Roger-Shephard and Zhang inequalities for general measures

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Interactions between Partial Differential Equations and Convex Geometry

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- What is presented is part of a joint work with D. Alonso-G., M.A. H. Cifre, J. Yepes N. and A. Zvavitch, and D. Langharst and A. Zvavitch.

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A brief history of convexity: convex bodies and the Minkowski sum

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The collection $\mathcal{K}^n$ is equipped with a natural addition, called **Minkowski addition**. That is, given $K, L \in \mathcal{K}^n$, one has

$$K + L = \{x + y : x \in K, y \in L\} = \{x \in \mathbb{R}^n : K \cap (x - L) \neq \emptyset\}.$$

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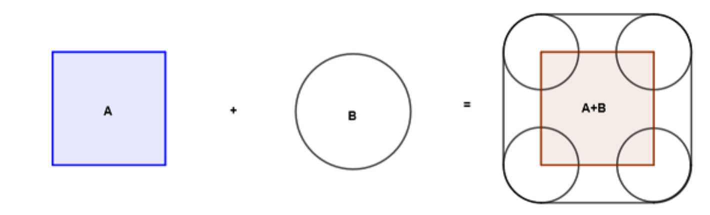

Brunn-Minkowski Inequality: Given $K, L \in \mathcal{K}^n$,

$$\mathrm{Vol}_n(K+L)^{1/n} \geq \mathrm{Vol}_n(K)^{1/n} + \mathrm{Vol}_n(L)^{1/n},$$

with equality if, and only if, $L = \lambda K + v$, with $\lambda > 0$ and $v \in \mathbb{R}^n$.

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We remark that from the AG-GM and the homogeniety of the volume, the Brunn-Minkowski inequality implies the weaker geometric inequality:

$$\mathrm{Vol}_n((1-t)K + tL) \geq \mathrm{Vol}_n(K)^{1-t}\mathrm{Vol}_n(L)^t, \quad t \in (0,1).$$

Minkowski defined the following notion of surface area of a convex body K:

$$\mathrm{Vol}_{n-1}(\partial K) := \lim_{\varepsilon\to 0^+} \frac{\mathrm{Vol}_n(K+\varepsilon B_2^n) - \mathrm{Vol}_n(K)}{\varepsilon},$$

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Finally, using the fact that $\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1}) = n\mathrm{Vol}_n(B_2^n)$, the above becomes:

$$\left(\frac{\mathrm{Vol}_{n-1}(\partial K)}{\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})}\right)^{\frac{1}{n-1}} \geq \left(\frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(B_2^n)}\right)^{\frac{1}{n}}$$

Theorem (Brunn-Minkowski for β-concave measures)

Let $s \in [-1/n, +\infty]$ and $t \in (0,1)$. Given any measure μ on $\mathbb{R}^n$ defined by $d\mu(x) = \phi(x)dx$, where $\phi\colon \mathbb{R}^n \to \mathbb{R}_+$ is s-concave on its support, and any pair of Borel sets $A, B \subset \mathbb{R}^n$, one has that μ is β-concave for $\beta = \frac{s}{1+ns} \in [-\infty, 1/n]$, i.e., that

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One way to obtain this inequality is with use of the Prékopa-Leindler inequality.

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A Rogers-Shephard type inequality for general measures

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Given any Borel measure μ on $\mathbb{R}^n$, can one expect to have

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Theorem (Alonso-G.-Cifre-R.-Yepes-Zvavitch)

Let μ be a measure on $\mathbb{R}^n$ given by $d\mu(x) = \phi(x)dx$, where $\phi \colon \mathbb{R}^n \to [0,\infty)$ is radially decreasing, i.e, for every $x \in \mathbb{R}^n$ and every $t \in (0,1)$, one has $\phi(tx) \geq \phi(x)$. Then, for any $K \in \mathcal{K}^n$, one has

$$\mu(K-K) \leq \binom{2n}{n} \min\{\overline{\mu}(K), \overline{\mu}(-K)\}$$

where

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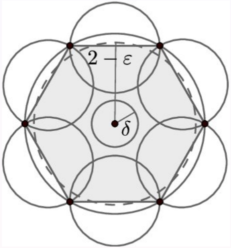
Moreover, if ϕ is continuous at the origin, then equality holds above if, and only if, μ is a positive multiple of the Lebesgue measure on $K-K$ and K is a simplex.

Let $0 < \epsilon < \delta < 2$, and consider the measure $d\mu(x) = \varphi(x)dx$ on $\mathbb{R}^2$ defined by $\varphi(x) = 1$ if $x \in \delta B_2^2 \cup (2B_2^2 \setminus (2-\epsilon)B_2^2)$, and $\varphi(x) = 0$ otherwise. Then one has

$$\mu(B_2^2 - B_2^2) > 6\sup\{\mu(x + B_2^2) \colon x \in \mathbb{R}^2\}.$$



Theorem (R., 2019)

Fix $p \in \mathbb{N}$. Let η be a measure on $\mathbb{R}^n$ given by $d\eta(x) = \psi(x)dx$, where $\psi\colon \mathbb{R}^n \to \mathbb{R}_+$ is $\left(\frac{1}{s}\right)$-concave, for some $s \in (0, \infty)$, and such that $\psi(0) = \|\psi\|_\infty$. For each $i = 1, \ldots, p$ let μ_i be measure on $\mathbb{R}^n$ with density $\phi_i\colon \mathbb{R}^n \to \mathbb{R}_+$ that is radially decreasing. Let $\nu = \prod_{i=1}^{p} \mu_i$ be the associated product measure on $(\mathbb{R}^n)^p$ having density ϕ. For each $i = 1, \ldots, p$ let $H_i \in G_{n,m_i}$ $H_i \in G_{n,m_i}$ be an m_i-dimensional subspace of the ith copy of $\mathbb{R}^n$, and set $\bar{H} = H_1 \times \cdots \times H_p$ be the associated product subspace of $(\mathbb{R}^n)^p$.

A generalization of Schneider's result

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$$\nu\left(D_p(K) \cap \bar{H}\right) \leq \frac{c(n,m,s)}{\eta(K)} \int_K \prod_{i=1}^p \mu_i[(y-K)\cap H_i]\,d\eta(y),$$

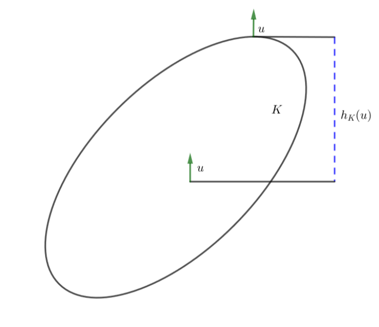
where $m = m_1 + \cdots + m_p$ and where

$$c(n,m,s) = \binom{n+m+s}{m+s}.$$

The support function of a convex body

Given any $K \in \mathcal{K}^n$, the support function of K is defined to be

$$h_K(u) := \max_{x \in K} \langle x, u \rangle, \quad u \in \mathbb{S}^{n-1}$$



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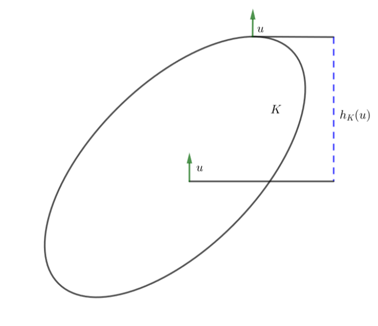

Moreover, we remark the one may redefine the Minkowski sum of two convex bodies K and L as $h_{K+L}(u) = h_K(u) + h_L(u), \quad u \in \mathbb{S}^{n-1}$.

Projection bodies and the covariogram

Given any $K \in \mathcal{K}^n$, the projection body of K is the convex body ΠK whose support function is given by

$$h_{\Pi K}(\theta) = \mathrm{Vol}_{n-1}(K|\theta^\perp), \quad \theta \in \mathbb{S}^{n-1}, \quad \theta^\perp = \{x \in \mathbb{R}^n \colon \langle x, \theta \rangle = 0\}.$$

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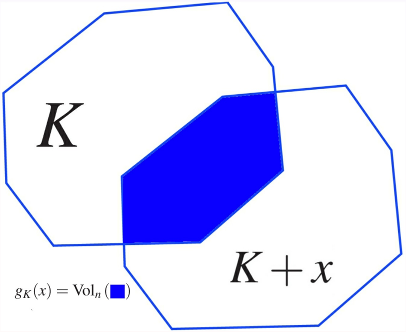
The covariogram of a convex body K is defined to be

$$g_K(x) = \mathrm{Vol}_n(K \cap (x+K)) = (\chi_K * \chi_{-K})(x).$$



The covariogram plays an essential role in many question in convex geometry. In particular, as was established by Matherian, its radial derivatives have a critical connection with ΠK:

$$\frac{\partial}{\partial r} g_K(r\theta)\,|_{r=0} = -\frac{1}{2}\int_{\partial K} |\langle \theta, n_K(y)\rangle| dy = -\mathrm{Vol}_{n-1}(K|\theta^{\perp}) = -h_{\Pi K}(\theta),$$

where $n_K(y)$ is the unit outer normal at the point $y \in \partial K$.

Projection bodies continued

Projection bodies were used independently by both Petty's and Schneider's solution to the so-called Shephard problem:

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A profound question of Petty from the 1960s asks whether ellipsoids minimize the affine invariant $PPI(K) := \text{Vol}_n(K)^{1-n}\text{Vol}_n(\Pi K)$ over all origin-symmetric $K \in \mathcal{K}^n$, $n \ge 3$.

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There is a deep history to Petty's problem, which is impossible to encapsulate into one slide, so for a comprehensive survey of the problem's history, significance, and contributions, see the books of Schneider and Gardner. Here we collection some key points:

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More progress has appeared due to Ivaki, and more recently, by O. Ortega-Moreno and F. Schuster concerning fixed points of Minkowski Valuations.

Petty's projection inequality

While Petty's conjecture still remains open, one may instead consider the affine invariant quantity

$$\mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}((\Pi K)^{\circ}),$$

where $L^{\circ} = \{x \in \mathbb{R}^n \colon h_L(x) \leq 1\}$ denotes the polar body of a convex body L containing the origin.

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Petty's Projection Inequality: Given $K \in \mathcal{K}^n$,

$$\mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n((\Pi K)^\circ) \leq \left(\frac{\mathrm{Vol}_n(B_2^n)}{\mathrm{Vol}_{n-1}(B_2^{n-1})}\right)^n$$

with equality if, and only if, K is an ellipsoid.

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It was shown by Zhang that Petty's Projection inequality not only implies but is, in fact, equivalent to an affine version of the Sobolev inequality.

Zhang's inequality

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Zhang's Inequality: Given $K \in \mathcal{K}^n$,

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Zhang's inequality continued

In the 1990s Gardner and Zhang found a family of convex bodies, called the radial mean bodies, which they used to connect the Rogers-Shephard inequality to Zhang's projection inequality in a continuous way. Their result reads:

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Theorem (Gardner-Zhang)

Let $K \in \mathcal{K}^n$. For each $-1 < p \leq q$, there exist convex bodies R_pK, R_qK such that

$$Vol_n(K-K) \leq c_{n,p} Vol_n(R_pK) \leq c_{n,q} Vol_n(R_qK) \leq n^n Vol_n(K)^n Vol_n((\Pi K)^\circ),$$

with equality if, and only if, K is a simplex. Here

$$c_{n,r} = (nB(r+1,n))^{-1/r}$$

whenever $r > -1$ and B denotes the Beta function.

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They showed that, when choosing $p = q = n$, then one has $\mathrm{Vol}_n(R_nK) = \mathrm{Vol}_n(K)$, and the far left side becomes the Rogers-Shephard inequality and the far right side Zhang's inequality.

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Corollary (Zhang's inequality for a general measure)

Let μ be a measure that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ and $K \in \mathcal{K}^n$. Then one has

$$\frac{1}{Vol_n(K)} \int_{\mathbb{R}^n} g_K(x) d\mu(x) \le \mu(n Vol_n(K)(\Pi K)^{\circ}).$$

This inequality is asymptotically sharp.

Given a measure μ on $\mathbb{R}^n$ defined by $d\mu(x) = \varphi(x)dx$, with $\varphi \colon \mathbb{R}^n \to \mathbb{R}_+$ and a convex body K, the μ-covariogram of K is defined by

$$g_{\mu,K}(x) = \mu(K \cap (x + K)) = \int_{K\cap(x+K)} \varphi(y)dy.$$

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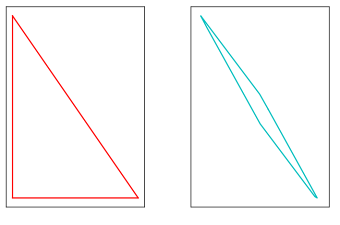

Figure: Left: A convex body $K \subset \mathbb{R}^2$ centered at the origin. Right: $\Pi^{\circ}_{\gamma_2} K$.

Measure theoretic projection bodies

Theorem (Langharst-.R.-Zvavitch, 2021)

Let $F:\mathbb{R}_+\to\mathbb{R}_+$ *be an increasing, invertiable, and let* μ *be a measure that if* $F(t)$*-concave on* $\mathbb{R}^n$ *have a non-negative density* φ*. Then, for any convex body* K *with* $\mu(K)>0$ *and such that* $\int_K \nabla\varphi(x)dx=0$*, one has*

$$K-K\subset\frac{F(\mu(K))}{F'(\mu(K))}\left(\Pi_\mu K\right)^\circ.$$

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$$\frac{\partial}{\partial r}g_{\mu,K}(r\theta)\mid_{r=0}=-\frac{1}{2}\int_{\partial K}|\langle\theta,n_K(y)\rangle|\varphi(y)dy+\eta_{\mu,K},$$

where $\eta_{\mu,K}=\frac{1}{2}\int_K\nabla\varphi(y)dy$.

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where $\eta_{\mu,K}=\frac{1}{2}\int_K\nabla\varphi(y)dy$.

With the above theorem in hand, it makes sense to define the μ-projection body of $\Pi_\mu K$ of K as the convex body whose support function is defined by

$$h_{\Pi_\mu K}(\theta) = \frac{1}{2}\int_{\partial K} |\langle \theta, n_K(y)\rangle| \varphi(y) dy, \quad \theta \in \mathbb{S}^{n-1}.$$

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These bodies were considered at an earlier time by G. Livshyts in her solution of a Shephard-type problem for general measures.

Lemma (L.-R.-Z., 2021)

Let ν be a measure with radially non-decreasing, continuous density φ, and let $f : \mathbb{R}^n \to \mathbb{R}^+$ be a compactly supported concave function such that $0 \in int(supp(f))$ and $f(0) = \max f(x)$. If $q \colon \mathbb{R}^+ \to \mathbb{R}$ is an increasing function, then we have

$$\int_{supp(f)} q(f(x))d\nu(x) \leq \beta \int_{\mathbb{S}^{n-1}} \int_0^{z(\theta)} \varphi(r\theta)r^{n-1}drd\theta,$$

where

$$z(\theta) = -\left(\left.\frac{df(r\theta)}{dr}\right|_{r=0}\right)^{-1} f(0) \quad \text{and} \quad \beta = n\int_0^1 q(f(0)t)(1-t)^{n-1}dt.$$

Equality occurs if, and only if, φ is a constant.

Theorem (L.-R.-Z., 2021)

Let $K \in \mathcal{K}^n$, μ, ν be measures on $\mathbb{R}^n$, with the density of μ locally Lipschitz, and the density of ν radially non-decreasing. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, invertiable, and differentiable function such that $F \circ g_{\mu,K}$ is concave. Then one has

$$\frac{1}{\mu(K)} \int_K \nu(y-K)d\mu(y) \leq \frac{n}{\mu(K)} \cdot \nu\left(\frac{F(\mu(K))}{F'(\mu(K))}(\Pi_\mu K - \eta_{\mu,K})^\circ\right) \times \int_0^1 F^{-1}(F(\mu(K))t)(1-t)^{n-1}dt.$$

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Zhang's inequality for two measures

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Suppose $s > 0$ and ν is a measure with radially non-decreasing density, and that μ is an s-concave measure. Then, for any $K \in \mathcal{K}^n$, one has

$$\frac{\binom{n+s^{-1}}{n}}{\mu(K)} \int_K \nu(y-K)d\mu(y) \leq \nu(s^{-1}\mu(K)(\Pi_\mu K - \eta_{\mu,K})^\circ).$$

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In particular, if μ and ν are taken to be the Lebesgue measure on $\mathbb{R}^n$, then we recover Zhang's inequality.

A functional version of the μ-covariogram.

Let $K \in \mathcal{K}^n$ and $d\mu(x) = \varphi(x)dx$ be a measure on $\mathbb{R}^n$ with non-negative density φ.

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$$g_{\mu,f}(K,x) := \int_{K\cap(K+x)} f(y-x)\varphi(y)dy.$$

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Theorem (L.-R.-Z., 2021)

Under the above assumptions, with the additional assumption of differentiablity of f and a Lipschitz condition on φ, one has

$$\frac{d}{dr} g_{\mu,f}(K,r\theta)|_{r=0} = \frac{1}{2}\int_K \langle f\nabla\varphi - \varphi\nabla f, \theta\rangle dy$$
$$-\frac{1}{2}\int_{\partial K} |\langle \theta, n_K(y)\rangle| f(y)d\mu(y).$$

Thanks

Thanks for listening, everyone! Questions?