Stability and index estimates of compact or noncompact capillary surfaces

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BIRS-IASM: Interaction Between PDE and Convex Geometry

Outline

Two topics:

- Index for **compact** capillary surfaces
- Stability rigidity results for **noncompact** capillary surfaces.

Compact capillary surfaces

Consider a compact 3-manifold M with boundary.

- Let Σ be a (compact, connected) capillary surface in M, i.e., it is a surface with constant mean curvature and it intersects ∂M at a constant angle $\theta \in (0, \pi)$.

Compact capillary surfaces

Consider a compact 3-manifold M with boundary.

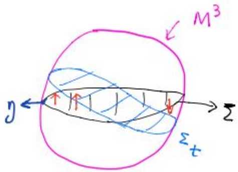
- Let Σ be a (compact, connected) capillary surface in M, i.e., it is a surface with constant mean curvature and it intersects ∂M at a constant angle $\theta \in (0, \pi)$.
- Σ is a critical point of the energy functional $E(\Sigma_t) = Area(\Sigma_t) - \cos\theta \cdot W(t)$ among surfaces that have same volume functional $V(\Sigma_t)$.

$$W(t) = \int_{\partial\Sigma\times[0,t]} X^* \ dA_{\partial M}$$

$$V(\Sigma_t) = \int_{\Sigma\times[0,t]} X^* \ dV_M$$

where X is the isometric immersion of Σ.

Capillary CMC surfaces

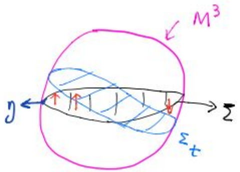


We calculate the first variational formula:

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi(\Sigma, t)) = -\int_{\Sigma} uH + \int_{\partial\Sigma} u\langle \nu, \eta - \cos\theta\, T\rangle,$$

where $\int_{\Sigma} u = 0$ since $V(t)' = \int_{\Sigma} u$.

Thus Σ is a (volume preserving) critical point iff $H \equiv c$ and Σ intersects $\partial\Omega$ at a costant angle θ.



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Thus Σ is a (volume preserving) critical point iff $H \equiv c$ and Σ intersects $\partial\Omega$ at a costant angle θ.

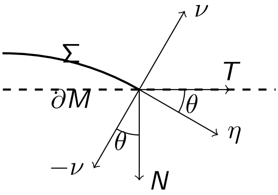
Such surfaces are *capillary surfaces.* In particular, when $\theta = \pi/2$, they are *free boundary surfaces*. There are infinitely many FBCMC surfaces (different topologies) in convex bodies, especially, in the unit ball.

Second variation

At a critical point, we have

$$Q(u,u) = \int_\Sigma |\nabla u|^2 - (\mathrm{Ric}_M(\nu,\nu) + |A_\Sigma|^2)u^2 - \int_{\partial\Sigma} qu^2$$
$$= -\int_\Sigma uJu + \int_{\partial\Sigma} u\left(\frac{\partial u}{\partial \eta} - qu\right),$$

where $q = \frac{1}{\sin\theta} h_{\partial M}(T,T) + \cot\theta A_\Sigma(\eta,\eta)$ and $J = \Delta + |A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)$



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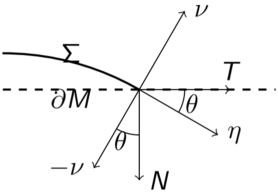

Along $\partial\Sigma$, $q + \kappa_{\partial\Sigma} = \frac{1}{\sin\theta} H_{\partial M} + \cot\theta H_\Sigma = \frac{1}{\sin\theta}(H_{\partial M} + \cos\theta H_\Sigma)$.

Weak index

- Denote by $\text{Ind}_w(\Sigma)$ the maximal dimension of a subspace of $\{u \in C^\infty(\Sigma) : \int_\Sigma u = 0\}$ in which the second variation of $E(\Sigma_t)$ is negative, i.e., $Q(u, u) < 0$. If $\text{Ind}_w(\Sigma) = 0$, we say the surface is weakly stable.

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- $\mathrm{Ind}_w(\Sigma)$ is equal to the number of negative eigenvalues of

$$\begin{cases} \tilde{J}u + \tilde{\lambda}u = 0 & \text{in } \Sigma \\ \frac{\partial u}{\partial \eta} = qu & \text{on } \partial\Sigma, \end{cases} \tag{1}$$

where $\tilde{J}u = Ju - \frac{1}{|\Sigma|}\int_\Sigma Ju$.

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- Eigenvalues have the min-max characterization:

$$\tilde{\lambda}_k = \inf\{Q(u,u) : \|u\|_2 = 1, u \perp 1, \varphi_1, \cdots, \varphi_{k-1}\}$$

where φ_i is the i-th eigenfunction.

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- We want to bound Morse index:

$$\mathrm{Ind}_w(\Sigma) \geq C_1 g + C_2 r$$

where g, r are the number of genus and boundary components of Σ.

Weak index bound

Theorem (H&Saturnino,21'; H&Aiex,20')

Let M be a 3-dimensional oriented Riemannian manifold with boundary isometrically embedded in $\mathbb{R}^d$ and let Σ be a compact capillary surface immersed in M at a constant angle θ with genus g and r boundary components. Suppose that every non-zero $\xi \in \mathcal{H}(\Sigma, \partial\Sigma)$ satisfies

$$\int_\Sigma \sum_{i=1}^{2} |\Pi_M(e_i, \xi)|^2 + |\Pi_M(e_i, \star\xi)|^2 \, dA - \int_\Sigma R_M |\xi|^2 \, dA < \int_\Sigma H_\Sigma^2 |\xi|^2 \, dA + \int_{\partial\Sigma} (2 \cot\theta H_\Sigma + \frac{2}{\sin\theta} H_{\partial M}) |\xi|^2 \, d\ell.$$

Then

$$\mathrm{Ind}_w(\Sigma) \geq \frac{2g + r - 1 - d}{2d}.$$

- $\{e_1, e_2\}$ is a local O.N. basis, $\star\xi$ is the dual harmonic vector field.

In Euclidean space

Let M be a smooth domain in $\mathbb{R}^3$ or $\mathbb{S}^3$.

Corollary

Suppose that $H_{\partial M} + H_\Sigma \cos\theta \geq 0$ along $\partial\Sigma$ and that one of the following holds:

$$H_\Sigma > 0, \; \textit{or} \quad H_{\partial M} > 0$$

at some point in $\partial\Sigma$. Then

$$\mathrm{Ind}_w(\Sigma) \geq \frac{2g + r - 4}{6}.$$

- When $\theta = \pi/2$, this corollary was obtained by Cavalcante and de Oliverira previously. Our result works for domain in general manifolds and without capillary boundary.
- It gives topological information for (weakly) stable capillary surfaces.

Idea of the proof

- Recall quadratic form:

$$Q(u,u) = \int_\Sigma |\nabla^\Sigma u|^2 - |A|^2 - \int_{\partial\Sigma} h^{\partial\Omega}(N,N)u^2$$

- Let $\mathcal{H}(\Sigma, \partial\Sigma) = \{$harmonic vector fields ξ tangential along boundary $\partial\Sigma\}$. Weitzenbock's formula:

$$\Delta_1 \xi = \nabla\nabla\xi + \mathrm{Ric}_\Sigma(\xi)$$

  Hodge theorem says:

$$\mathcal{H}(\Sigma, \partial\Sigma) \cong H_1(\Sigma, \partial\Sigma, \mathbb{R}) \text{ and } \dim(H_1(\Sigma, \partial\Sigma, \mathbb{R})) = 2g + r - 1$$

- We check coordinates $\xi \in \mathcal{H}^1_T(\Sigma)$ are admissible, that is, $\int_\Sigma \langle \xi, E_i \rangle = 0$ for $i = 1, 2, 3$.
- We calculate that

$$Q(\langle \xi, E_i \rangle, \langle \xi, E_i \rangle).$$

- Finally, using the Rank-Nullity theorem and a contradiction argument, we conclude the theorem.

II: Rigidity result for stable noncompact capillary surfaces.

Complete minimal surfaces with finite index

We review some classical theorems on complete, noncompact, minimal surfaces Σ in a 3-manifold M (by R. Schoen&Fischer-Colbrie). The index of Σ:

$$\mathrm{Ind}(\Sigma) = \lim_{R\to\infty} \mathrm{Ind}(D_R)$$

where $\mathrm{Ind}(D_R)$ is the number of negative eigenvalues of J on D_R with Dirichlet boundary condition. The associated quadratic form is

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Proposition (Fischer-Colbrie,85')

If Σ has finite index, then there exists $u > 0$ on Σ and a compact subset $C \subset \Sigma$ such that $Ju = 0$ in $\Sigma \setminus C$.

In particular, when index is zero, $C = \emptyset$. (Fischer-Colbrie&Schoen,82')

Finite index minimal surfaces

Theorem (Fischer-Colbrie,85')

Let Σ be a complete, finite index, oriented minimal surface in a 3-manifold M with $R_M \geq 0$, then Σ is conformally equivalent to a closed Riemann surface punctured at finite many points.

In particular, (Fischer-Colbrie&Schoen, 82')

- When index is zero, Σ is conformally equivalent to a complex plane C or a cylinder.
- Stable oriented minimal surface in $\mathbb{R}^3$ must be a plane.

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Theorem (Fischer-Colbrie,85')

Let Σ be a complete, finite index, oriented minimal surface in a 3-manifold M with $Ric_M \geq 0$, then it has finite total curvature. Moreover, when $M = \mathbb{R}^3$, finite index of Σ is equivalent to finite total curvature of Σ.

Some of these results are generalized by da Silveria to CMC case ($H = c$) later.

Index of capillary surfaces

Let Σ be a noncompact capillary surface in a 3-manifold M with boundary at a constant angle θ. Similarly, we define the index to be

$$\mathrm{Ind}(\Sigma) = \lim_{n\to\infty} \mathrm{Ind}(\Omega_n)$$

where $\Omega_1 \subset \cdots \subset \Omega_n \cdots$ exhaust Σ. Here, $\mathrm{Ind}(\Omega_n)$ is the number of negative eigenvalues of

$$\begin{cases} Ju + \lambda u = 0 & \text{in } \Omega_n \\ \frac{\partial u}{\partial \eta} - qu = 0 & \text{on } \Gamma = \partial\Omega_n \cap \partial M \\ u = 0 & \text{in } \partial\Omega_n \setminus \Gamma. \end{cases}$$

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In particular, we say Σ is stable if

$$Q(u,u) = \int_\Sigma |\nabla u|^2 - (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu))u^2 - \int_{\partial\Sigma} qu^2 \geq 0$$

for any compactly supported u.

Stability with boundary

Let $M = \mathbb{R}^3_+$.

A capillary Σ in $\mathbb{R}^3_+$ at a contact angle θ is weakly stable if

$$Q(u,u) = \int_\Sigma |\nabla u|^2 - |A_\Sigma|^2 u^2 - \int_{\partial\Sigma} \cot\theta A_\Sigma(\eta,\eta) u^2 \geq 0$$

for any $u \in C_c^\infty(\Sigma)$ such that $\int_\Sigma u = 0$.

- A half plane in half-space is weakly stable since

$$Q(u,u) = \int_\Sigma |\nabla u|^2 \geq 0.$$

- We'd like to show that

$$\text{weakly stable} \rightarrow \text{half-plane}$$

Main purpose

Our main target is to prove following theorem:

Theorem (H&Sartunino,21')

Let Σ be a noncompact capillary surface immersed in a half-space of $\mathbb{R}^3$ at constant angle θ. Assume that $H_\Sigma \cos\theta \geq 0$. Then Σ is weakly stable if and only if it is a half-plane.

Remark: This result is stronger than the one previously proved in our preprint arXiv:2105.12662:

Let .... Then Σ is strongly stable if and only if it is a half-plane.

Conformal structure of finite index surfaces

Theorem (H&Sartunino,21')

Let M be an oriented Riemannian 3-manifold with smooth boundary and let Σ be a noncompact capillary surface with finite index immersed in M at a constant angle θ. Assume that $R_M + H_\Sigma^2 \geq 0$ and that one of the following holds:

$$\partial\Sigma \text{ is compact,}$$

or

$$H_{\partial M} + H_\Sigma \cos\theta \geq 0 \text{ along } \partial\Sigma.$$

Then Σ is conformally equivalent to a compact Riemann surface $\bar{\Sigma}$ with boundary and finitely many points removed, each associated to an end of the surface. Moreover,

$$\int_\Sigma R_M + H_\Sigma^2 + |A_\Sigma|^2 + \int_{\partial\Sigma} H_{\partial M} + H_\Sigma \cos\theta < \infty.$$

Topological configurations of stable capillary surfaces

Corollary (H&Sartunino,21')

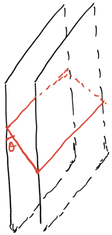
Let M be an oriented 3-manifold with smooth boundary and let Σ be a stable noncompact capillary surface immersed in M at a constant angle θ. Assume that $R_M + H_\Sigma^2 \geq 0$ in Σ and $H_{\partial M} + H_\Sigma \cos\theta \geq 0$ along $\partial\Sigma$. Then the compact Riemann surface $\overline{\Sigma}$ is a disk and the ends of Σ can only have one of the following configurations:

1. *There are two boundary ends and no interior ends.*
2. *There are no boundary ends and a single interior end.*
3. *There is a single boundary end and no interior ends.*

Moreover, if (1) or (2) holds, then Σ is totally geodesic, $R_M = 0$ in Σ and $H_{\partial M} = 0$ along $\partial\Sigma$.

Examples

- Let $M = \mathbb{R}^2 \times [0,1]$ and let Σ be an infinite flat strip in M meeting the boundary at a constant angle $\theta \in (0, \pi)$;



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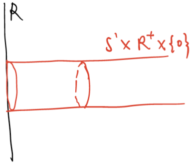

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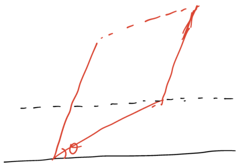
- Let $M = \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{R}$ and take $\Sigma = \mathbb{S}^1 \times \mathbb{R}^+ \times \{0\}$;
- Let M be a half-space of $\mathbb{R}^3$ and take Σ to be a half-plane.



Proof

Now let's prove the claim that

$$\text{weak stability} \longrightarrow \text{half-plane.}$$

On the contrary, suppose Σ is a non-flat weakly stable capillary surface in half-space. The claim follows if we construct a compactly supported piece-wise smooth function u such that

$$\int_\Sigma u = 0, \quad Q(u,u) = \int_\Sigma |\nabla u|^2 - |A_\Sigma|^2 u^2 - \int_{\partial\Sigma} \cot\theta A_\Sigma(\eta,\eta) u^2 < 0.$$

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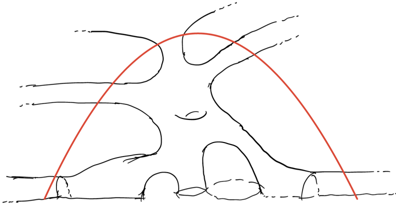
Construction:

- There exists a compact subset Σ_0 such that $\Sigma \setminus \Sigma_0 = (E_1 \cup \cdots \cup E_\ell) \cup (E_{\ell+1} \cup \cdots \cup E_k)$



Construct proper test function

- $E_1, \ldots, E_\ell$ are conformal equivalent to $\mathbb{S}^1 \times (0, \infty)$ and $E_{\ell+1}, \ldots, E_k$ are conformal equivalent to $\mathbb{S}^1_+ \times (0, \infty)$. Use coordinates on $(\theta, y) \in \mathbb{S}^1 \times (0, \infty)$.

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- Since $|A_\Sigma| \not\equiv 0$, let r be fixed such that

$$\int_{\Sigma_0} |A_\Sigma|^2 \geq \frac{12(k+\ell)\pi}{r(1-\cos\theta)}.$$

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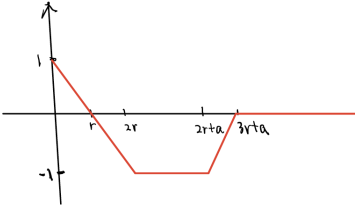
- For each $a > 0$, we can define functions $\varphi_i : E \to \mathbb{R}_+$ by

$$\varphi_i = \begin{cases} 1 - \frac{y}{r}, & 0 \leq y \leq 2r \\ -1, & 2r \leq y \leq 2r + a \\ -1 + \frac{y-(2r+a)}{r}, & 2r + a \leq y \leq 3r + a \\ 0, & 3r + a \leq y. \end{cases}$$



Let $\varphi_a : \Sigma \to \mathbb{R}_+$ be such that $\varphi = 1$ in Σ_0 and $\varphi = \varphi_i$ in the ends.

Construct proper test function

- Let $u = \frac{1}{\sin\theta} + \cot\theta\langle\nu, -E_3\rangle$. We can choose $a_0 > 0$ such that

$$\int_\Sigma u\varphi_{a_0} = 0$$

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- We show that $Q(u\varphi_{a_0}, u\varphi_{a_0}) < 0$.
- In fact,

$$\begin{cases} \Delta u + |A_\Sigma|^2 u = \frac{|A_\Sigma|^2}{\sin\theta}, & \text{in } \Sigma \\ \frac{\partial u}{\partial \eta} = \cot\theta A_\Sigma(\eta,\eta)u, & \text{on } \partial\Sigma. \end{cases}$$

Then

$$\begin{aligned} Q(u\varphi_{a_0}, u\varphi_{a_0}) &= \int_\Sigma u^2|\nabla\varphi_{a_0}|^2 - \varphi_{a_0}^2 u \frac{|A_\Sigma|^2}{\sin\theta} \\ &\leq \frac{4}{\sin^2\theta}\sum_{i=1}^n \int_{\Sigma_i} |\nabla\varphi_{a_0}|^2 - \frac{1-\cos\theta}{\sin^2\theta}\int_{\Sigma_0} |A_\Sigma|^2 \end{aligned}$$

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&= \frac{4}{\sin^2\theta}\left(3r\cdot 2\pi\cdot\frac{1}{r^2}\cdot\ell + 3r\cdot\pi\cdot\frac{1}{r^2}\cdot(n-\ell)\right) - \frac{1-\cos\theta}{\sin^2\theta}\int_{\Sigma_0}|A_\Sigma|^2 \\
&= \frac{12\pi^2(k+\ell)}{r\sin^2\theta} - \frac{1-\cos\theta}{\sin^2\theta}\int_{\Sigma_0}|A_\Sigma|^2 \\
&< 0.
\end{aligned}
$$

THANK YOU