Affine spectral inequalities and the affine Laplace operator

Julián Haddad (UFMG, Brazil)
joint work with H. Jiménez and M. Montenegro

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Interaction Between Partial Differential Equations and Convex Geometry

Rayleigh quotients

For an open and bounded $\Omega \subset \mathbb{R}^n$ and $1 \leq p, q \leq \infty$, let

$$\lambda(\Omega) = \inf \left\{ \frac{\| |\nabla f| \|_p}{\|f\|_q} \,\middle|\, f : \overline{\Omega} \to \mathbb{R} \text{ smooth and } f = 0 \text{ in } \partial\Omega \right\}$$

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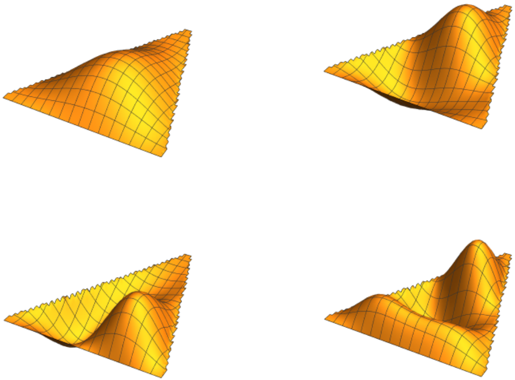
- There is a unique minimizer $f \in W_0^{1,2}(\Omega)$.
- It solves the differential equation

$$\begin{cases} \Delta f + \lambda_{2,\Omega}^2 f & = 0 \text{ in } \Omega \\ f & = 0 \text{ in } \partial\Omega. \end{cases}$$

- $f(x)\sin(\lambda_{2,\Omega} \cdot t)$ describes a vibrating membrane with the boundary fixed at $\partial\Omega$.



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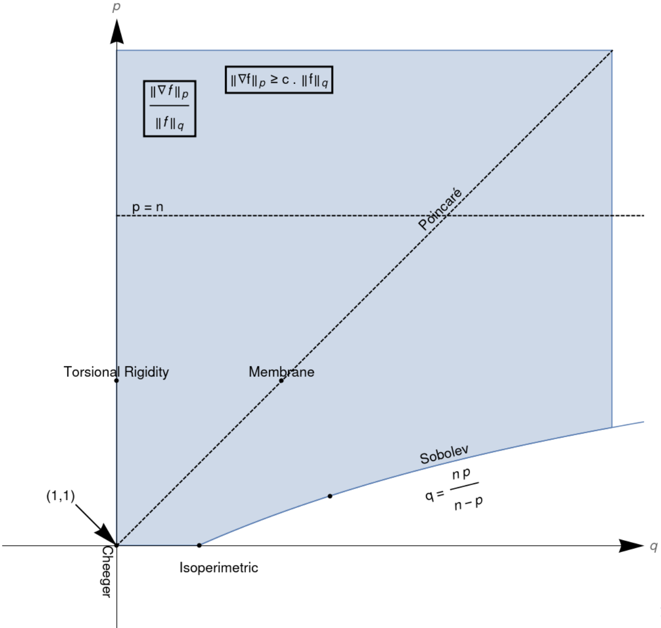

The Affine Invariant World

Definition

$$\mathcal{E}_p f = c_{n,p} \left(\int_{\mathbb{S}^{n-1}} \|\partial_\xi f\|_p^{-n} d\xi \right)^{-1/n}$$

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'03, Lutwak, Yang, Zhang - Sharp affine p-Sobolev Inequalities

'09, Cianchi, Lutwak, Yang, Zhang - Affine Moser-Trudinger...

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Equality case (Brothers-Ziemer result)

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For $f = \chi_K$, K convex and $p = 1$

$$\|\partial_\xi f\|_1 = 2|P_{\langle\xi\rangle^\perp} K|_{n-1}$$

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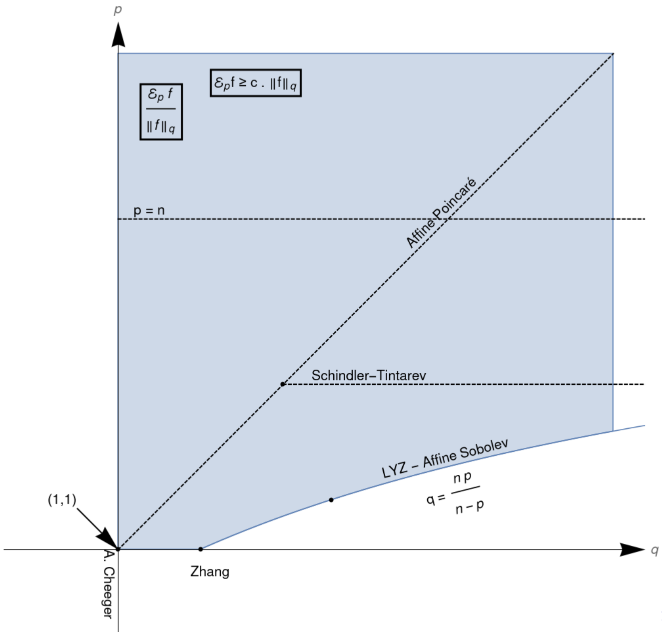

Results: bounds, compactness, existence and variation

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \geq 1$.

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   (Rellich-Kondrachov Theorem)

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   this set is unbounded in $W_0^{1,p}(\Omega)$.

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3. $\exists$ Extremal function $f_p \in W_0^{1,p}(\Omega)$ or $f_1 \in \mathrm{BV}(\Omega)$ for

$$\lambda_{p,\Omega}^{\mathcal{A}} = \inf \frac{\mathcal{E}_p f}{\|f\|_p}$$

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Let's call f_p the *p-affine eigenfunction*

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Yes! Let's call $\Delta_p^{\mathcal{A}}$ the *affine p-laplacian*

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equality only for ellipsoids.

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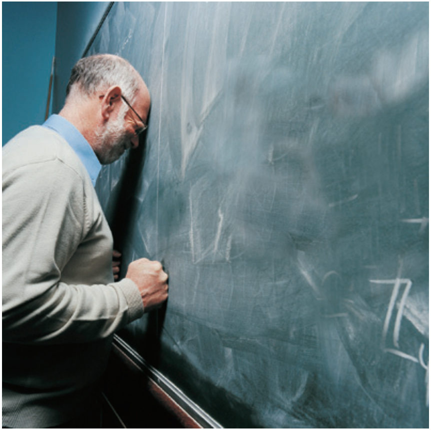
4. The differential equation is affine invariant.

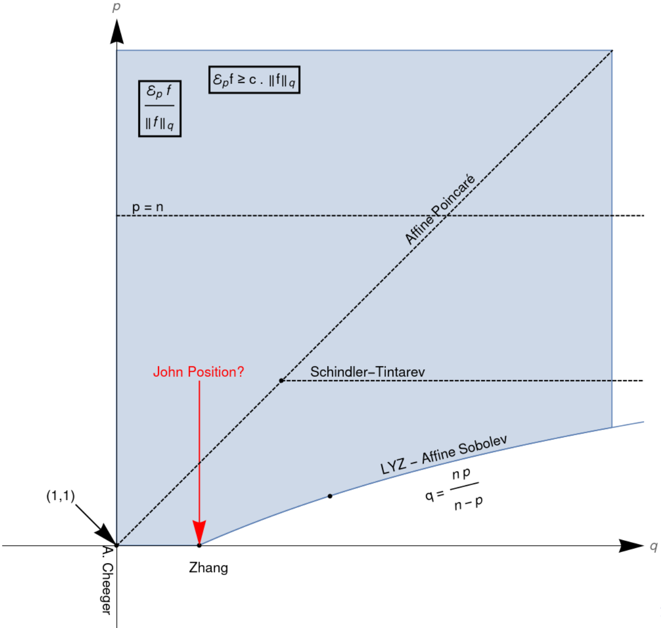
Open questions



Open questions

Existence of minimizers for mixed (p, q)-quotients?

Affine Rayleigh quotients



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Existence of minimizers for $p < q < \frac{np}{n-p}$

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$$\begin{aligned}\|\partial_\xi f\|_p^p &= \int_{\xi^\perp}\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial t}f(t\xi+x)\right|^p dt dx\\ &\ge t_p^p \int_{\xi^\perp}\int_{-\infty}^{\infty}|f(t\xi+x)|^p dt\; dx\,\mathrm{w}(\Omega,\xi)^{-p}\\ &= t_p^p\|f\|_p^p\,\mathrm{w}(\Omega,\xi)^{-p}.\end{aligned}$$

Open questions

Existence of minimizers for $1 \leq q < p$

$$\mathcal{E}_p f \geq C_{n,p}(\Omega)\|f\|_p^{\frac{n-1}{n}} \||\nabla f|\|_p^{1/n}$$
$$\geq C_{n,p}(\Omega)\|f\|_q^{\frac{n-1}{n}} \||\nabla f|\|_p^{1/n}$$

Existence of minimizers for $p < q < \frac{np}{n-p}$

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$$\|\nabla_\xi f\|_p^p = \int_{\xi^\perp}\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial t} f(t\xi + x)\right|^p dt dx$$
$$\geq t_p^p \int_{\xi^\perp}\left(\int_{-\infty}^{\infty} |f(t\xi + x)|^q dt\right)^{p/q} dx \, \mathrm{w}(\Omega,\xi)^{-p}$$
$$\geq t_p^p \|f\|_p^p \, \mathrm{w}(\Omega,\xi)^{-p}?.$$

Open questions

Is the affine eigenvalue simple?

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$$\Delta_{p,K} u = \lambda |u|^{p-2} u$$
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$$u_t(x) = \left(t v^p(x) + (1-t) u^p(x)\right)^{1/p}$$

Open questions

Is the affine eigenvalue simple?

$$\begin{aligned}
\Delta_{p,K} u &= \lambda |u|^{p-2} u \\
\Delta_{p,K} v &= \lambda |v|^{p-2} v \\
u_t(x) &= \left(t v^p(x) + (1-t) u^p(x)\right)^{1/p} \\
\|\nabla u_t\|_K^p &\leq t \|\nabla v(x)\|_K^p + (1-t) \|\nabla u(x)\|_K^p
\end{aligned}$$

Open questions

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$$\Delta_{p,K} u = \lambda |u|^{p-2} u$$

$$\Delta_{p,K} v = \lambda |v|^{p-2} v$$

$$u_t(x) = (t v^p(x) + (1-t) u^p(x))^{1/p}$$

$$\|\nabla u_t\|_K^p \leq t \|\nabla v(x)\|_K^p + (1-t) \|\nabla u(x)\|_K^p$$

with equality if and only if $u(x)\nabla v(x) = v(x)\nabla u(x)$

Open questions

Is the affine eigenvalue simple?

$$\Delta_p^{\mathcal{A}} u = \lambda |u|^{p-2} u$$
$$\Delta_p^{\mathcal{A}} v = \lambda |v|^{p-2} v$$
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Open questions

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Open questions

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$$\Delta_p^{\mathcal{A}} u = \Delta_{p,K_u} u = \lambda |u|^{p-2} u$$
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$$u_t(x) = (t v^p(x) + (1-t) u^p(x))^{1/p}$$
$$\|\nabla u_t\|_?{}^p \leq t \|\nabla v^p(x)\|_? + (1-t)\|\nabla u^p(x)\|_?$$

Open questions

A tough question

For $p = 1, q \in [1, \frac{n}{n-1})$ the eigenfunction is χ_K with $K \subseteq \Omega$ minimizing

$$\frac{\mathrm{vol}(\Pi^\circ K)^{-1/n}}{V(K)^{1/q}}$$

Open questions

A tough question

For $p = 1, q \in [1, \frac{n}{n-1})$ the eigenfunction is χ_K with $K \subseteq \Omega$ minimizing

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Open questions

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- False if Ω is not convex

Open questions

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- B. Kawohl, N. Kutev, Global behaviour of solutions to a parabolic mean curvature equation, '95

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- V. Caselles, A. Chambole, M. Novaga - Uniqueness of the cheeger set of a convex body, '07

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- V. Alter, V. Caselles, Uniqueness of the Cheeger set of a convex body, '08

Open questions

Related question: Brunn-Minkowsky for the quotient

$$\frac{V(K+L)}{S(K+L)} \geq \frac{V(K)}{S(K)} + \frac{V(L)}{S(L)}?$$

Open questions

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$$\frac{V(K+L)}{S(K+L)} \geq \frac{V(K)}{S(K)} + \frac{V(L)}{S(L)}?$$

No.

Open questions

Related question: Brunn-Minkowsky for the quotient

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No.

M. Fradelizi, A. Giannopoulos, M. Meyer, Some inequalities about mixed volumes, '03

Open questions

Related question: Brunn-Minkowsky for the quotient

$$\frac{V(K+L)}{\mathrm{vol}(\Pi^\circ(K+L))^{-1/n}} \geq \frac{V(K)}{\mathrm{vol}(\Pi^\circ K)^{-1/n}} + \frac{V(L)}{\mathrm{vol}(\Pi^\circ L)^{-1/n}}?$$

Open questions

- Brunn-Minkowsky type inequality?

$$\lambda^{\mathcal{A}}_{p,t\Omega_1+(1-t)\Omega_2}{}^{-1} \geq t\lambda^{\mathcal{A}}_{p,\Omega_1}{}^{-1} + (1-t)\lambda^{\mathcal{A}}_{p,\Omega_2}{}^{-1}$$

Open questions

- Brunn-Minkowsky type inequality?

$$\lambda_{p,t\Omega_1+(1-t)\Omega_2}^{\mathcal{A}}{}^{-1} \geq t\lambda_{p,\Omega_1}^{\mathcal{A}}{}^{-1} + (1-t)\lambda_{p,\Omega_2}^{\mathcal{A}}{}^{-1}$$

- Continuity of $\lambda_{p,\Omega}^{\mathcal{A}}$ with respect to p and Ω?

Open questions

- Brunn-Minkowsky type inequality?

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- Continuity of $\lambda^{\mathcal{A}}_{p,\Omega}$ with respect to p and Ω?
- Affine invariant flow

Open questions

- Brunn-Minkowsky type inequality?

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- Affine invariant flow
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Open questions

- Brunn-Minkowsky type inequality?

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- Continuity of $\lambda^{\mathcal{A}}_{p,\Omega}$ with respect to p and Ω?
- Affine invariant flow
- Neumann boundary conditions
- Characterize John position by solvability of a PDE?

The end

Thank you