The log-Minkowski Problem

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Standard Definitions in Convex Geometry

- $\mathcal{K}$ = all convex compact $K \subset \mathbb{R}^n$ ("body") with $0 \in \mathrm{int}(K)$.
- $h_K(\theta) = \sup_{x \in K} \langle \theta, x \rangle$ support function, 1-homogeneous on $\mathbb{R}^n$.
- $\mathfrak{n}_x^{\partial K}$ = unit outer normal at $x \in \partial K$ (exists $\mathcal{H}^{n-1}|_{\partial K} - a.e.$).
- $S_K = (\mathfrak{n}^{\partial K})_*(\mathcal{H}^{n-1}|_{\partial K})$ surface-area measure on $\mathbb{S} := S^{n-1}$.
- $\mathfrak{m}$ = induced Lebesgue measure on $\mathbb{S} \subset \mathbb{R}^n$.
- If $K \in \mathcal{K}^2_+$, i.e. C^2-smooth with $\kappa_x^{\partial K} := \det(\mathrm{II}_x^{\partial K}) > 0$, then:

$$S_K = \det{}_{n-1}(D^2 h_K)\mathfrak{m} \;, \;\; D^2 h_K := \bar{D}^2_{\mathbb{R}^n} h_K|_{T\mathbb{S}} = \nabla^2_{\mathbb{S}} h_K + h_K \mathrm{Id}_{T\mathbb{S}} .$$

Note that $\det_{n-1}(D^2 h_K)(\mathfrak{n}_x^{\partial K}) = \frac{1}{\kappa_x^{\partial K}}$ for all $x \in \partial K$.

Minkowski's Problem

Problem (Minkowski, Alexandrov): characterize all μ's on $\mathbb{S}$ so that:

$$\exists K \in \mathcal{K} \quad S_K = \mu.$$

If $\mu = f\mathfrak{m}$, $K \in \mathcal{K}^2_+$, this amounts to Monge–Ampère PDE:

$$\det(D^2 h_K) = \det(\nabla^2_{\mathbb{S}} h_K + h_K \mathrm{Id}) = f.$$

Existence $\Leftrightarrow$ $\int_{\mathbb{S}} \theta d\mu(\theta) = 0$ and μ not concentrated on hemispheres.

Uniqueness (up to translation) follows from Brunn–Minkowski inq:

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}} \text{ w/ equality iff } L = cK + b.$$

Regularity (Lewy, Nirenberg, Cheng–Yau, Pogorelov, Caffarelli, ...):

$$\mu = f\mathfrak{m} \;,\; 0 < f \in C^{m,\alpha} \;\Rightarrow\; K \in \mathcal{K}^{m+2,\alpha}_+.$$

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L^p-Firey–Minkowski Sum

Minkowski sum: $aK + bL = \{ax + by \; ; \; x \in K, y \in L\}$ $(a, b \geq 0)$:

$$h_{aK+bL} = ah_K + bh_L.$$

Firey 60's: L^p-Minkowski-sum ($p \geq 1$) $a{\cdot}K +_p b{\cdot}L$ defined by:

$$h_{a\cdot K +_p b\cdot L} := (ah_K^p + bh_L^p)^{\frac{1}{p}}.$$

Example: $\mathcal{E}_1 + \mathcal{E}_2$ is not an ellipsoid, but $\mathcal{E}_1 +_2 \mathcal{E}_2$ is.

Firey established the L^p-Brunn–Minkowski inequality ($p > 1$):

$V((1-\lambda){\cdot}K +_p \lambda{\cdot}L)^{\frac{p}{n}} \geq (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}$ w/ equality iff $L = cK$.

Since $p > 1$, consequence of classical BM and Jensen's inequality:

$$(1-\lambda){\cdot}K +_p \lambda{\cdot}L \supset (1-\lambda)K + \lambda L.$$

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L^p-Brunn–Minkowski Theory ($p \geq 1$)

Lutwak 90's developed the L^p-Brunn–Minkowski theory:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} V(K+\epsilon L) = \int_{\mathbb{S}} h_L dS_K \ , \quad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} V(K +_p \epsilon \cdot L) = \int_{\mathbb{S}} h_L^p dS_p K,$$

where:

$$S_p K := h_K^{1-p} S_K \text{ is the } L^p\text{-surface-area measure}.$$

L^p-Minkowski problem (Lutwak): characterize all μ's on $\mathbb{S}$ such that:

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- When $n \neq p \geq 1$ and μ even, existence by Lutwak.
- Scale invariant case $p = n$ by Lutwak–Yang–Zhang = LYZ.
- Extended to general μ not concentrated on hemispheres by Chou–Wang '06.
- Regularity by Lutwak–Oliker, Chou–Wang - same as $p = 1$ case.
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Case $p = 0$

Can ask the L^p-Minkowski problem $\mu = S_p K = h_K^{1-p} S_K$ for $p < 1$.
In particular, the case $p = 0$ is very natural:

- When $p = 0$, $V_K = \frac{1}{n} S_0 K = \frac{1}{n} h_K S_K$ is the cone-volume measure:

$$\mathrm{Leb}|_K \mapsto_{\frac{x}{\|x\|_K}} \partial K \mapsto_{\mathfrak{n}_{\partial K}} \mathbb{S}.$$

- Firey '74: "what is the ultimate shape of a worn stone?"

$$\frac{\partial x}{\partial t} = -\kappa_x^{\partial K(t)} \mathfrak{n}_x^{\partial K(t)} \quad \text{isotropic Gauss-curvature flow.}$$

Limiting shape given by self-similar solutions $x = -c \frac{\partial x}{\partial t}$, i.e.

$$h_K = \frac{c}{\det(D^2 h_K)} \iff \frac{dV_K}{d\mathfrak{m}} = \frac{1}{n} h_K \det(D^2 h_K) \equiv \frac{c}{n}.$$

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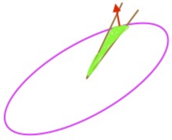

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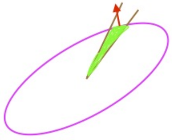

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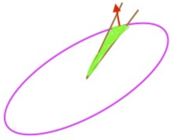

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General case $-n < p < 1$

Anisotropic power-of-Gauss-curvature flow
(Andrews, Chen, Chow, Guan, Ni, Tso, ...):

$$\frac{\partial x}{\partial t} = -\left(\kappa_x^{\partial K(t)} \rho(\mathfrak{n}_x^{\partial K(t)})\right)^{\alpha} \mathfrak{n}_x^{\partial K(t)} \; , \; \rho : \mathbb{S} \to \mathbb{R}_+ \; , \; \alpha > 0.$$

Self-similar solutions solve $S_p K = \rho \mathfrak{m}$ for $\alpha = \frac{1}{1-p}$. Uniqueness?

Critical exponent is $p = -n$. $\frac{dS_{-n}K}{d\mathfrak{m}}$ is centro-affine Gauss-curvature:

$$\forall T \in SL_n \quad \frac{dS_{-n}T(K)}{d\mathfrak{m}} = T_*^{(0)} \frac{dS_{-n}K}{d\mathfrak{m}} \quad T^{(0)} := \frac{T^{-t}\cdot}{|T^{-t}\cdot|} : \mathbb{S} \to \mathbb{S}.$$

In particular, $S_{-n}(\mathcal{E}) = c_n V(\mathcal{E})^2 \mathfrak{m}$ for any ellipsoid $\mathcal{E}$, no uniqueness.
Calabi 70's: $S_{-n}K = c\mathfrak{m} \Leftrightarrow K$ is affine sphere $\Leftrightarrow K$ is an ellipsoid.

Thm (Andrews–Guan–Ni '16 $p \in [0,1)$ and $K \in \mathcal{K}_e$, Brendle–Choi–Daskalopoulos '17 in general)

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Existence in L^p-Minkowski problem via Calc. Var.

To solve $S_p K = \mu$ ($p \in \mathbb{R}$), consider the L^p-Minkowski functional:

$$\mathcal{K} \ni K \mapsto F_{\mu,p}(K) := \frac{\frac{1}{p}\int h_K^p d\mu}{V(K)^{p/n}}.$$

Lutwak '93: for even μ, minimize $\mathcal{K}_e \ni K \mapsto F_{\mu,p}(K)$.
Chou–Wang '06: minimize $\mathcal{K} \ni K \mapsto \max_{a \in K} F_{\mu,p}(K - a)$ ($p \neq 1$).

Lutwak: any local minimizer of $\mathcal{K}_e \ni K \mapsto F_{\mu,p}(K)$ satisfies $S_p K = c\mu$.

For even μ, existence of $K \in \mathcal{K}_e$ is ensured when:
- $0 < p < 1$, iff μ not concentrated on hemispheres (Haberl–LYZ '10).
- $p = 0$, iff $\mu(\mathbb{S} \cap E) \leq \frac{\dim E}{n} \mu(\mathbb{S})$ (Böröczky–LYZ '12).

For general μ, Chen–Li–Zhu '17 showed these are sufficient.

Thm (Chou–Wang '06 for $0 < c \leq f \leq C$, Bianchi–Böröczky–Colesanti–Yang '19)

$$-n < p < 0 \,,\; \mu = f\mathfrak{m} \,,\; f \in L^{\frac{n}{n+p}}(\mathfrak{m}) \;\Rightarrow\; \exists K \in \bar{\mathcal{K}} \;\; S_p K = \mu.$$

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Lutwak: any local minimizer of $\mathcal{K}_e \ni K \mapsto F_{\mu,p}(K)$ satisfies $S_p K = c\mu$.

For even μ, existence of $K \in \mathcal{K}_e$ is ensured when:
- $0 < p < 1$, iff μ not concentrated on hemispheres (Haberl–LYZ '10).
- $p = 0$, iff $\mu(\mathbb{S} \cap E) \leq \frac{\dim E}{n} \mu(\mathbb{S})$ (Böröczky–LYZ '12).

For general μ, Chen–Li–Zhu '17 showed these are sufficient.

Thm (Chou–Wang '06 for $0 < c \leq f \leq C$, Bianchi–Böröczky–Colesanti–Yang '19)

$-n < p < 0$, $\mu = f\mathfrak{m}$, $f \in L^{\frac{n}{n+p}}(\mathfrak{m})$ $\Rightarrow$ $\exists K \in \bar{\mathcal{K}}$ $S_p K = \mu$.

Existence in L^p-Minkowski problem via Calc. Var.

To solve $S_p K = \mu$ ($p \in \mathbb{R}$), consider the L^p-Minkowski functional:

$$\mathcal{K} \ni K \mapsto F_{\mu,p}(K) := \frac{\frac{1}{p}\int h_K^p d\mu}{V(K)^{p/n}}.$$

Lutwak '93: for even μ, minimize $\mathcal{K}_e \ni K \mapsto F_{\mu,p}(K)$.
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Uniqueness when $-n < p < 1$?

Uniqueness in L^p–Minkowski problem is false for $p \in (-n, 1)$ in general (Andrews, Chen, Chou, He, Jian, Li, Lu, Stancu, Wang, Zhu):

$$\forall p \in (-n, 1) \quad \exists K, L \in \mathcal{K} \;\; K \neq L \;\text{ and }\; S_p K = S_p L.$$

However, for fixed $K \in \mathcal{K}$, uniqueness might hold for $p \in (p_K, 1)$.

All counterexamples in the range $p \in (0, 1)$ are not origin-symmetric.
For any $p \in (-n, 0)$, exist counterexamples in $\mathcal{K}_e$ (Kolesnikov–M. '17).

Why challenging? when $p < 1$ no good L^p-Brunn–Minkowski theory.

Not even clear how to define L^p-Minkowski sum $L = K_0 +_p K_1$.

$\Rightarrow$ use "Alexandrov body", i.e. largest L so that $h_L \leq \left(h_{K_0}^p + h_{K_1}^p\right)^{1/p}$.

Q: Is it true when $p < 1$ that $V(K_0 +_p K_1) \geq (V(K_0)^{\frac{p}{n}} + V(K_1)^{\frac{p}{n}})^{\frac{n}{p}}$?
A: In general NO! $K_0 =$ cube, $K_1 =$ translation of K_0.

Idea: let's rule out translations by assuming $K_i \in \mathcal{K}_e$ origin-symmetric.

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The even log-Brunn–Minkowski Conjecture

Even log-BM Conjecture (BLYZ = Böröczky–Lutwak–Yang–Zhang '12)

For all $p \in [0,1)$, $K_i \in \mathcal{K}_e$, even L^p-Brunn–Minkowski inq holds:

$$V((1-\lambda)\cdot K_0 +_p \lambda \cdot K_1)^{\frac{p}{n}} \geq (1-\lambda)V(K_0)^{\frac{p}{n}} + \lambda V(K_1)^{\frac{p}{n}} \quad \forall \lambda \in (0,1).$$

Case $p = 0$ interpreted in limiting sense:

$$V((1-\lambda)\cdot K_0 +_0 \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^{\lambda}.$$

By Jensen's inequality, conjectured inq gets stronger as $p \searrow 0$, so:

- Enough to establish the "logarithmic" case $p = 0$.
- If true, would be a strengthening of classical $p = 1$ case on $\mathcal{K}_e$.

False for $p < 0$ ($K_0, K_1 =$ two different centered cubes).

The Equivalent Conjectures

Thm (BLYZ '12, Kolesnikov–M. '17, Chen–Huang–Li–Liu '18)

The following statements are equivalent for fixed $p \in (-n, 1)$:

1. $\forall q \in (p, 1) \quad \forall K, L \in \mathcal{K}^{2,\alpha}_{+,e} \quad S_q K = S_q L \quad \Rightarrow \quad K = L.$
2. Even L^p-Brunn–Minkowski inq: $\forall K_i \in \mathcal{K}_e$, $\forall \lambda \in [0, 1]$,
$$V((1-\lambda) \cdot K_0 +_p \lambda \cdot K_1) \geq \left((1-\lambda) V(K_0)^{\frac{p}{n}} + \lambda V(K_1)^{\frac{p}{n}} \right)^{\frac{n}{p}} .$$
3. Even L^p-Minkowski inq: $\forall K \in \mathcal{K}_e$,
$$\forall L \in \mathcal{K}_e \quad \frac{1}{p} \int_{\mathbb{S}} h_L^p \, d S_p K \geq \frac{n}{p} V(K)^{1-\frac{p}{n}} V(L)^{\frac{p}{n}},$$
i.e. $\mathcal{K}_e \ni L \mapsto F_{S_p K, p}(L)$ attains a minimum at $L = cK$.

$\Rightarrow$ The BLYZ conjecture is that any (all) of the above hold for $p = 0$. "The even log-Brunn–Minkowski / log-Minkowski conjecture".

$\Rightarrow$ BLYZ '12, Ma '15, Xi–Leng '16: True for $n = 2$. Open for $n \geq 3$.

A sample of previously known results

Thm (Rotem '14, Saroglou '15)

Log-Minkowski conjectures (1)+(2)+(3) hold for K_0, K_1 complex / unconditional convex bodies.

Thm (Colesanti–Livshytz–Marsiglietti '16, Kolesnikov–M. '17, Chen–Huang–Li–Liu '18)

Log-Minkowski conjectures (1)+(3) hold for $K = C^2/C^0$ perturbation of Euclidean ball. (2) holds when both K_0, K_1 are perturbations.

Thm (Kolesnikov–M. '17, Chen–Huang–Li–Liu '18)

L^p-Minkowski conjectures (1)+(2)+(3) are true for $p = 1 - \frac{c}{n^{3/2}}$.

Advancement in KLS conjecture (Y. Chen '20) $\Rightarrow$ $p = 1 - \frac{c_\epsilon}{n^{1+\epsilon}}$.

Roughly: KM prove local uniqueness in even L^p-Minkowski problem, CHLL add local-to-global step.

Main Results 1

Thm (M. '21)

Let $K \in \mathcal{K}^{2,\alpha}_{+,e}$ s.t. $\exists T \in GL_n$, $\tilde{K} = T(K)$ satisfies $\frac{1}{R} \leq \mathrm{II}^{\partial \tilde{K}} \leq \frac{1}{r}$. Then:

$$3 - \frac{n-1}{2}\frac{r^2}{R^2} < p < 1 \;,\; L \in \mathcal{K}_e \;,\; S_p L = S_p K \;\Rightarrow\; L = K,$$

and $\forall L \in \mathcal{K}_e$ $\frac{1}{p}\int_{\mathbb{S}} h_L^p dS_p K \geq \frac{n}{p} V(K)^{1-\frac{p}{n}} V(L)^{\frac{p}{n}}$ w/ equality iff $L = cK$.

In particular, if $\frac{R^2}{r^2} < \frac{n-1}{6}$, applies to $p = 0$:

$$L \in \mathcal{K}_e \;\; V_L = V_K \;\Rightarrow\; L = K \text{ and } \frac{1}{V(K)}\int_{\mathbb{S}} \log\frac{h_L}{h_K} dV_K \geq \frac{1}{n}\log\frac{V(L)}{V(K)}.$$

Can be seen as extension of Brendle–Choi–Daskalopoulos to self-similar sols of pinched anisotropic power-of-Gauss-curvature flow.

Compare: $K = B_2^n$ (in fact $\mathcal{E}$), $r = R = 1$, $p > 3 - \frac{n-1}{2}$ (instead of $-n$).

Main Results 2

Can resolve the conjectures (uniqueness in L^p-Minkowski problem and L^p-Minkowski inequality) after appropriate perturbation of K.

$$\text{Geometric Distance: } d_G(K,L) := \min\{ab > 0 \; ; \; \frac{1}{b}K \subset L \subset aK\},$$

$$\text{Banach-Mazur Distance: } d_{BM}(K,L) := \min\{d_G(K, T(L)) \; ; \; T \in GL_n\}.$$

Thm (M. '21)

Let $\bar{K} \in \mathcal{K}_e$ and set $D := d_{BM}(\bar{K}, B_2^n)$ ($\leq \sqrt{n}$ by John's Thm).

Then $\forall 8 < \gamma < \frac{D}{2}$ $\exists \tilde{K} \in \mathcal{K}^\infty_{+,e}$ s.t. $d_G(\bar{K}, \tilde{K}) \leq \gamma$ and:

$\forall p \in \left(\frac{7}{3} - \frac{n-1}{24}\frac{\gamma^2}{D^2}, 1\right)$ L^p-conjectures hold for $K = T(\tilde{K})$ $\forall T \in GL_n$.

Thm (M. '21) - Isomorphic resolution of even log-Minkowski Conj.

$\forall \bar{K} \in \mathcal{K}_e$ $\exists \tilde{K} \in \mathcal{K}^\infty_{+,e}$ $d_G(\bar{K}, \tilde{K}) \leq 8$ s.t. log-Minkowski conjecture (case $p = 0$) holds for $K = T(\tilde{K})$ $\forall T \in GL_n$.

Remark: can improve 8 to $1 + C\frac{\sqrt{D}}{\sqrt[4]{n}}$ if $D := d_{BM}(\bar{K}, B_2^n) \ll \sqrt{n}$.

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Linearize $S_p K = h_K^{1-p}\det(D^2 h_K)$, $K \in \mathcal{K}^2_+$:

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Linearize $S_p K = h_K^{1-p}\det(D^2 h_K)$, $K \in \mathcal{K}^2_+$:

$h_{K_\epsilon} = h_K(1+\epsilon z) \Rightarrow \frac{d}{d\epsilon}\big|_{\epsilon=0} \log(h_{K_\epsilon}^{1-p}\det(D^2 h_{K_\epsilon})) = (\Delta_K + (n-p)\mathrm{Id})z$.

$$\Delta_K z := \left[(D^2 h_K)^{-1}\right]^{ij} D^2(z h_K)_{ij} - (n-1)z$$

$$= g_K{}^{ij}\left[(\nabla^2_{\mathbb{S}} z)_{ij} + (\log h_K)_i z_j + (\log h_K)_j z_i\right], \quad g_K = \frac{D^2 h_K}{h_K} > 0.$$

Δ_K = HBM operator, 2nd order, elliptic. $-\Delta_K \geq 0$ on $L^2(V_K)$:

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Uniqueness in $S_p K = \mu$ related to $n - p \notin \sigma(-\Delta_K)$.
Hilbert 1912: BM inq (case $p = 1$) $\Leftrightarrow \lambda_1(-\Delta_K) \geq n - 1$.

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Spectral Problem

Denote $\lambda_{1,e}(-\Delta_K) := \min \sigma(-\Delta_K|_{\text{even}\cap 1^\perp})$.
Recall that $\lambda_1(-\Delta_K) = \min \sigma(-\Delta_K|_{1^\perp}) = n-1$ (Hilbert).

Equivalent spectral formulation of BLYZ conjecture

$\forall K \in \mathcal{K}^2_{+,e}$ $\lambda_{1,e}(-\Delta_K) \geq n$. "Next eigenvalue" problem.

(Kolesnikov–M. '17: in appropriate sense, $\lambda_{1,e}(-\Delta_{\text{Cube}}) = n$).

Thm (Kolesnikov–M. '17) – spectrum is centro-affine invariant

$\sigma(-\Delta_{T(K)}) = \sigma(-\Delta_K)$ $\forall T \in GL_n$ (deeper reason soon).

Thm (M. '21) - Szëgo–Weinberger-type isospectral bound

$\forall K \in \mathcal{K}^2_{+,e}$ $\lambda_{1,e}(-\Delta_K) \leq 2n$ with equality iff K is a centered ellipsoid.

Cor (M. '21): $\forall$ non-ellipsoid $K \in \mathcal{K}^2_{+,e}$ $\exists q_K \in (-n,0)$ $\forall p \in (-n, q_K)$
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Affine Differential Geometry

Traditional: hypersurface $H \subset (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ induces Euclidean unit normal $\mathfrak{n}^H$ and ∇_H, the Levi-Civita connection of induced metric.

Affine Differential Geometry: start by equipping H with choice of normal ξ (“normalization"), which induces:

- Second-fundamental form (“metric g^ξ") and affine connection ∇^ξ via Gauss Equation $\bar{D}_U V = \nabla^\xi_U V - g^\xi(U,V)\xi$.
- Volume form ν_ξ. Good choice of ξ will yield $\nabla^\xi \nu_\xi = 0$.

In general, ∇^ξ not Levi-Civita connection for g^ξ, ν_ξ not Riem. volume.

If normalization is affine equivariant, then so are the resulting objects.

Prevalent choice is Blaschke–Berwald equiaffine normalization (used by Calabi, Cheng-Yau in classification of affine spheres).

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Centro-Affine Differential Geometry

We use the centro-affine normalization on ∂K: $\xi(x) = x$.
Invariant under origin-preserving affine transformations.
Denote the differential objects g_K, ∇_K, ν_K.

Trivial normalization! turns ∂K into a centro-affine unit-sphere:
Shape = Id, curvature is constant 1, $\mathrm{Ric}_{\nabla_K} \equiv (n-2)g_K$.

This is in stark contrast to the Ricci curvature of $(\mathbb{S}, g_K, V_K)$ as a metric-measure space (for which Δ_K is the weighted Riemannian Laplacian), which might be negative.

Observation (M. '21)

The Hilbert–Brunn–Minkowski operator Δ_K coincides with the centro-affine Laplacian $\mathrm{div}^{\nabla_K} \mathrm{grad}_{g_K} z$ (and therefore equivariant).

Can now employ classical arguments under positive Ricci (Lichnerowicz, Bochner Formula) to obtain eigenvalue estimates.
These also work for non-Levi-Civita connections.

Challenge: use the evenness to get an improved eigenvalue estimate.

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Thank you very much!