Wilf-Zeilberger Theory and Its Applications

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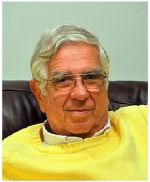
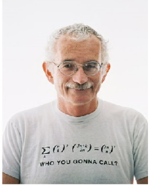


Herbert Wilf



Doron Zeilberger

Telescoping

Problem. For a sequence $f(k)$ in some class $\mathfrak{S}(k)$, decide whether there exists $g(k) \in \mathfrak{S}(k)$ s.t.

$$f(k) = g(k+1) - g(k)$$

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$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Delta_k\left(-\frac{1}{k}\right) = 1 - \frac{1}{n+1}$$

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- Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k\left(\frac{4k\binom{2k}{k}^2}{4^{2k}}\right) = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2n+2}}$$

Creative telescoping

Problem. For a sequence $f(n,k)$ in some class $\mathfrak{S}(n,k)$, find a linear recurrence operator $L \in \mathbb{F}[n, S_n]$ and $g \in \mathfrak{S}(n,k)$ s.t.

$$\underbrace{L(n,S_n)}_{\text{Telescoper}}(f) = \Delta_k(g)$$

Call g the certificate for L.

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Example. Let $f(n,k) = \binom{n}{k}^2$. Then a telescoper for f and its certificate are

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

Proving identities

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Since $f(n,k) = 0$ when $k < 0$ or $k > n$, we have

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2$$

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Taking sums on both sides of $L(f) = \Delta_k(g)$:

$$\sum_{k=-\infty}^{+\infty} L(f) = L\left(\sum_{k=-\infty}^{+\infty} f\right) = g(n,+\infty) - g(n,-\infty) = 0$$

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The sequence $F(n)$ satisfies

$$(n+1)F(n+1) - (4n+2)F(n) = 0$$

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Verify the initial condition:

$$F(1) = 2 = \binom{2}{1}$$

Then the identity is proved!

Example: Dixon's Identity

$$\sum_{k=-a}^{a} \underbrace{(-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}}_{F(b,k)} = \underbrace{\frac{(a+b+c)!}{a!b!c!}}_{f(b)}$$

1 Creative telescoping for $F(b,k)$ yields $L(b,S_b)(F) = \Delta_k(G)$ with

$$L = (-b-1)S_b + (a+b+c+1) \quad \text{and} \quad G = \frac{(a+k)(c+k)}{2(b-k+1)} \cdot F.$$

2 Summing both sides of $L(F) = \Delta_k(G)$ for k from $-a$ to a gets

$$\sum_{k=-a}^{a} L(F) = L\left(\sum_{k=1}^{n} F\right) = \sum_{k=1}^{n} \Delta_k(G)$$

$$= G(b,a+1) - G(b,-a) = 0 \quad \Rightarrow \quad L\left(\sum_{k=-a}^{a} F\right) = 0.$$

3 Note that $L(f(b)) = 0$ and the identity holds for $b = 0$.

Example: Identity about Harmonic Numbers

$$\sum_{k=1}^{n} \underbrace{(-1)^{k+1}\frac{1}{k}\binom{n}{k}}_{F(n,k)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \triangleq H_n.$$

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$$L = S_n - 1 \quad \text{and} \quad G = \frac{(-1)^k}{n+1}\binom{n}{k-1}.$$

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$$\sum_{k=1}^{n} L(F) = L\left(\sum_{k=1}^{n} F\right) - F(n+1,n+1) = \sum_{k=1}^{n} \Delta_k(G)$$

$$= G(n,n+1) - G(n,1) \quad \Rightarrow \quad L\left(\sum_{k=1}^{n} F\right) = \frac{1}{n+1}$$

An Identity from Representation Theory

$$\sum_{m=0}^{\min\{a,b\}} \sum_{n=0}^{\min\{c,d\}} F(m,n,a,b,c,d,t) = (-1)^t \frac{(a+c)!(a+d)!(b+c)!(b+d)!}{a!(b-t)!c!(d+t)!},$$

where

$$F = (-1)^{m+n}(m+n)!(a+b+c+d-m-n)!\binom{a-t}{m-t}\binom{b}{m}\binom{c+t}{n+t}\binom{d}{n}.$$

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Initial case: $t=0$

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Recurrence in t: creative telescoping for double summation

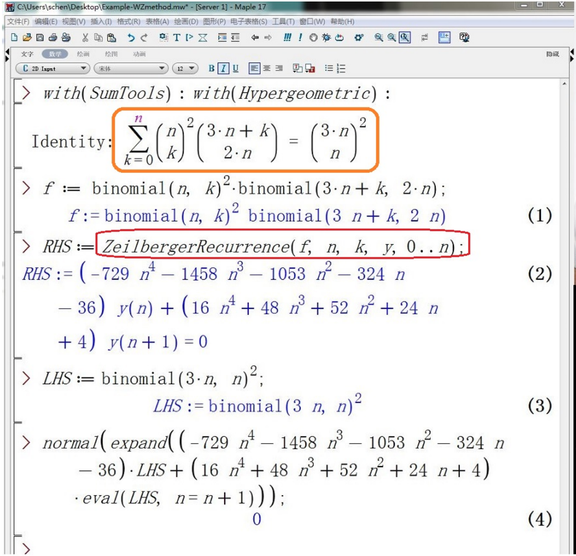
$$L(t,S_t)(F) = \Delta_m(G) + \Delta_n(H),$$

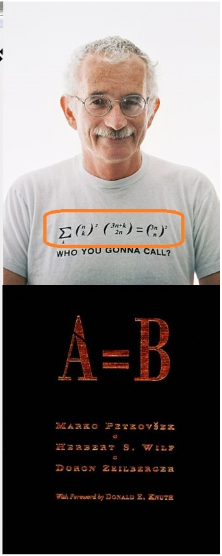
where $L = (d+t+1)S_t + (b-t)$ and $G = gF, \quad H = hF$ with

$$g = \frac{-m-am-bm-cm-dm-cdm+m^2-amn-bmn+m^2n}{(1+n)(a-t)}$$

$$h = \frac{-abn+amn+bmn-m^2n-t-at-bt-abt-ct-dt-cmt-dmt+nt+mnt+t^2+at^2+bt^2+ct^2+dt^2-mt^2-nt^2}{(a-t)(-1-m+t)}$$

Example: Identity on T-shirt





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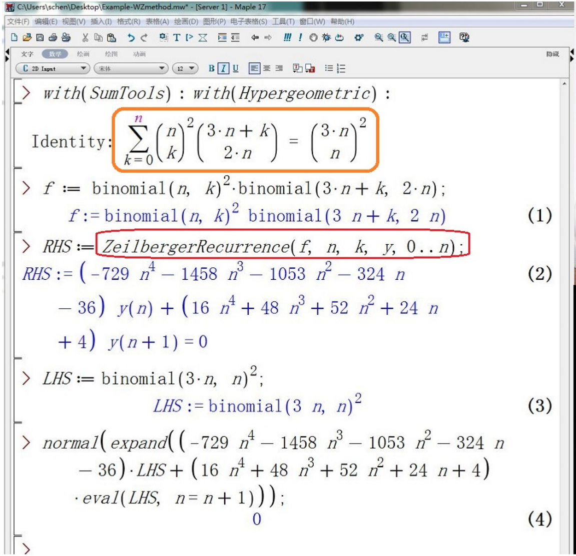

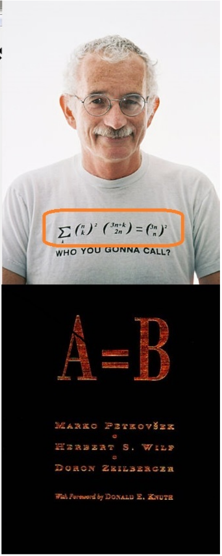

Handbooks of identities

Dixon's identity

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Hille-Hardy's identity

$$\sum_{n=0}^{\infty} \sum_{k_1} \sum_{k_2} \frac{u^n n!}{(a+1)_n} \binom{n+a}{n-k_1} \frac{(-x)^{k_1}}{k_1!} \binom{n+a}{n-k_2} \frac{(-y)^{k_2}}{k_2!}$$

$$= (1-u)^{-a-1} \exp\left\{ -\frac{(x+y)^u}{1-u} \right\} \sum_n \frac{1}{n!(a+1)_n} \left(\frac{xyu}{(1-u)^2} \right)^n$$

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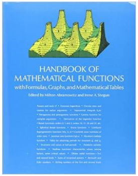

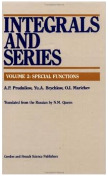

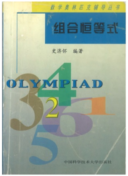

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Solving conjectures in combinatorics

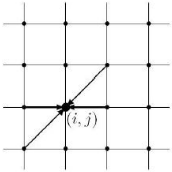
PNAS 2009

Proof of Ira Gessel's lattice path conjecture

Manuel Kauers[a], Christoph Koutschan[a], and Doron Zeilberger[b,1]

Theorem. *Let $f(n;i,j)$ denote the number of Gessel walks going in n steps from $(0,0)$ to (i,j). Then $f(n;0,0)=0$ if n is odd and*

$$f(2n;0,0)=16^n\frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}\quad (n\geq 0),$$



PNAS 2011

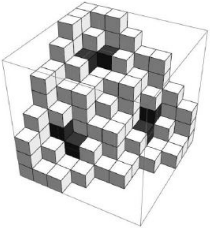
Proof of George Andrews's and David Robbins's *q*-TSPP conjecture

Christoph Koutschan[a,1], Manuel Kauers[b,2], and Doron Zeilberger[c]

Theorem 1. *Let π/S_3 denote the set of orbits of a totally symmetric plane partition π under the action of the symmetric group S_3. Then the orbit-counting generating function (ref. 3, p. 200, and ref. 2, p. 106) is given by*

$$\sum_{\pi\in T(n)} q^{|\pi/S_3|}=\prod_{1\leq i\leq j\leq k\leq n}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n.



7

Fundamental problems

Creative telescoping

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$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,y)) = D_y(g(x,y))$$

Fundamental problems

Creative telescoping

$$\int_{-\infty}^{+\infty} \exp(-x^2/y^2 - y^2)dy = \sqrt{\pi}\exp(-2x)$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,k)) = \Delta_k(g(x,k))$$

Fundamental problems

Creative telescoping

$$\sum_{k=0}^{+\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x, \partial_x)}_{\text{Telescoper}} (f(x, y_1, \ldots, y_m)) = \sum_{i=1}^{m} \partial_{y_i} (g_i(x, y_1, \ldots, y_m))$$

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Existence problem.

For a function $f(n,k)$, decide whether telescopers exist?

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For a function $f(n,k)$, how to computer a telescoper if it exists?

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Tools:

- Algebraic analysis (holonomic D-modules)
- Differential and difference algebra
- Non-commutative rings (Ore polynomials)
- Computational algebraic geometry
- ...

Existence of telescopers

Timeline of works on existence problem

Existence of telescopers

Timeline of works on existence problem

1990: Zeilberger proved that telescopers always exist for holonomic functions:

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence of telescopers

Timeline of works on existence problem

1992: Wilf and Zeilberger proved that telescopers always exist for proper hypergeometric terms:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence of telescopers

Timeline of works on existence problem

2002: Abramov and Le solved the existence problem for rational functions in two discrete variables:

N·H
ELSEVIER
Discrete Mathematics 259 (2002) 1–17
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A criterion for the applicability of Zeilberger's algorithm to rational functions ☆

S.A. Abramov[a], H.Q. Le[b,*]

Existence of telescopers

Timeline of works on existence problem

2003: Abramov solved the existence problem for bivariate hypergeometric terms:

ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

Advances in Applied Mathematics 30 (2003) 424–441

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/aam

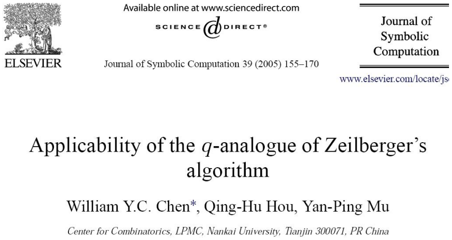
When does Zeilberger's algorithm succeed?

S.A. Abramov [1]

Existence of telescopers

Timeline of works on existence problem

2005: W.Y.C. Chen, Hou and Mu solved the existence problem for bivariate q-hypergeometric terms:


ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Symbolic Computation 39 (2005) 155–170
Journal of Symbolic Computation
www.elsevier.com/locate/js

Applicability of the q-analogue of Zeilberger's algorithm

William Y.C. Chen*, Qing-Hu Hou, Yan-Ping Mu
Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, PR China

Existence of telescopers

Timeline of works on existence problem

2012: S. Chen and Singer solved the existence problem for bivariate rational functions in the mixed cases:

Advances in Applied Mathematics 49 (2012) 111–133

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Residues and telescopers for bivariate rational functions ☆

Shaoshi Chen, Michael F. Singer *

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205, USA

Existence of telescopers

Timeline of works on existence problem

2015: Chen et al. solved the existence problem for bivariate mixed hypergeometric terms:

Journal of Symbolic Computation 68 (2015) 1–26

ELSEVIER

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

On the existence of telescopers for mixed hypergeometric terms ☆

CrossMark

Shaoshi Chen [a], Frédéric Chyzak [b], Ruyong Feng [a], Guofeng Fu [a], Ziming Li [a]

Existence of telescopers

Timeline of works on existence problem

2016: Chen et al. solved the existence problem for rational functions in three discrete variables:

Existence Problem of Telescopers: Beyond the Bivariate Case *

Shaoshi Chen[1,2], Qing-Hu Hou[3], George Labahn[2], Rong-Hua Wang[4]

Existence of telescopers

Timeline of works on existence problem

2020: Chen et al. solved the existence problem for rational functions in three variables:


Journal of Symbolic Computation
Available online 20 August 2020
In Press, Corrected Proof

On the existence of telescopers for rational functions in three variables

Shaoshi Chen [a, b], Lixin Du [a, b, c], Rong-Hua Wang [d], Chaochao Zhu [a, b]

Mixed hypergeometric terms

Let $\mathbb{F}$ be a field of char. zero and algebraically closed.

$$\mathbf{t} = (t_1, \ldots, t_m), \qquad \mathbf{x} = (x_1, \ldots, x_n)$$

$$D_i : \underbrace{\partial/\partial t_i}_{\text{derivations}}, \qquad S_j : \underbrace{x_j \to x_j + 1}_{\text{shifts}}$$

Definition. $h(\mathbf{t}, \mathbf{x})$ is mixed hypergeometric over $\mathbb{F}(\mathbf{t}, \mathbf{x})$ if

all $\dfrac{D_i(h)}{h}$ and $\dfrac{S_j(h)}{h}$ are rational functions in $\mathbb{F}(\mathbf{t}, \mathbf{x})$.

Remark. Mixed hypergeometric terms are solutions of systems of first-order homogeneous differential and difference equations.

Examples

- Rational functions:

$$t_1+t_2+x_1, \quad \frac{1}{(t_1+t_2)}, \quad \frac{t_1+x_1+1}{t_1+t_2+x_1^2+3}, \quad \ldots$$

Examples

- Rational functions:

$$t_1 + t_2 + x_1, \quad \frac{1}{(t_1 + t_2)}, \quad \frac{t_1 + x_1 + 1}{t_1 + t_2 + x_1^2 + 3}, \quad \ldots$$

- Hyperexponential functions:

$$\exp(t_1 + t_2^2), \quad (t_1^2 + t_2 + 1)^{\sqrt{5}}, \quad \exp\left(\int \frac{1}{t_1 + t_2}\right), \quad \ldots$$

Examples

- Rational functions:

$$t_1 + t_2 + x_1, \quad \frac{1}{(t_1 + t_2)}, \quad \frac{t_1 + x_1 + 1}{t_1 + t_2 + x_1^2 + 3}, \quad \ldots$$

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- Symbolic powers:

$$t_1^{x_1}, \quad (t_1 + t_2)^{x_1} \cdot (t_2 + t_3^2)^{x_2}, \quad \ldots$$

Examples

- Rational functions:

$$t_1 + t_2 + x_1, \quad \frac{1}{(t_1 + t_2)}, \quad \frac{t_1 + x_1 + 1}{t_1 + t_2 + x_1^2 + 3}, \quad \ldots$$

- Hyperexponential functions:

$$\exp(t_1 + t_2^2), \quad (t_1^2 + t_2 + 1)^{\sqrt{5}}, \quad \exp\left(\int \frac{1}{t_1 + t_2}\right), \quad \ldots$$

- Symbolic powers:

$$t_1^{x_1}, \quad (t_1 + t_2)^{x_1} \cdot (t_2 + t_3^2)^{x_2}, \quad \ldots$$

- Hypergeometric terms:

$$2^{x_1}, \quad x_1!, \quad (x_1 + 2x_2 + \sqrt{3})!, \quad \ldots$$

Structure theorem

Theorem. Any mixed hypergeometric term $h(\mathbf{t},\mathbf{x})$ is of the form

$$f(\mathbf{t},\mathbf{x}) \cdot \prod_{j=1}^{n} \beta_j(\mathbf{t})^{x_j} \cdot \exp(g_0(\mathbf{t})) \cdot \prod_{\ell=1}^{L} g_\ell(\mathbf{t})^{c_\ell} \cdot \prod_{\lambda} (\mathbf{v}_\lambda \cdot \mathbf{x} + p_\lambda)!^{e_\lambda}$$

where f is a rational function in $\mathbb{F}(\mathbf{t},\mathbf{x})$.

Structure theorem

Theorem. Any mixed hypergeometric term $h(\mathbf{t}, \mathbf{x})$ is of the form

$$f(\mathbf{t}, \mathbf{x}) \cdot \prod_{j=1}^{n} \beta_j(\mathbf{t})^{x_j} \cdot \exp(g_0(\mathbf{t})) \cdot \prod_{\ell=1}^{L} g_\ell(\mathbf{t})^{c_\ell} \cdot \prod_{\lambda} (\mathbf{v}_\lambda \cdot \mathbf{x} + p_\lambda)!^{e_\lambda}$$

where f is a rational function in $\mathbb{F}(\mathbf{t}, \mathbf{x})$.

Proper terms. A mixed hypergeometric term $h(\mathbf{t}, \mathbf{x})$ is proper if it is of the form

$$P(\mathbf{t}, \mathbf{x}) \cdot \prod_{j=1}^{n} \beta_j(\mathbf{t})^{x_j} \cdot \exp(g_0(\mathbf{t})) \cdot \prod_{\ell=1}^{L} g_\ell(\mathbf{t})^{c_\ell} \cdot \prod_{\lambda} (\mathbf{v}_\lambda \cdot \mathbf{x} + p_\lambda)!^{e_\lambda}$$

where P is a polynomial in $\mathbb{F}[\mathbf{t}, \mathbf{x}]$.

Holonomic terms

Let $H(\mathbf{z})$ be a function of continuous variables $\mathbf{z} = (z_1, \ldots, z_s)$.
Notation: $\mathscr{A}_s := \mathbb{F}[z_1, \ldots, z_s]\langle D_{z_1}, \ldots, D_{z_s}\rangle$, and

$$\mathrm{ann}_{\mathscr{A}_s}(H(\mathbf{z})) := \{L \in \mathscr{A}_s \mid L(H) = 0\}.$$

Holonomic terms

Let $H(\mathbf{z})$ be a function of continuous variables $\mathbf{z} = (z_1, \ldots, z_s)$.
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$$\mathrm{ann}_{\mathscr{A}_s}(H(\mathbf{z})) := \{L \in \mathscr{A}_s \mid L(H) = 0\}.$$

Definition.

- $H(\mathbf{z})$ is holonomic if the Hilbert dimension of $\mathrm{ann}_{\mathscr{A}_s}(H(\mathbf{z}))$ as a left ideal of $\mathscr{A}_s$ is s.
- A function $h(\mathbf{t}, \mathbf{x})$ is holonomic if the generating function

$$H(\mathbf{t}, \mathbf{z}) = \sum_{x_1, \ldots, x_n \geq 0} h(\mathbf{t}, \mathbf{x}) z_1^{x_1} \cdots z_n^{x_n}$$

is holonomic over $\mathscr{A}_{m+n} := \mathbb{F}(\mathbf{t}, \mathbf{z})\langle D_{t_1}, \ldots, D_{t_m}, D_{z_1}, \ldots, D_{z_n}\rangle$.

Holonomic terms

Let $H(\mathbf{z})$ be a function of continuous variables $\mathbf{z} = (z_1, \ldots, z_s)$.
Notation: $\mathscr{A}_s := \mathbb{F}[z_1, \ldots, z_s]\langle D_{z_1}, \ldots, D_{z_s}\rangle$, and

$$\text{ann}_{\mathscr{A}_s}(H(\mathbf{z})) := \{L \in \mathscr{A}_s \mid L(H) = 0\}.$$

Definition.

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$$H(\mathbf{t}, \mathbf{z}) = \sum_{x_1, \ldots, x_n \geq 0} h(\mathbf{t}, \mathbf{x}) z_1^{x_1} \cdots z_n^{x_n}$$

  is holonomic over $\mathscr{A}_{m+n} := \mathbb{F}(\mathbf{t}, \mathbf{z})\langle D_{t_1}, \ldots, D_{t_m}, D_{z_1}, \ldots, D_{z_n}\rangle$.

Remark. No algorithm for verifying holonomicity:-(

Wilf–Zeilberger conjecture: Holonomic ⇔ Proper

In the fundamental paper by Wilf and Zeilberger:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
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Wilf–Zeilberger conjecture: Holonomic ⇔ Proper

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Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

In Page 585, they said:

Our examples are all proper-hypergeometric. We conjecture that a hypergeometric term is proper-hypergeometric if and only if it is holonomic.

Wilf–Zeilberger conjecture: Holonomic ⇔ Proper

In the fundamental paper by Wilf and Zeilberger:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Chen and Koutschan recently proved the conjecture:

Proof of the Wilf–Zeilberger Conjecture for Mixed Hypergeometric Terms

Shaoshi Chen[a,b], Christoph Koutschan[c]

Construction of telescopers

Four approaches:

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
Algebraic-Geometry Approach	Elimination-Based Approach	Gosper-Based Approach	Redution-Based Approach
- Picard 1902	- Fasenmyer 1947	- Zeilberger 1990	- BostanChenChyzakLi 2010
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....	....	....	- ChenKaKoutschan 2016

1902: Picard proved the existence of Picard-Fuchs equations for parameterized integrals of algebraic functions:

ÉMILE PICARD

Sur les périodes des intégrales doubles dans la théorie des fonctions algébriques de deux variables

Annales scientifiques de l'É.N.S. 3e série, tome 19 (1902), p. 65-73.

Construction of telescopers

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....	....	....	- ChenKaKoutschan 2016

1958: Manin gave a constructive method for finding Picard-Fuchs equations:

ALGEBRAIC CURVES OVER FIELDS WITH DIFFERENTIATION

Ju. I. MANIN

A differential-algebraic homomorphism is constructed from the group of divisor classes of degree zero on a curve defined over a constant field with differentiation into the additive group of a finite-dimensional vector space over the constant field. A partial study of the kernel of this homomorphism is made.

Construction of telescopers

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1958: Manin gave a constructive method for finding Picard-Fuchs equations:

$$\alpha(x) = \oint_{\Gamma} \frac{dy}{\sqrt{y(y-1)(y-x)}} \quad \rightsquigarrow \quad y'' + \frac{2x-1}{x(x-1)}y' + \frac{1}{4x(x-1)}y = 0$$

Construction of telescopers

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1969: Griffiths developed the Dwork-Griffiths reduction, which later is used to compute telescopers for multivariate rational functions:

Annals of Mathematics

On the Periods of Certain Rational Integrals: I
Author(s): Philip A. Griffiths
Source: *Annals of Mathematics*, Second Series, Vol. 90, No. 3 (Nov., 1969), pp. 460-495

Construction of telescopers

Four approaches:

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2012: Chen, Kauers and Singer gave a method for computing telescopers for algebraic functions via residues:

Telescopers for Rational and Algebraic Functions via Residues

Shaoshi Chen* Manuel Kauers† Michael F. Singer*

Construction of telescopers

Four approaches:

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....	....	....	- ChenKaKoutschan 2016

1947: Fasenmyer gave a method, so-called Sister Celine's method, to find recurrence relations satisfied by hypergeometric sums:

SOME GENERALIZED HYPERGEOMETRIC POLYNOMIALS

SISTER MARY CELINE FASENMYER

1. Introduction. We shall obtain some basic formal properties of the hypergeometric polynomials

$$f_n(a_i; b_j; x) \equiv f_n(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x)$$

(1)

Construction of telescopers

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1990: Zeilberger's algorithm for computing telescopers for holonomic functions via non-commutative elimination in Weyl algebra:

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Construction of telescopers

Four approaches:

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- ChKauersSinger 2012	- ChyzakSalvy 1998	- Koutschan 2010	- ChHuangKaLi 2015
....	....	....	- ChenKaKoutschan 2016

1990: Zeilberger's algorithm for computing telescopers for holonomic functions via non-commutative elimination in Weyl algebra:

$$\begin{cases} P(x,y,D_x)(h)=0 \\ Q(x,y,D_y)(h)=0 \end{cases} \rightsquigarrow A(x,D_x,D_y)(h)=0 \rightsquigarrow A(x,D_x,0) \text{ is telescoper}$$

Construction of telescopers

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1992: Takayama improved the non-commutative elimination in Weyl algebra by Groebner bases computation:

J. Symbolic Computation (1992) **14**, 265–282

An Approach to the Zero Recognition Problem by Buchberger Algorithm

NOBUKI TAKAYAMA

Department of Mathematics, Kobe University, Rokko, Kobe,657, Japan

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
Algebraic-Geometry Approach	Elimination-Based Approach	Gosper-Based Approach	Redution-Based Approach
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1998: Chyzak and Salvy applied non-commutative elimination in Ore algebra to identities proofs :

J. Symbolic Computation (1998) **26**, 187–227
Article No. sy980207

Non-commutative Elimination in Ore Algebras Proves Multivariate Identities

FRÉDÉRIC CHYZAK[†] AND BRUNO SALVY[‡]

INRIA-Rocquencourt and École polytechnique, France

Construction of telescopers

Four approaches:

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Algebraic-Geometry Approach	Elimination-Based Approach	Gosper-Based Approach	Redution-Based Approach
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- ChKauersSinger 2012	- ChyzakSalvy 1998	- Koutschan 2010	- ChHuangKaLi 2015
....	....	....	- ChenKaKoutschan 2016

1990: Based on Gosper's algorithm, Zeilberger developed an algorithm for computing telescoping for bivariate hypergeometric terms:

Discrete Mathematics 80 (1990) 207–211
North-Holland

COMMUNICATION

A FAST ALGORITHM FOR PROVING TERMINATING HYPERGEOMETRIC IDENTITIES

Doron ZEILBERGER*
Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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....	....	....	- ChenKaKoutschan 2016

1990: Almkvist and Zeilberger extends Zeilberger's algorithm to the hyperexponential case:

J. Symbolic Computation (1990) **10**, 571–591

The Method of Differentiating under the Integral Sign

GERT ALMKVIST[1] AND DORON ZEILBERGER[2]†

Construction of telescopers

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2000: Chyzak extends Zeilberger's algorithm to the high-order case:

ELSEVIER Discrete Mathematics 217 (2000) 115–134

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

An extension of Zeilberger's fast algorithm to general holonomic functions☆

Frédéric Chyzak

Construction of telescopers

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2010: Koutschan improved Chyzak's algorithm via advanced ansatz and applied to solve many conjectures in combinatorics:

Math.Comput.Sci. (2010) 4:259–266
DOI 10.1007/s11786-010-0055-0
Mathematics in Computer Science

A Fast Approach to Creative Telescoping

Christoph Koutschan

Construction of telescopers

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....	....	....	- ChenKaKoutschan 2016

2010: Bostan et al. design a fast algorithm for creative telescoping for bivariate rational functions using classical Hermite reduction:

Complexity of Creative Telescoping for Bivariate Rational Functions*

Alin Bostan, Shaoshi Chen, Frédéric Chyzak
Algorithms Project-Team, INRIA
Paris-Rocquencourt
78153 Le Chesnay (France)

Ziming Li
Key Laboratory of Mathematics Mechanization,
Academy of Mathematics and System Sciences
100190 Beijing (China)

Construction of telescopers

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- ChKauersSinger 2012	- ChyzakSalvy 1998	- Koutschan 2010	- ChHuangKaLi 2015
....	....	....	- ChenKaKoutschan 2016

2010: Bostan et al. design a fast algorithm for creative telescoping for bivariate rational functions using classical Hermite reduction:

$$f(x) = D_x(g) + \frac{p}{q}$$

where $p, q \in \mathbb{F}[x]$ with q squarefree and $\deg_x(p) < \deg_x(q)$.

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
Algebraic-Geometry Approach	Elimination-Based Approach	Gosper-Based Approach	Redution-Based Approach
- Picard 1902 - Manin 1958 - Griffiths 1969 - ChKauersSinger 2012	- Fasenmyer 1947 - Zeilberger 1990 - Takayama 1992 - ChyzakSalvy 1998	- Zeilberger 1990 - AlmkvistZeilberger 1990 - Chyzak 2000 - Koutschan 2010	- BostanChenChyzakLi 2010 - BoChChLiXin 2013 - BoLairezSalvy 2013 - ChHuangKaLi 2015 - ChenKaKoutschan 2016

2010: Bostan et al. design a fast algorithm for creative telescoping for bivariate rational functions using classical Hermite reduction:

$$\int f(x)dx = \text{rational part} + \text{logarithmic part}$$

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2013: Bostan et al. generalize the Hermite reduction to hyperexponential case and design a reduction-based telescoping algorithm:

Hermite Reduction and Creative Telescoping for Hyperexponential Functions*

Alin Bostan[1], Shaoshi Chen[2], Frédéric Chyzak[1], Ziming Li[3], Guoce Xin[4]

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2013: Bostan, Lairez and Salvy design a telescoping algorithm for multivariate rational function based on Dwork-Griffiths reduction:

Creative Telescoping for Rational Functions Using the Griffiths–Dwork Method*

Alin Bostan
Inria (France)
alin.bostan@inria.fr

Pierre Lairez
Inria (France)
pierre.lairez@inria.fr

Bruno Salvy
Inria (France)
bruno.salvy@inria.fr

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2015: Chen et al. design a telescoping algorithm for bivariate hypergeometric terms based on modified Abramov-Petkovsek reduction:

A Modified Abramov-Petkovšek Reduction and Creative Telescoping for Hypergeometric Terms*

Shaoshi Chen[1], Hui Huang[1,2], Manuel Kauers[2], Ziming Li[1]

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2016: Chen, Kauers and Koutschan design a telescoping algorithm for bivariate algebraic functions based on Trager's reduction and polynomial reduction:

Reduction-Based Creative Telescoping for Algebraic Functions*

Shaoshi Chen[1,2], Manuel Kauers[3], Christoph Koutschan[4]

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2017: Chen, Hoeij, Kauers and Koutschan design a telescoping algorithm for fuchsian D-finite functions:

Reduction-based Creative Telescoping for Fuchsian D-finite Functions.
with Mark van Hoeij , Manuel Kauers, Christoph Koutschan: [PDF]
To appear in Journal of Symbolic Computation.

Gosper's algorithm

In 1978, Gosper solved the telescoping problem for hypergeometric terms.

Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
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Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

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Example. $k! = \Delta_k($No solution!$)$

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Example. $k \cdot k! = \Delta_k(k!)$

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Example. $(-1)^k \dfrac{n!}{2^n \binom{n}{k}} = \Delta_k \left(\dfrac{(k-n-1)(-1)^k n!}{(n+2)2^n \binom{n}{k}} \right)$

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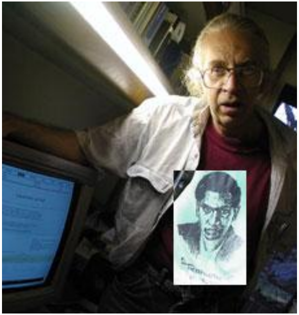

B. Gosper

Gosper's algorithm

Let $f = S_k(H)/H \in \mathbb{E}(k)$. Find a rational solution of

$$f \cdot S_k(u(k)) - u(k) = 1.$$

1 Compute Gosper's form

$$f = \frac{S_k(p)}{p} \cdot \frac{q}{r},$$

where $p, q, r \in \mathbb{E}[k]$ and q, r satisfies

$$\gcd(q(k), r(k+j)) = 1 \quad \text{for all } j \in \mathbb{N}.$$

2 Find a polynomial solution of

$$p = q \cdot S_k(v(k)) - S_k^{-1}(r) \cdot v(y)$$

3 If $v \in \mathbb{E}[k]$ exists, return $u := S_k^{-1}(r)v/p$.

Zeilberger's algorithm

Input: A proper hypergeometric term $H(n,k)$
Output: A telescoper $L \in \mathbb{F}[n, S_n]$ s.t.

$$L(n, S_n)(H) = \Delta_k(G)$$

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- Consider the hypergeometric term

$$L_r(H) := \sum_{i=0}^{r} c_r H(n+i, k)$$

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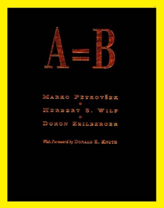

Petkovsek, Wilf & Zeilberger

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Telescoper

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The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Guess the certificate of L?

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2 - 84n^3 - 68460k^6n^4 - 840n^4 - 3720n^5 + 140700k^4n^6 - 9480n^6 - 15024n^7 - 10500k^2n^8 - 14808n^8 - 8400n^9 - 79590n^2k^7 + 284235n^4k^5 - 143640n^6k^3 + 210nk^8 - 26250n^3k^6 + 133035n^5k^4 - 35700n^7k^2 + 252k^{11} + 18900k^9n - 213780k^7n^3 + 368340k^5n^5 - 110460k^3n^7 - 2100n^{10} + 1890k^9 - 1764k^7 + 1260k^5 - 378k^3 - 1260k^{10} - 294nk^2 + 700nk^4 - 588nk^6 + 63504k^{11}n^5 + 52920k^{11}n^4 + 30240k^{11}n^3 + 11340k^{11}n^2 - 2940n^2k^2 - 13080n^3k^2 - 33780n^4k^2 - 55116n^5k^2 - 57348n^6k^2 - 17360k^3n^2 - 48860k^3n^3 - 94920k^3n^4 - 135156k^3n^5 - 55440k^3n^8 - 13860k^3n^9 - 3780k^3n + 7000n^2k^4 + 31185n^3k^4 + 80850n^4k^4 + 90090n^7k^4 + 27720n^8k^4 + 57141k^5n^2 + 155610k^5n^3 + 347886k^5n^6 + 238392k^5n^7 + 110880k^5n^8 + 27720k^5n^9 + 12600k^5n - 5880n^2k^6 - 114114n^5k^6 - 123816n^6k^6 - 83160n^7k^6 - 27720n^8k^6 - 379830k^7n^4 - 469128k^7n^5 - 411840k^7n^6 - 257400k^7n^7 - 110880k^7n^8 - 27720k^7n^9 - 17640k^7n + 9405n^3k^8 + 24750n^4k^8 + 42075n^5k^8 + 47520n^6k^8 + 34650n^7k^8 + 13860n^8k^8 + 85085k^9n^2 + 398475k^9n^4 + 23100k^9n^9 + 480480k^9n^5 + 92400k^9n^8 + 235620k^9n^7 + 227150k^9n^3 + 404250k^9n^6 - 12628k^{10}n - 13860k^{10}n^9 - 152460k^{10}n^3 - 60060k^{10}n^8 - 267960k^{10}n^4 - 157080k^{10}n^7 - 271656k^{10}n^6 - 56980k^{10}n^2 - 323400k^{10}n^5 + 2520k^{11}n + 2520k^{11}n^9 + 11340k^{11}n^8 + 30240k^{11}n^7 + 52920k^{11}n^6)$$

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2-84n^3-68460k^6n^4-840n^4-3720n^5+140700k^4n^6-9480n^6-15024n^7-10500k^2n^8-14808n^8-8400n^9-79590n^2k^7+284235n^4k^5-143640n^6k^3+210nk^8-26250n^3k^6+133035n^5k^4-35700n^7k^2+252k^{11}+18900k^9n-213780k^7n^3+368340k^5n^5-110460k^3n^7-2100n^{10}+1890k^9-1764k^7+1260k^5-378k^3-1260k^{10}-294nk^2+700nk^4-588nk^6+63504k^{11}n^5+52920k^{11}n^4+30240k^{11}n^3+11340k^{11}n^2-2940n^2k^2-13080n^3k^2-33780n^4k^2-55116n^5k^2-57348n^6k^2-17360k^3n^2-48860k^3n^3-94920k^3n^4-135156k^3n^5-55440k^3n^8-13860k^3n^9-3780k^3n+7000n^2k^4+31185n^3k^4+80850n^4k^4+90090n^7k^4+27720n^8k^4+57141k^5n^2+155610k^5n^3+347886k^5n^6+238392k^5n^7+110880k^5n^8+27720k^5n^9+12600k^5n-5880n^2k^6-114114n^5k^6-123816n^6k^6-83160n^7k^6-27720n^8k^6-379830k^7n^4-469128k^7n^5-411840k^7n^6-257400k^7n^7-110880k^7n^8-27720k^7n^9-17640k^7n+9405n^3k^8+24750n^4k^8+42075n^5k^8+47520n^6k^8+34650n^7k^8+13860n^8k^8+85085k^9n^2+398475k^9n^4+23100k^9n^9+480480k^9n^5+92400k^9n^8+235620k^9n^7+227150k^9n^3+404250k^9n^6-12628k^{10}n-13860k^{10}n^9-152460k^{10}n^3-60060k^{10}n^8-267960k^{10}n^4-157080k^{10}n^7-271656k^{10}n^6-56980k^{10}n^2-323400k^{10}n^5+2520k^{11}n+2520k^{11}n^9+11340k^{11}n^8+30240k^{11}n^7+52920k^{11}n^6)$$

Very often, certificates are not needed!

Telescoping without certificates

Problem. Can we compute the telescopers without also computing the certificates?

Telescoping without certificates

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Algorithms: $L(x, \partial_x)(f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_m}(g_m)$

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- Bivariate algebraic case:

  Trager's reduction + polynomial reduction

Softwares

Softwares

- MAPLE:
  1. `EKHAD` by Zeilberger
  2. `DEtools:-Zeilberger` by Le
  3. `SumTools[Hypergeometric]:-Zeilberger` by Le
  4. `Mgfun:-creative_telescoping` by Chyzak
  5. `HermiteCT:-Telescoper` by S.C.
  6. ...

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  1. `fastZeil: Zb` by Paule and Schorn
  2. `HolonomicFunctions: CreativeTelescoping` by Koutschan

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  2. HolonomicFunctions: CreativeTelescoping by Koutschan
- Maxima: Zeilberger by Fabrizio Caruso
- Reduce: zeilberg by Wolfram Koepf
- Kan: sm1 by Nobuki Takayama
- ...

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  Given hypergeometric $F(n,k)$, find hypergeometric $G(n,k)$ and $H(n,k)$ s.t.

$$F = \Delta_n(G) + \Delta_k(H).$$

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Thank you!