{qrs} + \cdots}^{N \text{ terms}})$$

$$\frac{\partial}{\partial\,\text{corner}}\text{Square}_{rs} = \mathbb{Z}\,\frac{\partial}{\partial\,\text{corner}}(\text{Cube}_{prs} + T_p\text{Cube}_{qrs} + T_q T_p\text{Cube}_{qrs} + \cdots$$

Periodic Multiform Reduction of Gel'fand Dikii N=4

- Multicomponent Gel'fand Dikii N=4 quad 2-form
- 5 Quad equations with 5 components (u, v, w, x, y)
- Quad Lagrangian contains only 3 components (u, v, w)

$$\mathcal{L}_{ij} = \left(j^4 - i^4\right)\log(i - j + u_j - u_i) \\ + \text{polynomial}(u, u_i, u_j, u_{ij}, v_i, v_j, v_{ij}, w; i, j)$$

- We omit many details and just note dependencies for brevity
- Take $i, j, k \in \{p, q, r, s\}$ and let shifted equations also vanish

$$Q^{(1)}_{ij}(u, u_i, u_j, u_{ij}, v_{ij}, w, x_{ij}, y; i, j) = 0\,,$$
$$Q^{(2)}_{ij}(u_i, u_j, v_i, v_j, v_{ij}, x_i, x_j; i, j) = 0\,,$$
$$Q^{(3)}_{ij}(u_i, u_j, u_{ij}, v_i, v_j; i, j) = 0\,,$$
$$Q^{(4)}_{ij}(u_i, u_j, w, w_i, w_j, y_i, y_j; i, j) = 0\,,$$
$$Q^{(5)}_{ij}(u, u_i, u_j, w_i, w_j; i, j) = 0$$

$$\Longrightarrow \quad \text{Cube}_{ijk} = 0\,, \qquad \frac{\partial}{\partial\,\text{corner}}\text{Cube}_{ijk} = 0$$

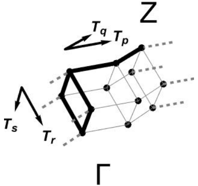
Periodic Multiform Reduction of Gel'fand Dikii N=4

- The periodic reduction leads to 10 Gel'fand Dikii-like equations on each edge of the square
- Implicit commuting maps T_r and T_s on $(u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2, y_1, y_2)$

$$\mathcal{L}_r := \mathbb{Z}(\mathcal{L}_{pr} + T_p\mathcal{L}_{qr}) \qquad \mathcal{L}_s := \mathbb{Z}(\mathcal{L}_{ps} + T_p\mathcal{L}_{qs})$$

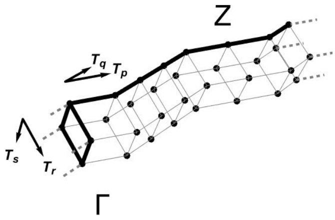
$$Q_r^{(1,1)}(u_1, u_2, T_r u_1, T_r u_2, T_r v_2, w_1, T_r x_2) := \mathbb{Z}Q_{pr}^{(1)}(u, u_p, u_r, u_{pr}, v_{pr}, w, x_{pr}, y; p, r) = 0$$

$$Q_r^{(1,2)}, \quad \cdots, \quad T_r Q_s^{(5,2)} = 0$$



$$\Longrightarrow \quad \text{Square}_{rs} = 0, \qquad \frac{\partial}{\partial\,\text{corner}}\text{Square}_{rs} = 0$$

Concluding Remarks



- Framework is natural and elevates periodic reductions onto the multiform level
- In future work strong form equations will arise from 1-form
- Variable transformations needed to avoid degenerate Poisson brackets
- This framework produces a wealth of novel 1-Form structures
- Combining the 2-form MDC structure and periodic staircase in 4D is a key insight
- Perpendicular flows are fundamental and can be reduced to simpler flows in plane of the staircase