Non semi-simple vertex tensor categories

V VOA.

Questions:

1.) Can one classify modules of a certain type?

2.) Are there vertex tensor categories of modules?

3.) More structure? Applications?

A. "Nice VOAs"

- Let V be a conical (lower bounded, $V_0 = \mathbb{C}|0\rangle$) VOA
- $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V)[[z^{\pm 1}]]$ fields $(v \in V)$
- $C_2(V) = \mathrm{span}_{\mathbb{C}}\{v_{-2}w \mid v, w \in V\}$
- M V-module $\underset{\underset{n}{\uparrow}}{C_2}(M) = \mathrm{span}_{\mathbb{C}}\{\underset{\underset{n}{\uparrow}}{v_{-2}} m \mid v \in V, m \in M\}$

M is C_2-cofinite if $M/C_2(M)$ is finite.

Theorem: V conical and C_2-cofinite and $V = V'$

1) V-mod is a vertex tensor category (Huang, 2009)

2) V-mod has finitely many inequivalent simple objects and every object is of finite Jordan-Hölder length (Huang, 2009; Miyamoto, 2004)

- M in V-mod $\Rightarrow$

$$M = \bigoplus_{n \in \mathbb{C}} M_n \quad , \quad M_n \text{ finite}$$

$$M_{n+N} = 0 \quad \text{for every } n \text{ and } N \gg 0.$$

$\Rightarrow$ M satisfies nice finiteness conditions.

- If every object in V-mod is in addition completely reducible then V is strongly rational and V-mod is a modular tensor category (Huang, 2008)
- Simple lower bounded modules are classified by Zhu's algebra, some assoc. algebra.

Applications:

1.) Strongly rational VOAs are chiral algebras of rational 2-dim conformal field theories, which in turn appear as world-sheet theories of strings.

2.) MTC's lead to TFT's, i.e. Chern-Simons theories

3.) C_2-cofinite VOAs are chiral algebras of "nice" logarithmic 2-dim CFTs.
Module categories of their tensor categories conjecturally describe categories of line operators in TFT's coupled to flat G-connections.
(see Nathan Geer's talk on related categories)

In general VOAs and their categories appear as meaningful invariants of interesting higher-dimensional manifolds. VOAs are usually not C_2-cofinite.

B. The moduli space of VOAs

$\mathfrak{g}$ Lie superalgebra

$B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant bilinear form

$V^B(\mathfrak{g})$ associated affine VOA

$L_B(\mathfrak{g})$ simple quotient

$f \in \mathfrak{g}_{even}$ nilpotent

$W^B(\mathfrak{g}, f)$ W-algebra (quantum Hamiltonian reduction)

Standard operations:

1.) V, W VOA $\Rightarrow$ $V \otimes W$ VOA

2.) $W \subset V$ VOA $\Rightarrow$ $\mathrm{Com}(W, V)$ VOA (coset)

3.) $G \subset \mathrm{Aut}(V)$ $\Rightarrow$ V^G VOA (orbifold)

4.) V VOA, A in V-mod, s.t. $V \subset A$ and A extends VOA structure on V.

Categorically: A is a comm. assoc. algebra in V-mod

I am not aware of any finitely generated VOA that is not obtained from an affine VOA or W-algebra via iterated standard operations. Maybe Haagerup VOA ??

Related categoricall question (MTC's)

Is there a Witt class that is not represented by an integrable affine VOA?

C. $\hat{sl}_2$

$\mathfrak{g} = sl_2 = \mathrm{span}_{\mathbb{C}}\{e, h, f\}$

$[h, e] = 2e \qquad [e, f] = h \qquad [h, f] = -2f$

$B(h, h) = 2 \qquad B(e, f) = 1$

Modules:

dim $< \infty$: L_r:  dim $L_r = r$

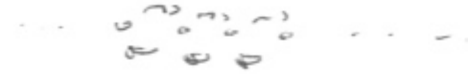
h.w. $= r-1$

dim $= \infty$:

a.) highest-weight:  D^+_λ, h.w $= \lambda$

b.) lowest-weight: 

D^-_λ, l.w. $= \lambda$

c.) relaxed-weight: $E_{\lambda, \Delta}$: 

Δ = Casimir eigenvalue

h-eigenvalues on $E_{\lambda,\Delta}$ are $\cancel{\lambda + A}$ $\lambda + 2\mathbb{Z}$

$L_k(sl_2) := L_{kB}(sl_2)$

$k = -2 + \frac{u}{v}$, $(u,v) = 1$, $v \in \mathbb{Z}_{\geq 1}$, $u \in \mathbb{Z}_{\geq 2}$

admissible levels.

Let $\hat{M}$ be the almost simple $V^k(sl_2)$ - module whose top level is M.

$$\Delta_{r,s} := \frac{(vr-us)^2 - v^2}{4uv}$$

$$\lambda_{r,s} := r - 1 - \frac{u}{v} s$$

<u>Theorem</u> (Adamovic - Milas, 1995 + recent)

A complete list of almost simple lower bounded $L_k(sl_2)$-modules is

a.) $\hat{L}_r$, $r = 1, \ldots, u-1$

b.) $\hat{D}^{\pm}_{\pm\lambda_{r,s}}$, $r = 1, \ldots, u-1$, $s = 1, \ldots, v-1$

c.) $\hat{E}_{\lambda, \Delta_{r,s}}$, $r = 1, \ldots, u-1$, $s = 1, \ldots, v-1$

$\lambda \neq \lambda_{r,s}, \lambda_{u-r,v-s}$

d.) $\hat{E}^{\pm}_{\lambda, \Delta_{r,s}}$ $\lambda \in \{\lambda_{r,s}, \lambda_{u-r,v-s}\}$

<u>where</u>, e.g. (Adamovic 2017, Ridout - Kawasetsu 2018)

$$0 \to \hat{D}^+_{r,s} \to \hat{E}^+_{r,s} \to (\sigma^{-1})^* (\hat{D}^+_{r,s-1}) \to 0$$

$$0 \to \hat{D}^+_{r,s} \to \sigma^*(\hat{E}^-_{u-r,v-s-1}) \to \sigma^*(\hat{D}^+_{r,s+1}) \to 0$$

spectral flow:

$\ell \in \mathbb{Z}$ (think of $\mathbb{Z}$ as the weight lattice of sl_2, better co-weight

$\sigma^\ell(e_n) = e_{n-\ell}$ $\sigma^\ell(h_n) = h_n - \delta_{n,0}\,\ell k$

$\sigma^\ell(f_n) = f_{n+\ell}$

here e_n, f_n, h_n generate $\hat{sl}_2$ at level k

$(\sigma^\ell)^*(\hat{M})$ spectrally flow twist of $\hat{M}$, i.e.

$X.(\sigma^\ell)^*(m) := (\sigma^\ell)^*(\sigma^{-\ell}(X).m)$ for $m \in \hat{M}$.

and $X \in \mathcal{U}(\hat{sl_2})$

Logarithmic modules $\hat{P}^+_{r,s}$ (Adamovic, 2017)

Loewy diagram

$$\begin{tikzcd}
 & D^+_{r,s} \arrow[dl, dash] \arrow[dr, dash] & \\
(\sigma^{-1})^*(D^+_{r,s-1}) \arrow[dr, dash] & & (\sigma^*)(D^+_{r,s+1}) \arrow[dl, dash] \\
 & D^+_{r,s} &
\end{tikzcd}$$

$s = 2, \dots, v-2$

($s \in \{1, v-1\}$ is similar)

Theorem (Arakawa - C. Kawasetsu)

a.) $P^+_{r,s}$ is projective and injective and a projective cover and injective hull of $D^+_{r,s}$ in the category of smooth weight modules of $L_k(sl_2)$.

b.) A complete list of projective and injective modules is

$(\sigma^\ell)^*(\mathcal{E}_{\lambda,\Delta_{r,s}})$, $(\sigma^\ell)^*(P^+_{r,s})$ $r = 1, \dots, u-1$

$\lambda \neq \lambda_{r,s}, \lambda_{u-r,v-s}$ $s = 1, \dots, v-1$

$\lambda \in [0,2)$

D. Categories

$\mathfrak{g}$ Lie superalgebra with $B = \kappa$ Killing form, s.t. long roots have norm 2. $L_k(\mathfrak{g}) := L_{kB}(\mathfrak{g})$.

A module is then a $\hat{\mathfrak{g}}$-module at level k that is also an $L_k(\mathfrak{g})$-module.

1.) $\mathcal{C}_k(\mathfrak{g})$ smooth weight modules

M smooth, $M = \bigoplus_{\lambda, \Delta} M_{\lambda, \Delta}$ and $M_{\lambda, \Delta + N} = 0$ for $N \gg 0$

λ $\mathfrak{g}$-weight, Δ conformal weight

2.) $\mathcal{C}_k^{fin}(\mathfrak{g}) \subset \mathcal{C}_k(\mathfrak{g})$, $\dim M_{\lambda, \Delta} < \infty$ $\forall (\lambda, \Delta)$

3.) $\mathcal{R}_k(\mathfrak{g}) \subset \mathcal{C}_k(\mathfrak{g})$, M lower-bounded

4.) $\mathcal{R}_k^{fin}(\mathfrak{g}) = \mathcal{R}_k(\mathfrak{g}) \cap \mathcal{C}_k^{fin}(\mathfrak{g})$

5.) $\mathcal{O}_k(\mathfrak{g}) \subset \mathcal{R}_k^{fin}(\mathfrak{g})$, $\bigoplus_{\lambda} M_{\lambda, \Delta}$ finite dimensional $\forall \Delta$

Classification for general $\mathfrak{g}$ ($\mathfrak{g}$ Lie algebra)

- simples in $\mathcal{R}_k^{fin}(\mathfrak{g})$ (Kawasetsu - Ridout)
- Any simple in $\mathcal{C}_k^{fin}(\mathfrak{g})$ is a spectral flow twist of a simple in $\mathcal{R}_k^{fin}(\mathfrak{g})$ (Arakawa - C - Kawasetsu)

  Also efficient criterion on vanishing of Ext^1

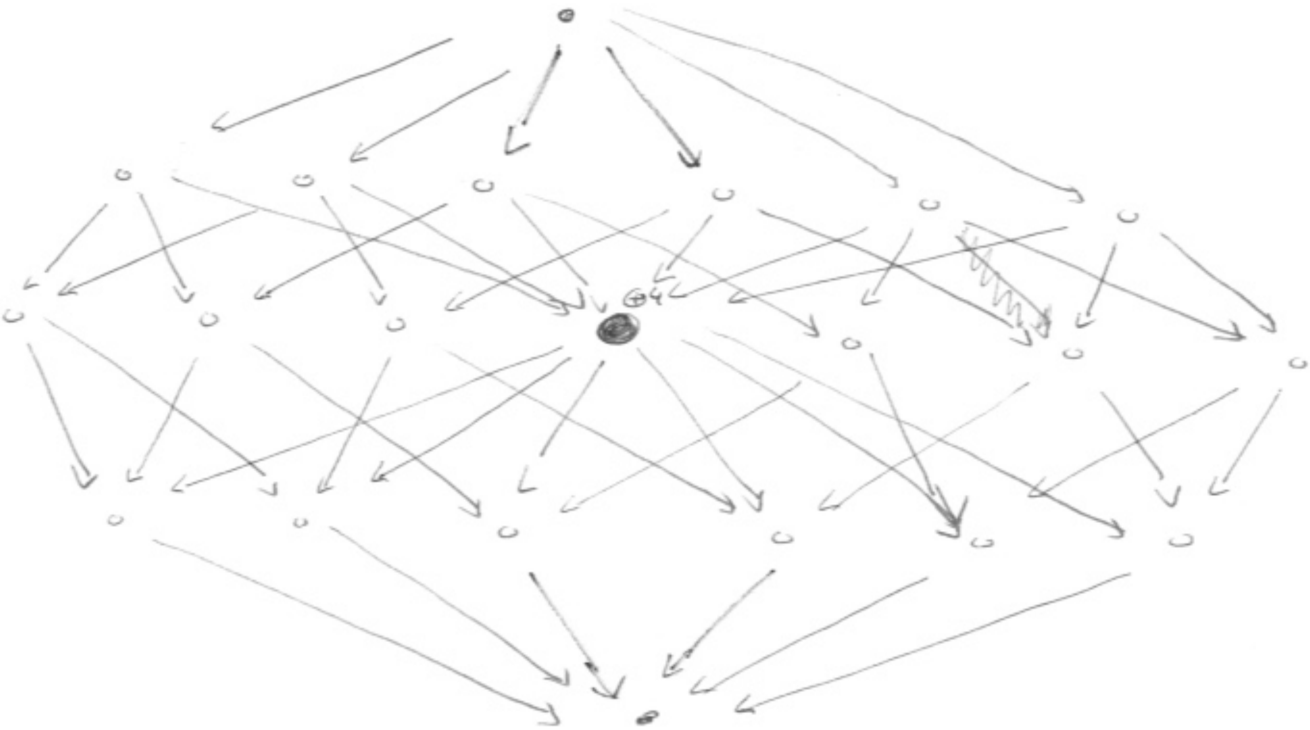
Big problem: construct logarithmic modules.

Adamovic idea: Embed $L_k(g)$ in larger structure and use screening charges to construct logarithmic modules.

Progress for $g = sl_3$ (Adamovic - C - Genra)

Conjecture: (C-Ridout-Rupert)

$L_{-\frac{3}{2}}(sl_3)$ principal block projective



E. Vertex Tensor Category

Theorem (C-Yang, 2020)

V VOA, any simple ordinary module C_1-cofinite and all grading restricted generalized Verma modules of finite length, then the category of generalized modules is a vertex tensor category.

$\Rightarrow$ $\mathcal{O}_k(g)$ is a vertex tensor category for

a.) k irrational and g_{even} semi-simple

b.) k admissible and g Lie algebra or $g =$ (C-Huang-Yang) $osp(1|2n)$

c.) $k \neq 0$ and $g = gl(1|1)$ (C-McRae-Yang)

d.) various non-admissible examples

$\mathcal{O}_k(g)$ is rigid for

a.) k admissible and $g \in ADE$ or $osp(1|2n)$

b.) $g = gl(1|1)$, $k \neq 0$.

$R_k^{fin}(g)$ will not be closed under tensor product

$C_k^{fin}(g)$ is expected to be a rigid vertex tensor category

$L_{-\frac{1}{2}}(sl_2)$, $L_{-\frac{4}{3}}(sl_2)$:

Are extensions of singlet VOAs times Heisenberg VOAs.
Vertex tensor category for singlet + hopefully rigidity will appear (C - Kanade - McRae - Yang)

$\Rightarrow$ same holds for $L_{-\frac{1}{2}}(sl_2)$, $L_{-\frac{4}{3}}(sl_2)$ via VOA-extension theory.

In general we need new technology (see Shigenori Nakatsuka's talk)