Quantum Fourier Analysis

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Quantum Fourier Analysis is a subject that combines an algebraic Fourier transform (pictorial in the case of subfactor theory) with analytic estimates. This provides interesting tools to investigate phenomena such as quantum symmetry.

Jaffe-Jiang-L-Ren-Wu, PNAS 2020

Arthur Jaffe
Harvard

Chunlan Jiang
Hebei Normal University

Yunxiang Ren
Harvard

Jinsong Wu
Harbin Institute of Technology

Classical Fourier Duality

In the early 1800's, Joseph Fourier introduced his transformation to solve differential equations describing heat.
The Fourier transform $\mathcal{F}$ on measurable functions f on $\mathbb{R}$ is

$$\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi itx}dt \ .$$

Convolution for such functions is:

$$(f_1 * f_2)(s) = \int_{-\infty}^{\infty} f_1(t)f_2(s-t)dt \ ,$$

yielding the Fourier duality

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2) \ . \tag{1}$$

Inequalities on $\mathbb{R}$

Take

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt\right)^{1/p}, \ 0 < p < \infty.$$

For $p \geqslant 1$, $\|\cdot\|_p$ is the p-norm of measurable functions and $\|f\|_\infty$ is the essential maximum of f.

Plancherel formula (1910):

$$\|\mathcal{F}(f)\|_2 = \|f\|_2.$$

Interpolating with the elementary inequality $\|\mathcal{F}(f)\|_\infty \leqslant \|f\|_1$, one obtains the Hausdorff-Young inequality,

$$\|\mathcal{F}(f)\|_q \leqslant \|f\|_p \ , \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/q = 1 \ .$$

Young's inequality for convolution (1912):

$$\|f_1 * f_2\|_r \leqslant \|f_1\|_p \|f_2\|_q \ .$$

for $p, q, r \geqslant 1$, $1/p + 1/q - 1/r = 1$.

Inequalities on Finite Groups

From a finite group G, we have two C^* algebras:
$\mathcal{A} = L^\infty(G)$ with a discrete measure d.
$\mathcal{B} = \mathcal{L}(G)$ with a trace τ, $\tau(g) = \delta_{g,1}$.
The linear extension of the identity map on G induces a Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$. Under the Fourier duality,

- the multiplication on $\mathcal{B}$ induces the classical convolution $*$ on $\mathcal{A}$;
- the multiplication on $\mathcal{A}$ induces the "convolution" $*$ on $\mathcal{B}$, which is the Hadamard product of matrices:

$$(A * B)_{i,j} = A_{i,j} B_{i,j}.$$

Then for any $f, f_1, f_2 \in \mathcal{A}$: $\|\mathcal{F}(f)\|_2 = \|f\|_2$;

$$\|\mathcal{F}(f)\|_q \leqslant \|f\|_p \,, \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/q = 1 \,;$$
$$\|f_1 * f_2\|_r \leqslant \|f_1\|_p \|f_2\|_q \,, \quad p, q, r \geqslant 1, \quad 1/p + 1/q - 1/r = 1 \,.$$

It is true, but less obvious that the inequalities hold for operators in $\mathcal{B}$.

Fusion Rings

Irreducible representations of a finite group G forms a fusion ring under $\otimes$.

A fusion ring R is a ring which is free as a $\mathbb{Z}$-module, with a basis $\{x_1, x_2, \ldots, x_m\}, m \in \mathbb{N}$, with $x_1 = 1$, and such that

- $x_j x_k = \sum_{s=1}^{m} N_{j,k}^{s} x_s$, with $N_{j,k}^{s} \in \mathbb{N}$, and
- there exists an involution $*$ on $\{1, 2, \ldots, m\}$ such that $N_{j,k}^{1} = \delta_{j,k^*}$ inducing an anti-isomorphism of $\mathfrak{A}$, given by $x_k^* := x_{k^*}$ and $x_k^* x_j^* = (x_j x_k)^*$.

For a fusion ring R, let d be the Perron-Frobenius dimension,

$$d(x_j)d(x_k) = \sum_{s=1}^{m} N_{j,k}^{s} d(x_s).$$

Then $\{g_j = x_j/d_j : j = 1, 2, \cdots, m\}$ forms a *probability group* G.

$$g_j g_k = \sum_{s=1}^{m} p_{j,k}^{s} g_s.$$

Fusion Bialgebras

For a probability group, we similarly define $\mathcal{A} = L^\infty(G)$ with a discrete measure $d(g_j) = 1$. $\mathcal{B} = \mathcal{L}(G)$ with a trace τ, $\tau(g) = \delta_{g,1}$.
This induces a *fusion bialgebra* $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ of the fusion ring R:
C^*-algebra $\mathcal{A}$ has a basis $\{x_1, x_2, \ldots, x_m\}$ with multiplication $\diamond$, adjoint $\#$,

- $x_j \diamond x_k = \delta_{j,k}\, d(x_j)^{-1} x_j$,
- $x_j^\# = x_j$.

C^*-algebra $\mathcal{B} = \mathcal{L}(R) = \mathbb{C} \otimes_{\mathbb{Z}} R$ with a trace τ, $\tau(x_i) = \delta_{i,1}$.

The identity map on x_i induces a Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, and convolutions $*$ on $\mathcal{A}$ and $\mathcal{B}$.

(The convolution on $\mathcal{B}$ is no longer the Hadamard product in general.)

QFA on Fusion Bialgebras

Quantum Fourier Analysis (QFA) on fusion bialgebras are studied in L-Palcoux-Wu 2021:

Theorem (L-Palcoux-Wu 2021)

For any $f, f_1, f_2 \in \mathcal{A}$*:* $\|\mathcal{F}(f)\|_2 = \|f\|_2$;

$$\|\mathcal{F}(f)\|_q \leqslant \|f\|_p \,, \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/q = 1 \,;$$
$$\|f_1 * f_2\|_r \leqslant \|f_1\|_p \|f_2\|_q \,, \quad p, q, r \geqslant 1, \quad 1/p + 1/q - 1/r = 1 \,.$$

The quantum Hausdorff-Young inequality holds on $\mathcal{B}$. However, the quantum Young's inequality does NOT hold on $\mathcal{B}$.

Analytic Obstruction of Unitary Categorifications

Question: For which kind of fusion rings R, quantum Young's inequality holds on $\mathcal{B}$?

It holds for the fusion ring R of $Rep(G)$, when G is a finite group;

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Algebra: a quantum group; a weak C^*-Hopf algebras;

Functional Analysis: a subfactor $\mathcal{N} \subseteq \mathcal{M}$;

Topology: R has a unitary categorification, $\rightarrow$ 3-manifold invariant

i.e. R is the Grothendieck ring of a unitary fusion category $\mathscr{C}$.

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i.e. R is the Grothendieck ring of a unitary fusion category $\mathscr{C}$.

Therefore, the failure of quantum Young's inequality is an analytic obstruction of unitary categorification of fusion rings.

Besides the quantum Young's inequality, the *quantum Schur product theorem*, *quantum sumset estimates* etc provide such analytic obstructions of unitary categorification as well.

Quantum Schur Product Theorem

Theorem (Liu 2016, quantum Schur Product theorem)

If $x, y > 0$ in $\mathcal{A}$ of a subfactor, then their convolution $x * y > 0$.

Applying this theorem to the Drinfeld center of a unitary fusion category $\mathcal{C}$:

Theorem (L-Palcoux-Wu 2021, Schur Product Criterion)

Suppose a unitary fusion category $\mathcal{C}$ has a commutative Grothendieck ring with a character table $\Lambda = (\lambda_{i,j})$. Then

$$\sum_i \frac{\lambda_{i,j_1}\lambda_{i,j_2}\lambda_{i,j_3}}{\lambda_{i,1}} \geq 0, \ \forall\, j_1, j_2, j_3.$$

This inequality may not hold on a fusion ring, therefore it is an analytic obstruction unitary categorification of fusion rings.

The simple integral fusion ring of rank 7, Frobenius-Perron dim 210, type $[[1,1],[5,3],[6,1],[7,2]]$ and fusion matrices:

$$\begin{smallmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{smallmatrix},\ \begin{smallmatrix}0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&0&3\\0&1&1&1&1&3&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&3&1\\1&1&1&1&2&1&2\end{smallmatrix}$$

Its character table is the following:

$$\begin{bmatrix}1&1&1&1&1&1&1\\5&-1&-\zeta_7-\zeta_7^6&-\zeta_7^5-\zeta_7^2&-\zeta_7^4-\zeta_7^3&0&0\\5&-1&-\zeta_7^5-\zeta_7^2&-\zeta_7^4-\zeta_7^3&-\zeta_7-\zeta_7^6&0&0\\5&-1&-\zeta_7^4-\zeta_7^3&-\zeta_7-\zeta_7^6&-\zeta_7^5-\zeta_7^2&0&0\\6&0&-1&-1&-1&1&1\\7&1&0&0&0&0&-3\\7&1&0&0&0&-1&2\end{bmatrix}$$

Schur product Criterion: $\frac{1^3}{1}+\frac{0^3}{5}+\frac{0^3}{5}+\frac{0^3}{5}+\frac{1^3}{6}+\frac{(-3)^3}{7}+\frac{2^3}{7}=-\frac{65}{42}<0.$

L-Palcoux-Wu 21+: We find 34 simple integral fusion rings subject to

rank	≤ 5	6	7	8	9	10	all
$FPdim <$	1000000	150000	15000	4080	504	240	132

4 of which are group-like and 28 out of 30 can be eliminated by applying the Schur product criterion. ($\sim$ 93%!)

Subfactors

A factor is a von-Neumann algebra with trivial center. It is of type II_1, if it is infinite dimensional and has a trace τ.

Example: $\mathcal{R} := \bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$ with a trace $\tau(I) = 1$. (GNS-construction)
Remark: $\mathcal{R} = M_{[0,1]}(\mathbb{C})$.

A subfactor is an inclusion of (type II_1) factors $\mathcal{N} \subseteq \mathcal{M}(\text{“} = \mathcal{N} \rtimes G\text{”})$.
Jones index theorem (1983 Inv. Math.):

$$\{\dim_{\mathcal{N}} \mathcal{M} \text{“} = |G|\text{”}\} = \{4\cos^2 \frac{\pi}{n}, n = 3, 4, \cdots\} \cup [4, \infty].$$

From a subfactor, we obtain bimodules $X = {}_{\mathcal{N}}\mathcal{M}_{\mathcal{M}}$ and $\overline{X} = {}_{\mathcal{M}}\mathcal{M}_{\mathcal{N}}$ as well as two tracial C^*-algebras of bimodule maps
$\mathcal{A} = \hom(X \otimes \overline{X})$ and $\mathcal{B} = \hom(\overline{X} \otimes X)$, where $\otimes$ is Connes' fusion.

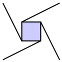
Example: For $\mathcal{R} \subseteq \mathcal{R} \rtimes G$, $\mathcal{A} = L^\infty(G)$, $\mathcal{B} = \mathcal{L}(G)$.

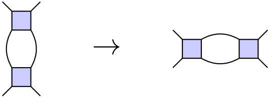
Pictorial Fourier Duality

2D pictorial interpretations in *subfactor planar algebras*: A morphism in $\mathcal{A}$ or $\mathcal{B}$ is represented by a square-like picture .
It intertwines the multiplication and the convolution as illustrated.

$$\mathfrak{F}_s(f_1 * f_2) = \mathfrak{F}_s(f_1)\mathfrak{F}_s(f_2)\ . \qquad (2)$$

String Fourier Transform   Multiplication   Convolution

 :

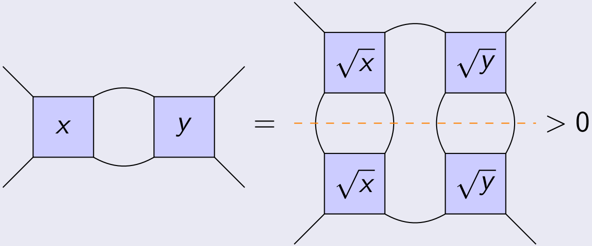


Proof of Quantum Schur Product Theorem

Theorem (L 2016, quantum Schur Product theorem)

If $x, y > 0$ *in* $\mathcal{A}$ *of a subfactor, then their convolution* $x * y > 0$.

Proof.

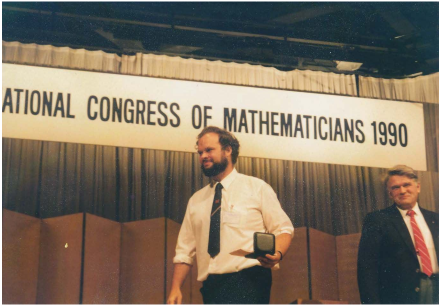


□

Subfactor Theory

Subfactor theory has wide connections in mathematics and physics: Operator Algebras, Quantum Groups, Representation theory, Knot Theory, Lower Dimensional Topology, Category Theory, Statistical Physics, Quantum Field Theory etc.

QFA adds an extra dimension to these connections.



Vaughan Jones won the Fields metal at the 1990 ICM at Kyoto.

Quantum Symmetries

Quantum Fourier Analysis (QFA) on Various Quantum Symmetries

Central type	Subfactors
Infinite type	Kac Algebras Locally Compact Quantum Groups
Topological Type	Subfactor Planar Algebras Subfactor Surface Algebras Unitary TQFTs
Categorical Type	Unitary Fusion Categories Unitary Modular Tensor Categories Unitary 2-Categories
Quantum Information Manybody System	Multiple Qubits/Quons Lattice Modes Tensor Networks

Quantum Fourier Analysis

In a series of papers joint with L. Huang, A. Jaffe, C. Jiang, S. Palcoux, J. Wu etc, we formalized and proved quantum analogues of

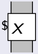
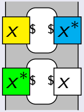
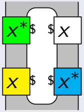
- Schur-product theorem
- Hausdorff-Young inequality
- Young's inequality
- Hirschman-Beckner uncertainty principle
- Donoho-Stark uncertainty principle
- Sum set estimate
- The characterization of operators which attain the equality of the above inequalities
- Hardy uncertainty principle
- Entropic uncertainty principle (von Neumann, Rényi, relative etc)
- ...

2D Central Limit Theorem

Jiang-L-Wu 2019, Sci. China Math.

Definition

For an irreducible subfactor $\mathcal{N} \subseteq \mathcal{M}(" = \mathcal{N} \rtimes G")$ with index μ, $x \in \mathcal{A}" = L^\infty(G)"$, we define the block map B_λ, $0 \leqslant \lambda \leqslant 1$,

$B_\lambda\Big($  $\Big) = \dfrac{\delta^2}{\|x\|_2^2}\Big(\dfrac{\lambda}{\|x\|_1}$  $+ \dfrac{(1-\lambda)}{\|x\|_\infty}$  $\Big).$

Theorem (2D central limit theorem)

For any $x \in \mathcal{A}$ of an irreducible subfactor,

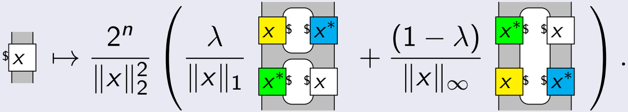
$$\lim_{n\to\infty} B_\lambda^n(x)$$

is a multiple of a biprojection, (constant on a "subgroup $H < G$".)

The 2D central limit theorem is proved for finite-index irreducible subfactors, which is new for $G = \mathbb{Z}_2$. We conjectured for $G = \mathbb{R}^n$:

Conjecture

For any $x \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, x converges to a multiple of a Gaussian function, under the action of the iteration of the block map



Remark: Gaussian Functions are fixed points.

Conjectures for $G = \mathbb{R}^n$

Conjecture

For any $f \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the Hirschman-Beckner entropy

$$\int_{\mathbb{R}} -|f|^2 \log |f|^2(x)dx + \int_{\mathbb{R}} -|\hat{f}|^2 \log |\hat{f}|^2(\xi)d\xi$$

decreases under the action of the block map.

Remark: Gaussian Functions are minimizers of the Hirschman-Beckner entropy.

The same question remains open for finite cyclic groups.

Applications to Subfactor Theory

QFA led to various characterizations of intermediate subfactors $N \subseteq Q \subseteq M$ (corresponding to biprojections), see Section 4 in [L 16].

Theorem (L 16)

*For a finite-index, irreducible subfactor $N \subseteq M$, if Q is an intermediate algebra, then Q is an intermediate subfactor, in particular Q is a *-algebra.*

Both finite-index and irreducible conditions are necessary.

Non-irreducible cases: $R \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \subset R \otimes \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset R \otimes \begin{bmatrix} * & * \\ * & * \end{bmatrix}$

Infinite index case: $\mathbb{Z}_+ \subset \mathbb{Z}$

Applications to Subfactor Theory

Let L be the lattice of intermediate subfactors of an irreducible subfactor with index μ, and $|L|$ be the cardinality.

- Watatani 96: $|L| < \infty$.
- Longo 03: $|L| < \mu^{2\mu^2}$. Conjecture: $|L| < \mu^\mu$.
- Kasheb-Das-L-Ren 19: $|L| < \min\{9^\mu, \mu^\mu\}$. (Applying QFA to L.)

Perspectives of QFA

Brascamp-Lieb inequalities

In 1976, Brascamp and Lieb proposed a fundamental inequality:
Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$, $1 \leqslant j \leqslant m$, be linear maps. Let f_j be a non-negative, measurable function on $\mathbb{R}^{n_j}$, and let $p_j > 0$ satisfy $\sum_{j=1}^{m} n_j/(p_j n) = 1$. Then

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \circ B_j \leqslant C \prod_{j=1}^{m} \|f_j\|_{p_j} \ , \tag{3}$$

This includes Young's inequality, Hölder's inequality, and the Loomis-Whitney inequality as special cases.
In 2008, Bennett, Carbery, Christ and Tao found the optimal constant C, which is obtained at certain Gaussian functions.

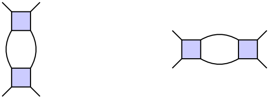
Example: Young's inequality

Take $n=2$, $B_1(x,y)=x$, $B_2(x,y)=y$, $B_3(x,y)=x+y$. Then $\displaystyle\sum_{j=1}^{3}\frac{1}{p_j}=2$.

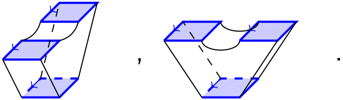
$$\begin{aligned}
&\int_{\mathbb{R}^n}\prod_{j=1}^{m} f_j\circ B_j\\
=&\int_{\mathbb{R}^2} f_1(x)f_2(y)f_3(x+y)dxdy\\
=&\int_{\mathbb{R}^2} f_1(x)f_2(t-x)f_3(t)dxdt\\
=&\int_{\mathbb{R}} f_1*f_2(t)f_3(t)dt\\
\leqslant& A_{p_1}A_{p_2}A_{p_3}\prod_{j=1}^{m}\|f_j\|_{p_j}\ .
\end{aligned}$$

2D to 3D Pictures

Recall that multiplication and convolution can be represented pictorially in planar algebras:



3D pictorial interpretations in *surface algebras* (Liu 2019 CMP):



Surface tangles $\longleftrightarrow$ 2+1D triangulated manifolds.

Topological Brascamp-Lieb inequality

Brascamp-Lieb inequality:

$$\|\prod_{j=1}^{m} f_j \circ B_j\|_1 \leqslant C \prod_{j=1}^{m} \|f_j\|_{p_j} \ , \tag{4}$$

(1) $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$, (2) $f_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$, (3) $p_j > 0$, $\sum_{j=1}^m n_j/(p_j n) = 1$.
Topological Brascamp-Lieb inequality:

$$tr\left(\left|\prod_{j=1}^{m} T_j(x_j)\right|^2\right) \leqslant C \prod_{j=1}^{m} \||x_j|^2\|_{p_j}, \tag{5}$$

where C is the best constant.
(1) $B_j^* \to$ a surface tangle T_j with k_j input discs and n output discs
(2) $\mathbb{R} \to \mathcal{A}$, $f_j \to |x_j|^2$ operators in $\mathcal{A}^{\otimes n_j}$ (Quantum Symmetry $\mathcal{A}$)

Topological identity for p_j's.

$$(3)\quad r_+ - m_- - \sum_{j=1}^{m_+} p_{j,+}^{-1} + \sum_{j=1}^{m_-} p_{k,-}^{-1} - 1 = 0\ ;$$

$$(3')\quad r_- - m_+ - \sum_{j=1}^{m_-} p_{k,-}^{-1} + \sum_{j=1}^{m_+} p_{j,+}^{-1} - 1 = 0\ .$$

for certain (genus-zero) surface tangle T, with r_+ unshaded regions, r_- shaded regions, m_+ unshaded inputs parameterized by $p_{j,+}$, $1 \leqslant j \leqslant m_+$, and m_- shaded inputs parameterized by $p_{k,-}$, $1 \leqslant k \leqslant m_-$,

$$(3) \quad r_+ - m_- - \sum_{j=1}^{m_+} p_{j,+}^{-1} + \sum_{j=1}^{m_-} p_{k,-}^{-1} - 1 = 0\ ;$$

$$(3') \quad r_- - m_+ - \sum_{j=1}^{m_-} p_{k,-}^{-1} + \sum_{j=1}^{m_+} p_{j,+}^{-1} - 1 = 0\ .$$

for certain (genus-zero) surface tangle T, with r_+ unshaded regions, r_- shaded regions, m_+ unshaded inputs parameterized by $p_{j,+}$, $1 \leqslant j \leqslant m_+$, and m_- shaded inputs parameterized by $p_{k,-}$, $1 \leqslant k \leqslant m_-$,

(3)+(3') is Euler's formula!

QFA leads to brand new connections between analysis and topology,
additional to the existing rich connections with algebras,
with potential applications in quantum informations.

Thank you!

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  and further references therein