

# A unified treatment for $L_p$ Brunn-Minkowski type inequalities

DU ZOU AND GE XIONG

A unified approach used to generalize classical Brunn-Minkowski type inequalities to  $L_p$  Brunn-Minkowski type inequalities, called the  $L_p$  transference principle, is refined in this paper. As illustrations of the effectiveness and practicability of this method, several new  $L_p$  Brunn-Minkowski type inequalities concerning the mixed volume, moment of inertia, quermassintegral, projection body and capacity are established.

## 1. Introduction

The classical Brunn-Minkowski inequality is a marvelous result of combining two basic notions: vector addition and volume, which reads as follows: If  $K$  and  $L$  are convex bodies (compact convex sets with nonempty interiors) in Euclidean  $n$ -space  $\mathbb{R}^n$  and  $\alpha \in (0, 1)$ , then

$$(1.1) \quad V_n((1-\alpha)K + \alpha L)^{\frac{1}{n}} \geq (1-\alpha)V_n(K)^{\frac{1}{n}} + \alpha V_n(L)^{\frac{1}{n}},$$

where

$$(1-\alpha)K + \alpha L = \{(1-\alpha)x + \alpha y : x \in K, y \in L\},$$

and  $V_n$  denotes the  $n$ -dimensional volume. Equality holds in (1.1) if and only if  $K$  and  $L$  are homothetic (i.e., they coincide up to a translation and a dilate). In brief, the functional  $V_n^{1/n}$  from  $\mathcal{K}^n$ , the class of convex bodies in  $\mathbb{R}^n$ , to  $[0, \infty)$  is concave.

As one of the cornerstones of convex geometry (see Gardner [11], Gruber [16], and Schneider [36]), the Brunn-Minkowski inequality is a powerful tool for solving many extremum problems dealing with important geometric quantities such as volume and surface area. It is also related closely with many other fundamental inequalities, such as the classical isoperimetric inequality, the Prékopa-Leindler inequality, the Sobolev inequality, and the

---

Research of the authors was supported by NSFC No. 11471206 and No. 11601399.

Brascamp-Lieb inequality. See, e.g., Barthe [1] and Bobkov and Ledoux [2]. For a more comprehensive understanding, we refer to the excellent survey of Gardner [10].

Nearly half a century ago, Firey [9] (see also Gardner, Hug, and Weil [14, p. 2311], Lutwak, Yang, and Zhang [30], and Schneider [36, Section 9.1]) introduced the  $L_p$  addition of convex bodies and established the  $L_p$  Brunn-Minkowski inequality for this new operation: If  $K$  and  $L$  are convex bodies in  $\mathbb{R}^n$  containing the origin in their interiors,  $p \in (1, \infty)$  and  $\alpha \in (0, 1)$ , then

$$(1.2) \quad V_n((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^{\frac{p}{n}} \geq (1 - \alpha)V_n(K)^{\frac{p}{n}} + \alpha V_n(L)^{\frac{p}{n}},$$

where  $(1 - \alpha) \cdot_p K = (1 - \alpha)^{1/p}K$ ,  $\alpha \cdot_p L = \alpha^{1/p}L$ , and

$$(1 - \alpha) \cdot_p K +_p \alpha \cdot_p L = \left\{ (1 - \beta)^{\frac{p-1}{p}}(1 - \alpha)^{\frac{1}{p}}x + \beta^{\frac{p-1}{p}}\alpha^{\frac{1}{p}}y : x \in K, y \in L, 0 \leq \beta \leq 1 \right\}.$$

Equality holds in (1.2) if and only if  $K$  and  $L$  are dilates. Write  $\mathcal{K}_o^n$  for the class of convex bodies with the origin in their interiors. In brief, the functional  $V_n^{p/n}$  from  $\mathcal{K}_o^n$  to  $[0, \infty)$  is concave.

Further developments of the  $L_p$  Brunn-Minkowski theory were greatly impelled by Lutwak [24, 25], who nearly set up a broad framework for the theory. A series of fundamental notions, geometric objects, and central results in the classical Brunn-Minkowski theory evolved into their  $L_p$  analogs. See, e.g., [26–30, 32, 34, 37–39].

In retrospect, we observe that to establish new Brunn-Minkowski type inequalities, we encounter essentially the following two general situations.

First, if a functional  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous of order  $j$ ,  $j \neq 0$ , is it the case that functional  $F^{1/j}$  is concave? Precisely, for  $K, L \in \mathcal{K}^n$  and  $\alpha \in (0, 1)$ , is it the case that

$$F((1 - \alpha)K + \alpha L)^{\frac{1}{j}} \geq (1 - \alpha)F(K)^{\frac{1}{j}} + \alpha F(L)^{\frac{1}{j}}?$$

Incidentally, we can list some beautiful confirmed examples within the classical Brunn-Minkowski theory, such as the classical mixed volumes (see Schneider [36, p. 406]), Hadwiger’s harmonic quermassintegrals (see, e.g., Hadwiger [17, p. 268] and Schneider [36, p. 514]), and Lutwak’s affine quermassintegrals (see, e.g., Gardner [10, p. 393], [13, p. 361] and Schneider [36, p. 515]). Also, some instances were discovered in other disciplines. For example, Brascamp and Lieb [4] established a Brunn-Minkowski type inequality for the first eigenvalue of the Laplace operator. Borell [3] proved a Brunn-Minkowski type inequality for the Newtonian capacity. See also Caffarelli,

Jerison, and Lieb [5], Colesanti and Salani [6], Gardner and Hartenstine [12], and the references within.

Second, assume that a functional  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous and concave, for  $K, L \in \mathcal{K}_o^n$  and  $\alpha \in (0, 1)$ . Is it the case that

$$F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \geq (1 - \alpha)F(K)^p + \alpha F(L)^p?$$

Obviously, if  $F = V_n^{1/n}$ , for  $p > 1$ , the  $L_p$  Brunn-Minkowski inequality is a confirmed case.

The prime motivation of this paper is to formulate and prove the following  $L_p$  transference principle.

**Theorem 1.1.** *Suppose that  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous, increasing and concave, and  $p \in (1, \infty)$ . If  $K, L \in \mathcal{K}_o^n$ , then*

$$F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \geq (1 - \alpha)F(K)^p + \alpha F(L)^p, \quad \text{for all } \alpha \in (0, 1).$$

*Furthermore, if  $F : \mathcal{K}_o^n \rightarrow (0, \infty)$  is strictly increasing, equality holds if and only if  $K$  and  $L$  are dilates.*

See Section 2 for the definitions of positive homogeneity and increasing property of a functional  $F$ .

In Section 3, we prove Theorem 1.1 and then dwell on the equality condition. It is observed that equality holds in the classical Brunn-Minkowski inequality if and only if the convex bodies are *homothetic*, while equality holds in the  $L_p$  Brunn-Minkowski inequality if and only if the convex bodies are *dilates*. We reveal the reason and characterize this phenomenon. See Theorem 3.4 and Theorem 3.5.

In Section 4, as illustrations of the effectiveness and practicability of our  $L_p$  transference principle, several new  $L_p$  Brunn-Minkowski type inequalities are established, which are concerned with the classical mixed volume, moment of inertia, affine quermassintegral, harmonic quermassintegral, projection body and capacity. For example, by using the  $L_p$  transference principle, we obtain the  $L_p$  capacity Brunn-Minkowski inequality directly.

**Theorem 1.2.** *Suppose that  $K, L \in \mathcal{K}_o^n$ ,  $1 \leq q < n$ , and  $1 < p < \infty$ . Then*

$$\text{Cap}_q(K+_p L)^{\frac{p}{n-q}} \geq \text{Cap}_q(K)^{\frac{p}{n-q}} + \text{Cap}_q(L)^{\frac{p}{n-q}},$$

*with equality if and only if  $K$  and  $L$  are dilates.*

## 2. Preliminaries

As usual,  $S^{n-1}$  denotes the unit sphere in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , and  $B^n$  denotes the unit ball in  $\mathbb{R}^n$ . If  $x, y \in \mathbb{R}^n$ , then  $x \cdot y$  denotes the inner product of  $x$  and  $y$ . If  $u \in S^{n-1}$ , then  $u^\perp$  denotes the  $(n-1)$ -dimensional subspace orthogonal to  $u$ . Write  $V_j$  for  $j$ -dimensional volume, where  $j = 1, \dots, n$ . As usual,  $\omega_j$  denotes the volume of  $j$ -dimensional unit Euclidean ball.

Write  $G_{n,j}$  for the Grassmann manifold of all  $j$ -dimensional linear subspaces of  $\mathbb{R}^n$ , which is equipped with Haar probability measure  $\mu_j$ . For  $K \in \mathcal{K}^n$ , let  $K|\xi$  be the orthogonal projection of  $K$  onto  $\xi \in G_{n,j}$ .

Each convex body  $K$  in  $\mathbb{R}^n$  is uniquely determined by its support function  $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ , which is defined by

$$h_K(x) = \max \{x \cdot y : y \in K\},$$

for  $x \in \mathbb{R}^n$ . For  $\alpha > 0$ , the body  $\alpha K = \{\alpha x : x \in K\}$  is called a dilate of  $K$ .

For  $K, L \in \mathcal{K}^n$ , their Minkowski sum is the convex body

$$K + L = \{x + y : x \in K, y \in L\}.$$

Let  $1 < p < \infty$ . The  $L_p$  sum of  $K, L \in \mathcal{K}_o^n$  is the convex body  $K +_p L$ , defined by

$$h_{K+_p L}(u)^p = h_K(u)^p + h_L(u)^p,$$

for  $u \in S^{n-1}$ . If  $p = \infty$ , the convex body  $K +_\infty L$  is defined by

$$h_{K+_\infty L}(u) = \max\{h_K(u), h_L(u)\},$$

for  $u \in S^{n-1}$ .

For  $\alpha > 0$  and  $K \in \mathcal{K}_o^n$ , the  $L_p$  scalar multiplication  $\alpha \cdot_p K$  is the convex body  $\alpha^{\frac{1}{p}} K$ .

Given a functional  $F : \mathcal{K}^n \rightarrow [0, \infty)$ , we say that  $F$  is

(1) *positively homogeneous*, provided

$$F(\alpha K) = \alpha F(K),$$

for  $K \in \mathcal{K}^n$  and  $\alpha > 0$ .

(2) *increasing*, provided

$$F(K) \leq F(L),$$

for  $K, L \in \mathcal{K}^n$  with  $K \subseteq L$ . Moreover, if the strict inclusion  $K \subsetneq L$  implies  $F(K) < F(L)$ , then  $F$  is *strictly increasing*.

(3)  $p$ -concave, provided

$$F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L) \geq ((1 - \alpha)F(K)^p + \alpha F(L)^p)^{\frac{1}{p}},$$

for  $K, L \in \mathcal{K}^n$  and  $\alpha \in (0, 1)$ . As usual, 1-concave is called concave for brevity.

Associated with a functional  $F : \mathcal{K}_o^n \rightarrow [0, \infty)$ ,  $p \in [1, \infty)$  and  $K, L \in \mathcal{K}_o^n$ , it is convenient to introduce a function  $F_{p;K,L} : [0, 1] \rightarrow [0, \infty)$  defined by

$$F_{p;K,L}(\alpha) = F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p,$$

for  $\alpha \in [0, 1]$ .

The next lemma shows that the  $p$ -concavity of  $F$  and the concavity of  $F_{p;K,L}$  are actually equivalent.

**Lemma 2.1.** *Suppose that  $K, L \in \mathcal{K}_o^n$  and  $1 \leq p < \infty$ . Then,  $F$  is  $p$ -concave, if and only if  $F_{p;K,L}$  is concave.*

*Proof.* Let  $\lambda, \alpha, \beta \in [0, 1]$ . Assume that  $F_{p;K,L}$  is concave. Then

$$F_{p;K,L}((1 - \lambda)\alpha + \lambda\beta) \geq (1 - \lambda)F_{p;K,L}(\alpha) + \lambda F_{p;K,L}(\beta).$$

Taking  $\alpha = 0$  and  $\beta = 1$ , we obtain

$$F_{p;K,L}(\lambda) \geq (1 - \lambda)F_{p;K,L}(0) + \lambda F_{p;K,L}(1),$$

i.e.,

$$F((1 - \lambda) \cdot_p K +_p \lambda \cdot_p L)^p \geq (1 - \lambda)F(K)^p + \lambda F(L)^p,$$

which shows that  $F$  is  $p$ -concave.

Conversely, assume that  $F$  is  $p$ -concave. Let

$$\begin{aligned} K_\alpha &= (1 - \alpha) \cdot_p K +_p \alpha \cdot_p L, \\ K_\beta &= (1 - \beta) \cdot_p K +_p \beta \cdot_p L, \\ Q &= (1 - ((1 - \lambda)\alpha + \lambda\beta)) \cdot_p K +_p ((1 - \lambda)\alpha + \lambda\beta) \cdot_p L. \end{aligned}$$

Then

$$\begin{aligned} h_Q^p &= (1 - ((1 - \lambda)\alpha + \lambda\beta)) h_K^p + ((1 - \lambda)\alpha + \lambda\beta) h_L^p \\ &= ((1 - \lambda)(1 - \alpha) + \lambda(1 - \beta)) h_K^p + ((1 - \lambda)\alpha + \lambda\beta) h_L^p \\ &= (1 - \lambda) ((1 - \alpha) h_K^p + \alpha h_L^p) + \lambda ((1 - \beta) h_K^p + \beta h_L^p) \\ &= (1 - \lambda) h_{K_\alpha}^p + \lambda h_{K_\beta}^p, \end{aligned}$$

which implies that  $Q = (1 - \lambda) \cdot_p K_\alpha +_p \lambda \cdot_p K_\beta$ .

Thus, from the  $p$ -concavity of  $F$ , it follows that

$$\begin{aligned} F_{p;K,L}((1-\lambda)\alpha + \lambda\beta) &= F(Q)^p \\ &= F((1-\lambda)\cdot_p K_\alpha + \lambda\cdot_p K_\beta)^p \\ &\geq (1-\lambda)F(K_\alpha)^p + \lambda F(K_\beta)^p \\ &= (1-\lambda)F_{p;K,L}(\alpha) + \lambda F_{p;K,L}(\beta), \end{aligned}$$

which shows that  $F_{p;K,L}$  is concave. □

The following lemma will be used in Section 3.

**Lemma 2.2.** *Suppose  $K, L \in \mathcal{K}_O^n$ ,  $1 < p < \infty$ , and  $0 < \alpha < 1$ . Then*

$$(1-\alpha)\cdot_p K +_p \alpha\cdot_p L \supseteq (1-\alpha)K + \alpha L,$$

*with equality if and only if  $K = L$ .*

*Proof.* From the definition of  $(1-\alpha)\cdot_p K +_p \alpha\cdot_p L$  and the strict convexity of  $f(t) = t^p$  in  $t \in (0, \infty)$ , it follows that for  $u \in S^{n-1}$ ,

$$\begin{aligned} h_{(1-\alpha)\cdot_p K +_p \alpha\cdot_p L}(u) &= ((1-\alpha)h_K(u)^p + \alpha h_L(u)^p)^{\frac{1}{p}} \\ &\geq (1-\alpha)h_K(u) + \alpha h_L(u) \\ &= h_{(1-\alpha)K + \alpha L}(u). \end{aligned}$$

Equality holds in the second line if and only if  $h_K(u) = h_L(u)$ , for all  $u \in S^{n-1}$ , and therefore if and only if  $K = L$ . □

The next lemma will be needed in Section 3 and Section 4.

**Lemma 2.3.** *Suppose  $K, L \in \mathcal{K}^n$  and  $j \in \{1, \dots, n-1\}$ . If  $K \subsetneq L$ , then there exists a  $\mu_j$ -measurable subset  $G \subseteq G_{n,j}$  such that  $\mu_j(G) > 0$  and*

$$V_j(K|\xi) < V_j(L|\xi), \quad \text{for all } \xi \in G.$$

*Proof.* Recall that  $h_K$  and  $h_L$  are continuous on  $S^{n-1}$ . So, the assumption  $K \subsetneq L$  implies that there exists an open geodesic ball  $U$  in  $S^{n-1}$  such that

$\mathcal{H}^{n-1}(U) > 0$  and  $h_K(u) < h_L(u)$ , for all  $u \in U$ . Let

$$G = \{\xi \in G_{n,j} : \xi \cap U \neq \emptyset\}.$$

Next, we aim to show that

$$\mu_j(G) > 0.$$

Note that when its points are antipodally identified, the sphere  $S^{n-1}$  is identified with an  $(n-1)$ -dimensional elliptic space of constant curvature one. The measure  $\mu_j(G)$  can be represented as

$$\mu_j(G) = \frac{(n-j)!j!\omega_j \cdots \omega_1}{n!\omega_n \cdots \omega_{n-j+1}} \int_{G \cap L_{j-1} \neq \emptyset} dL_{j-1},$$

where  $dL_{j-1}$  denotes the kinematic density of a moving  $(j-1)$ -dimensional plane  $L_{j-1}$  in the elliptic space  $S^{n-1}$ . For more details, see Santaló [35, pp. 299–310]. Let  $r$  be the geodesic radius of  $U$ . Then equation (17.52) in [35] shows that

$$\int_{G \cap L_{j-1} \neq \emptyset} dL_{j-1} = \frac{(n-1)!\omega_{n-1} \cdots \omega_{n-j}}{(j-1)!(n-j-1)!\omega_{j-1} \cdots \omega_1} \int_0^r (\cos t)^{j-1} (\sin t)^{n-j-1} dt.$$

Thus,  $\mu_j(G) > 0$ .

Finally, let  $\xi \in G$  and  $u \in \xi \cap U$ . Since  $\xi \cap U \neq \emptyset$ , the definition of  $U$  implies that

$$h_{K|\xi}(u) = h_K(u) < h_L(u) = h_{L|\xi}(u).$$

Since  $h_{K|\xi}$  and  $h_{L|\xi}$  are continuous on  $S^{n-1} \cap \xi$ , and  $h_{K|\xi} \leq h_{L|\xi}$ , we have  $K|\xi \subsetneq L|\xi$ . This, combined with the convexity of  $K|\xi$  and  $L|\xi$ , implies  $V_j(K|\xi) < V_j(L|\xi)$ .  $\square$

### 3. The $L_p$ transference principle

#### 3.1. Statement

In the following, we prove the  $L_p$  transference principle.

**Theorem 3.1.** *Suppose that  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous, increasing and concave, and  $p \in (1, \infty)$ . If  $K, L \in \mathcal{K}_o^n$ , then*

$$(3.1) \quad \begin{aligned} & F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \\ & \geq (1 - \alpha)F(K)^p + \alpha F(L)^p, \quad \text{for all } \alpha \in (0, 1). \end{aligned}$$

Furthermore, if  $F : \mathcal{K}_o^n \rightarrow [0, \infty)$  is strictly increasing, equality holds in (3.1) if and only if  $K$  and  $L$  are dilates.

*Proof.* If  $F(K)F(L) = 0$ , then inequality (3.1) holds. To see this, assume  $F(K) = 0$ . Then the definitions of  $(1 - \alpha) \cdot_p K +_p \alpha \cdot_p L$  and  $\alpha \cdot_p L$  directly imply that

$$(1 - \alpha) \cdot_p K +_p \alpha \cdot_p L \supseteq \alpha \cdot_p L = \alpha^{\frac{1}{p}} L.$$

From the monotonicity and positive homogeneity of  $F$ , we have

$$F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \geq F\left(\alpha^{\frac{1}{p}} L\right)^p = \alpha F(L)^p.$$

So, we assume that  $F(K)F(L) > 0$ . Then inequality (3.1) is equivalent to

$$\frac{F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)}{[(1 - \alpha)F(K)^p + \alpha F(L)^p]^{\frac{1}{p}}} \geq 1.$$

By the positive homogeneity of  $F$ , this is equivalent to

$$F\left(\frac{(1 - \alpha) \cdot_p K +_p \alpha \cdot_p L}{[(1 - \alpha)F(K)^p + \alpha F(L)^p]^{\frac{1}{p}}}\right) \geq 1.$$

From the definition of  $L_p$  scalar multiplication, this is equivalent to

$$(3.2) \quad \begin{aligned} & F\left(\left(\frac{1 - \alpha}{(1 - \alpha)F(K)^p + \alpha F(L)^p}\right) \cdot_p K \right. \\ & \quad \left. +_p \left(\frac{\alpha}{(1 - \alpha)F(K)^p + \alpha F(L)^p}\right) \cdot_p L\right) \\ & \geq 1. \end{aligned}$$



Now, again using the definition of  $L_p$  scalar multiplication, we have

$$\begin{aligned} & \left( \frac{1-\alpha}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p K +_p \left( \frac{\alpha}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p L \\ &= \left( \frac{(1-\alpha)F(K)^p}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p \left( \frac{K}{F(K)} \right) \\ & \quad +_p \left( \frac{\alpha F(L)^p}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p \left( \frac{L}{F(L)} \right) \\ &= (1-\alpha') \cdot_p \left( \frac{K}{F(K)} \right) +_p \alpha' \cdot_p \left( \frac{L}{F(L)} \right), \end{aligned}$$

where

$$\alpha' = \frac{\alpha F(L)^p}{(1-\alpha)F(K)^p + \alpha F(L)^p}.$$

By Lemma 2.2, we have

$$(1-\alpha') \cdot_p \left( \frac{K}{F(K)} \right) +_p \alpha' \cdot_p \left( \frac{L}{F(L)} \right) \supseteq (1-\alpha') \left( \frac{K}{F(K)} \right) + \alpha' \left( \frac{L}{F(L)} \right).$$

Hence, from the definition of  $\alpha'$ , the above inclusion together with the monotonicity of  $F$ , the concavity of  $F$ , and the positive homogeneity of  $F$ , it follows that

$$\begin{aligned} & F \left( \left( \frac{1-\alpha}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p K +_p \left( \frac{\alpha}{(1-\alpha)F(K)^p + \alpha F(L)^p} \right) \cdot_p L \right) \\ &= F \left( (1-\alpha') \cdot_p \left( \frac{K}{F(K)} \right) +_p \alpha' \cdot_p \left( \frac{L}{F(L)} \right) \right) \\ &\geq F \left( (1-\alpha') \left( \frac{K}{F(K)} \right) + \alpha' \left( \frac{L}{F(L)} \right) \right) \\ &\geq (1-\alpha') F \left( \frac{K}{F(K)} \right) + \alpha' F \left( \frac{L}{F(L)} \right) \\ &= (1-\alpha') + \alpha' \\ &= 1. \end{aligned}$$

This establishes inequality (3.2). Therefore, inequality (3.1) holds.

Finally, under the additional assumption that  $F$  is strictly increasing on  $\mathcal{K}_o^n$ , we aim to prove the equality condition. Note that the strict monotonicity and positive homogeneity of  $F$  imply that  $F$  is positive on  $\mathcal{K}_o^n$ .

Assume that equality holds in (3.1). Then

$$\begin{aligned} & F\left((1 - \alpha') \cdot_p \left(\frac{K}{F(K)}\right) +_p \alpha' \cdot_p \left(\frac{L}{F(L)}\right)\right) \\ &= F\left((1 - \alpha') \left(\frac{K}{F(K)}\right) + \alpha' \left(\frac{L}{F(L)}\right)\right). \end{aligned}$$

By the strict monotonicity of  $F$ , this implies that

$$(1 - \alpha') \cdot_p \left(\frac{K}{F(K)}\right) +_p \alpha' \cdot_p \left(\frac{L}{F(L)}\right) = (1 - \alpha') \left(\frac{K}{F(K)}\right) + \alpha' \left(\frac{L}{F(L)}\right).$$

This equation, combined with Lemma 2.2, implies that

$$\frac{K}{F(K)} = \frac{L}{F(L)},$$

which shows that  $K$  and  $L$  are dilates.

Conversely, assume that  $K$  and  $L$  are dilates, say  $K = \beta L$ , for some constant  $\beta > 0$ . From the definition of  $L_p$  combination of convex bodies,

$$\begin{aligned} (1 - \alpha) \cdot_p K +_p \alpha \cdot_p L &= (1 - \alpha) \cdot_p (\beta L) +_p \alpha \cdot_p L \\ &= (1 - \alpha)^{\frac{1}{p}} \beta L +_p \alpha^{\frac{1}{p}} L \\ &= ((1 - \alpha)\beta^p + \alpha)^{\frac{1}{p}} L. \end{aligned}$$

From this and the positive homogeneity of  $F$ , it follows that

$$\begin{aligned} F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p &= F\left(\left((1 - \alpha)\beta^p + \alpha\right)^{\frac{1}{p}} L\right)^p \\ &= (1 - \alpha)\beta^p F(L)^p + \alpha F(L)^p \\ &= (1 - \alpha)F(\beta L)^p + \alpha F(L)^p \\ &= (1 - \alpha)F(K)^p + \alpha F(L)^p, \end{aligned}$$

which shows that equality holds in (3.1). □

Theorem 3.1 immediately yields the following corollary.

**Corollary 3.2.** *Suppose that  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous, increasing and concave, and  $p \in (1, \infty)$ . If  $K, L \in \mathcal{K}_o^n$ , then*

$$(3.3) \quad F(K +_p L)^p \geq F(K)^p + F(L)^p.$$

*Furthermore, if  $F : \mathcal{K}_o^n \rightarrow [0, \infty)$  is strictly increasing, equality holds in (3.3) if and only if  $K$  and  $L$  are dilates.*

*Proof.* Let  $\alpha \in (0, 1)$ . From the definition of  $L_p$  scalar multiplication, Theorem 3.1 and the positive homogeneity of  $F$ , it follows that

$$\begin{aligned} F(K+_pL)^p &= F\left((1-\alpha)\cdot_p((1-\alpha)^{-\frac{1}{p}}K)+_p\alpha\cdot_p(\alpha^{-\frac{1}{p}}L)\right)^p \\ &\geq (1-\alpha)F\left((1-\alpha)^{-\frac{1}{p}}K\right)^p + \alpha F\left(\alpha^{-\frac{1}{p}}L\right)^p \\ &= F(K)^p + F(L)^p, \end{aligned}$$

which is precisely (3.3). If the monotonicity of  $F$  is strict, equality holds in the second line if and only if  $(1-\alpha)^{-1/p}K$  and  $\alpha^{-1/p}L$  are dilates, and therefore if and only if  $K$  and  $L$  are dilates.  $\square$

For convex bodies  $K, L \in \mathcal{K}^n$ , their Hausdorff distance is

$$\delta_H(K, L) = \max\{|h_K(u) - h_L(u)| : u \in S^{n-1}\}.$$

From the definition of  $L_p$  addition, we have  $K+_pL \rightarrow K+_∞L$ , as  $p \rightarrow \infty$ . Assume functional  $F$  in Theorem 3.1 (or Corollary 3.2) is continuous with respect to  $\delta_H$ . Then by the monotonicity of  $F$  and the definition of  $K+_∞L$ , letting  $p \rightarrow \infty$ , inequality (3.1) (or (3.3)) yields

$$F(K+_∞L) \geq \max\{F(K), F(L)\}.$$

Furthermore, if  $F$  is strictly increasing, equality holds if and only if either  $K$  or  $L$  is a subset of the other set.

By the  $L_p$  transference principle, we can immediately obtain the  $L_p$  Brunn-Minkowski type inequality for quermassintegrals first established by Firey [9].

**Example 3.3.** For a convex body  $K \in \mathcal{K}^n$ , its quermassintegrals  $W_0(K)$ ,  $W_1(K), \dots, W_{n-1}(K)$  are defined by  $W_0(K) = V_n(K)$ , and

$$W_{n-j}(K) = \frac{\omega_n}{\omega_j} \int_{G_{n,j}} V_j(K|\xi) d\mu_j(\xi), \quad j = 1, \dots, n-1.$$

Now, the functional

$$W_{n-j}^{\frac{1}{j}} : \mathcal{K}^n \rightarrow (0, \infty), \quad K \mapsto W_{n-j}(K)^{\frac{1}{j}}$$

is positively homogeneous and increasing. A Brunn-Minkowski inequality for  $W_{n-j}$  reads as follows: If  $K, L \in \mathcal{K}^n$  and  $0 < \alpha < 1$ , then

$$W_{n-j}((1-\alpha)K + \alpha L)^{\frac{1}{j}} \geq (1-\alpha)W_{n-j}(K)^{\frac{1}{j}} + \alpha W_{n-j}(L)^{\frac{1}{j}},$$

with equality if and only if  $K$  and  $L$  are homothetic. So,  $W_{n-j}^{1/j}$  is concave. See, e.g., Gardner [10, p. 393].

Thus, by Corollary 3.2, we directly obtain Firey’s  $L_p$  Brunn-Minkowski inequality for quermassintegrals: If  $K, L \in \mathcal{K}_o^n$  and  $1 < p < \infty$ , then

$$(3.4) \quad W_{n-j}(K +_p L)^{\frac{p}{j}} \geq W_{n-j}(K)^{\frac{p}{j}} + W_{n-j}(L)^{\frac{p}{j}}.$$

From the definition of  $W_{n-j}$  and Lemma 2.3, we know that  $W_{n-j}^{1/j}$  is strictly increasing. Hence, equality holds in (3.4) if and only if  $K$  and  $L$  are dilates.

### 3.2. Characterizations of equality conditions

For many  $L_p$  Brunn-Minkowski type inequalities, equality only occurs when the convex bodies are dilates. This phenomenon can be completely characterized.

**Theorem 3.4.** *Suppose that  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is positively homogeneous, increasing and concave, and  $p \in (1, \infty)$ . Then the following assertions are equivalent.*

- (1) *For  $K, L \in \mathcal{K}_o^n$ , the function  $F_{p;K,L}$  is affine if and only if  $K$  and  $L$  are dilates.*
- (2) *When restricted to  $\mathcal{K}_o^n$ , the functional  $F$  is strictly increasing.*

*Proof.* The implication “(2)  $\Rightarrow$  (1)” is shown by Theorem 3.1. Next, we prove the implication “(1)  $\Rightarrow$  (2)” by contradiction. Assume that there exist  $K_0, L_0 \in \mathcal{K}_o^n$  such that  $K_0 \subsetneq L_0$  but  $F(K_0) = F(L_0)$ .

For any  $\alpha \in (0, 1)$ , let  $K_\alpha = (1 - \alpha) \cdot_p K_0 +_p \alpha \cdot_p L_0$ . Then

$$K_0 \subset K_\alpha \subset L_0.$$

By the monotonicity of  $F$ , we have

$$F(K_0) \leq F(K_\alpha) \leq F(L_0).$$

This, together with the assumption  $F(K_0) = F(L_0)$ , yields

$$F(K_\alpha)^p = (1 - \alpha)F(K_0)^p + \alpha F(L_0)^p.$$

Thus, assertion (1) implies that  $K_0$  and  $L_0$  are dilates, say  $K_0 = \beta L_0$ , for some  $\beta > 0$ . From the positive homogeneity of  $F$  and the assumption

$F(K_0) = F(L_0)$  again, we have

$$F(K_0) = F(\beta L_0) = \beta F(L_0) = F(L_0).$$

Note that  $F$  is strictly positive. So,  $\beta = 1$ , and therefore

$$K_0 = L_0,$$

which contradicts the assumption that  $K_0 \neq L_0$ . □

We say  $F$  is *translation invariant* if  $F(K + x) = F(x)$  for all  $x \in \mathbb{R}^n$ .

**Theorem 3.5.** *Suppose that  $F : \mathcal{K}^n \rightarrow [0, \infty)$  is translation invariant, positively homogeneous, increasing and concave, and  $p \in (1, \infty)$ . Then the following assertion (1) implies assertion (2).*

(1) *For  $K, L \in \mathcal{K}^n$ , the function  $F_{1;K,L}$  is affine if and only if  $K$  and  $L$  are homothetic.*

(2) *For  $K, L \in \mathcal{K}_o^n$ , the function  $F_{p;K,L}$  is affine if and only if  $K$  and  $L$  are dilates.*

*Proof.* Suppose that (1) holds but (2) does not hold, specifically that there exists an  $\alpha_0 \in (0, 1)$  and  $K_0, L_0 \in \mathcal{K}_o^n$ , which are not dilates, such that

$$F((1 - \alpha_0) \cdot_p K_0 +_p \alpha_0 \cdot_p L_0)^p = (1 - \alpha_0)F(K_0)^p + \alpha_0 F(L_0)^p.$$

Let

$$\begin{aligned} \alpha_1 &= \frac{\alpha_0 F(L_0)^p}{(1 - \alpha_0)F(K_0)^p + \alpha_0 F(L_0)^p}, \\ A_0 &= \frac{K_0}{F(K_0)}, \\ A_1 &= \frac{L_0}{F(L_0)}, \end{aligned}$$

and

$$A^{(r)} = (1 - \alpha_1) \cdot_r A_0 +_r \alpha_1 \cdot_r A_1, \quad \text{for } 1 \leq r \leq p.$$

Clearly,  $F(A_0) = F(A_1) = 1$ .

From Lemma 2.2 and the monotonicity of  $F$ , we have

$$\begin{aligned} F((1 - \alpha_1) \cdot_p A_0 + \alpha_1 \cdot_p A_1) &\geq F((1 - \alpha_1)A_0 + \alpha_1 A_1) \\ &\geq (1 - \alpha_1)F(A_0) + \alpha_1 F(A_1). \end{aligned}$$

Meanwhile, the assumptions yield

$$(3.5) \quad F((1 - \alpha_1) \cdot_p A_0 + \alpha_1 \cdot_p A_1) = (1 - \alpha_1)F(A_0) + \alpha_1 F(A_1).$$

Hence,

$$(3.6) \quad F((1 - \alpha_1)A_0 + \alpha_1 A_1) = (1 - \alpha_1)F(A_0) + \alpha_1 F(A_1).$$

Thus, from assertion (1), there exist  $\lambda_0 > 0$  and  $x_0 \in \mathbb{R}^n$  such that

$$A_0 = \lambda_0 A_1 + x_0.$$

But then, from the positive homogeneity, translation invariance and strict positivity of  $F$ , we have

$$\begin{aligned} F(A_1) &= F(A_0) \\ &= F(\lambda_0 A_1 + x_0) \\ &= \lambda_0 F(A_1). \end{aligned}$$

Thus,  $\lambda_0 = 1$ .

With  $A_0 = A_1 + x_0$  in hand, for  $u \in S^{n-1}$  and  $1 \leq r \leq p$ , we have

$$h_{A^{(r)}}(u) = [(1 - \alpha_1)(h_{A_1}(u) + x_0 \cdot u)^r + \alpha_1 h_{A_1}(u)^r]^{\frac{1}{r}}$$

and

$$h_{A_1}(u) + x_0 \cdot u > 0.$$

Thus, for  $u \in S^{n-1}$ , two observations are in order.

First, if  $u \cdot x_0 = 0$ , then for  $1 \leq r_1 \leq r_2 \leq p$ ,

$$h_{A^{(1)}}(u) = h_{A^{(r_1)}}(u) = h_{A^{(r_2)}}(u) = h_{A^{(p)}}(u).$$

Second, if  $u \cdot x_0 \neq 0$ , then the strict convexity of power functions implies that for  $1 < r_1 < r_2 < p$ ,

$$h_{A^{(1)}}(u) < h_{A^{(r_1)}}(u) < h_{A^{(r_2)}}(u) < h_{A^{(p)}}(u).$$

Consequently, for  $1 \leq r_1 < r_2 \leq p$ , we conclude that

- (a)  $A^{(1)} \subseteq A^{(r_1)} \subseteq A^{(r_2)} \subseteq A^{(p)}$  and
- (b)  $A^{(r_1)}$  and  $A^{(r_2)}$  are not homothetic.

Now, from (a) it follows that for  $\beta \in (0, 1)$ ,

$$A^{(1)} \subseteq (1 - \beta)A^{(1)} + \beta A^{(p)} \subseteq A^{(p)}.$$

Meanwhile, by (3.5) and (3.6), we have

$$F((1 - \alpha_1)A_0 + \alpha_1 A_1) = F((1 - \alpha_1) \cdot_p A_0 + \alpha_1 \cdot_p A_1),$$

i.e.,

$$F(A^{(1)}) = F(A^{(p)}).$$

Thus, by the monotonicity of  $F$ , we obtain

$$\begin{aligned} F((1 - \beta)A^{(1)} + \beta A^{(p)}) &= F(A^{(1)}) \\ &= F(A^{(p)}) \\ &= (1 - \beta)F(A^{(1)}) + \beta F(A^{(p)}). \end{aligned}$$

Hence, from assertion (1),  $A^{(1)}$  and  $A^{(p)}$  are homothetic. However, this contradicts (b). □

**Example 3.6.** The implication “(2)  $\Rightarrow$  (1)” stated in Theorem 3.5 does not always hold. This is demonstrated by the following example, which deals with mean width of convex bodies.

Let  $n \geq 2$  and  $r < 1, r \neq 0$ . Define  $F : \mathcal{K}^n \rightarrow [0, \infty)$  by

$$F(K) = \left( \int_{S^{n-1}} w_K(u)^r d\mathcal{H}^{n-1}(u) \right)^{\frac{1}{r}},$$

where  $w_K(u) = h_K(u) + h_K(-u)$ , for  $u \in S^{n-1}$ , is the width function of  $K$ .

It is obvious that  $F$  is translation invariant and positively homogeneous. Minkowski’s integral inequality (see [19, Theorem 198]) directly yields

$$F((1 - \alpha)K + \alpha L) \geq (1 - \alpha)F(K) + \alpha F(L), \quad \text{for } \alpha \in (0, 1),$$

with equality if and only if  $w_K = \lambda w_L$  for some constant  $\lambda > 0$ . Note that this may hold without  $K$  and  $L$  being homothetic, for example if  $K = B^n$  and  $L$  is a non-spherical convex body of the same constant width.

Thus, by the  $L_p$  transference principle, we obtain

$$(3.7) \quad F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \geq (1 - \alpha)F(K)^p + \alpha F(L)^p,$$

for any  $K, L \in \mathcal{K}_o^n$ ,  $\alpha \in (0, 1)$ , and  $p \in (1, \infty)$ .

Finally, we prove that equality holds in (3.7) if and only if  $K$  and  $L$  are dilates. By Theorem 3.4, it suffices to prove that  $F$  is strictly increasing.

Note that  $w_K = h_{K-K}$ , for  $K \in \mathcal{K}_o^n$ , and  $(K - K) \subset (L - L)$ , for any  $L \in \mathcal{K}_o^n$  containing  $K$ . Hence, if  $K \subsetneq L$ , then from continuity of support functions, there is a nonempty open subset  $U \subseteq S^{n-1}$ , such that  $w_K(u) < w_L(u)$ , for all  $u \in U$ . Thus,  $F(K) < F(L)$ , for  $K, L \in \mathcal{K}_o^n$  with  $K \subsetneq L$ . That is, the functional  $F$  is strictly increasing on  $\mathcal{K}_o^n$ .

Hence, the functional  $F$  has the required properties.

### 4. Applications of the $L_p$ transference principle

In this section, we aim to demonstrate the effectiveness and practicability of the  $L_p$  transference principle. As illustrations, several new  $L_p$  Brunn-Minkowski type inequalities are established.

#### 4.1. An application to mixed volumes

The mixed volume  $V : \mathcal{K}^n \rightarrow [0, \infty)$  is a nonnegative and symmetric functional such that

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \dots, i_m=1}^m V(K_{i_1}, \dots, K_{i_m}) \lambda_{i_1} \cdots \lambda_{i_m},$$

for  $K_1, \dots, K_m \in \mathcal{K}^n$  and  $\lambda_1, \dots, \lambda_m > 0$ . See, e.g., Schneider [36, Chapter 5].

Write  $V(K, j; K_{j+1}, \dots, K_n)$  for mixed volume  $V(K, \dots, K, K_{j+1}, \dots, K_n)$  with  $j$  copies of  $K$ . If  $j = n$ , then  $V(K, j; K_{j+1}, \dots, K_n)$  is  $V_n(K)$ . The classical Brunn-Minkowski inequality has a natural extension to mixed volumes as follows (see, e.g., Schneider [36, Theorems 7.4.5, 7.4.6 and 7.6.9]).

**Proposition 4.1.** *Suppose  $K, L, K_j, \dots, K_n \in \mathcal{K}^n$ , and  $j \in \{2, \dots, n\}$ . Then*

$$V(K + L, j; K_{j+1}, \dots, K_n)^{\frac{1}{j}} \geq V(K, j; K_{j+1}, \dots, K_n)^{\frac{1}{j}} + V(L, j; K_{j+1}, \dots, K_n)^{\frac{1}{j}}.$$



If  $j = n$ , or if  $2 \leq j \leq n - 1$  and  $K_{j+1}, \dots, K_n$  are smooth, then equality holds if and only if  $K$  and  $L$  are homothetic.

From Proposition 4.1 and the  $L_p$  transference principle, we obtain the following result.

**Theorem 4.2.** *Suppose  $K, L \in \mathcal{K}_o^n$ ,  $K_j, \dots, K_n \in \mathcal{K}^n$ ,  $p \in (1, \infty)$ , and  $j \in \{2, \dots, n\}$ . Then*

$$V(K+_p L, j; K_{j+1}, \dots, K_n)^{\frac{p}{j}} \geq V(K, j; K_{j+1}, \dots, K_n)^{\frac{p}{j}} + V(L, j; K_{j+1}, \dots, K_n)^{\frac{p}{j}}.$$

If  $j = n$ , or if  $2 \leq j \leq n - 1$  and  $K_{j+1}, \dots, K_n$  are smooth, then equality holds if and only if  $K$  and  $L$  are dilates.

If  $j = n$ , then the previous inequality reduces to (1.2). If  $1 \leq j \leq n - 1$  and  $K_{j+1} = \dots = K_n = B^n$ , then the previous inequality becomes (3.4).

*Proof.* We only need consider the case  $1 \leq j \leq n - 1$ . For  $K \in \mathcal{K}^n$ , define

$$F(K) = V(K, j; K_{j+1}, \dots, K_n)^{\frac{1}{j}}.$$

Then  $F$  is positively homogeneous and increasing (see, e.g., Schneider [36, (5.25), p. 282]). Since convex bodies are  $n$ -dimensional, from Theorem 5.1.8 of Schneider [36, p. 283],  $F$  is strictly positive. Meanwhile, Proposition 4.1 implies that  $F$  is concave.

Hence, from the  $L_p$  transference principle, it follows that

$$(4.1) \quad F((1 - \alpha) \cdot_p K +_p \alpha \cdot_p L)^p \geq (1 - \alpha)F(K)^p + \alpha F(L)^p,$$

for  $K, L \in \mathcal{K}_o^n$  and  $0 < \alpha < 1$ .

Assume  $2 \leq j \leq n - 1$ , and the bodies  $K_{j+1}, \dots, K_n$  are smooth. Note that  $F$  is translation invariant. Thus, by Proposition 4.1 and Theorem 3.5, equality holds in (4.1) if and only if  $K$  and  $L$  are dilates.  $\square$

### 4.2. An application to moments of inertia

From classic mechanics, we know that for each convex body  $K$  in  $\mathbb{R}^n$ , its moment of inertia,  $I(K)$ , is defined by

$$I(K) = \int_K |x - c_K|^2 dx,$$

where  $c_K$  denotes the centroid of  $K$ .

Proposition 4.3 was originally established by Hadwiger [17].

**Proposition 4.3.** *Suppose  $K, L \in \mathcal{K}^n$ . Then*

$$I(K + L)^{\frac{1}{n+2}} \geq I(K)^{\frac{1}{n+2}} + I(L)^{\frac{1}{n+2}}.$$

From Proposition 4.3 and the  $L_p$  transference principle, we obtain the following result.

**Theorem 4.4.** *Suppose that  $K, L \in \mathcal{K}_o^n$  are origin-symmetric and  $1 < p < \infty$ . Then*

$$(4.2) \quad I(K +_p L)^{\frac{p}{n+2}} \geq I(K)^{\frac{p}{n+2}} + I(L)^{\frac{p}{n+2}},$$

with equality if and only if  $K$  and  $L$  are dilates.

*Proof.* For  $K \in \mathcal{K}^n$ , define

$$F(K) = I(K)^{\frac{1}{n+2}}.$$

Obviously,  $F$  is positively homogeneous. From Proposition 4.3,  $F$  is concave. Moreover, if  $K$  is origin-symmetric, then the centroid  $c_K$  of  $K$  is at the origin, and then

$$F(K) = \left( \int_K |x|^2 dx \right)^{\frac{1}{n+2}}.$$

When the domain of  $F$  is restricted to the subset  $\mathcal{K}_{o,s}^n \subseteq \mathcal{K}_o^n$ , the class of origin-symmetric convex bodies, then  $F : \mathcal{K}_{o,s}^n \rightarrow (0, \infty)$  is strictly increasing.

Hence, from the  $L_p$  transference principle, we obtain the theorem.  $\square$

For an origin-symmetric convex body  $K$ , its isotropic constant  $L_K$  is defined by

$$(4.3) \quad L_K^2 = \frac{1}{n} \min \left\{ \frac{I(TK)}{V_n(K)^{\frac{n+2}{n}}} : T \in \text{SL}(n) \right\}.$$

For more information on isotropic constants, we refer to Milman and Pajor [31].

From (4.2) and (4.3), we obtain the following result.

**Corollary 4.5.** *Suppose that  $K_0, K_1 \in \mathcal{K}_o^n$  are origin-symmetric, and  $1 \leq p < \infty$ . Then*

$$V_n(K_0+_pK_1)^{\frac{p}{n}}L_{K_0+_pK_1}^{\frac{2p}{n+2}} \geq V_n(K_0)^{\frac{p}{n}}L_{K_0}^{\frac{2p}{n+2}} + V_n(K_1)^{\frac{p}{n}}L_{K_1}^{\frac{2p}{n+2}}.$$

### 4.3. An application to affine quermassintegrals

For a convex body  $K \in \mathcal{K}^n$ , Hadwiger [18, p. 267] introduced the harmonic quermassintegrals  $\hat{W}_0(K), \hat{W}_1(K), \dots, \hat{W}_{n-1}(K)$ , defined by  $\hat{W}_0(K) = V_n(K)$ , and

$$\hat{W}_j(K) = \frac{\omega_n}{\omega_{n-j}} \left( \int_{G_{n,n-j}} V_{n-j}(K|\xi)^{-1} d\mu_{n-j}(\xi) \right)^{-1}, \quad j = 1, \dots, n-1.$$

See also Gardner [11, p. 382], Schneider [36, p. 514], and Lutwak [20, 22]. Nearly thirty years later, Lutwak [20, 22] introduced the affine quermassintegrals  $\Phi_0(K), \Phi_1(K), \dots, \Phi_{n-1}(K)$ , defined by  $\Phi_0(K) = V_n(K)$ , and

$$\Phi_j(K) = \frac{\omega_n}{\omega_{n-j}} \left( \int_{G_{n,n-j}} V_{n-j}(K|\xi)^{-n} d\mu_{n-j}(\xi) \right)^{-\frac{1}{n}}, \quad j = 1, \dots, n-1.$$

Note that all the  $\Phi_j(K)$  are affine invariant, i.e.,  $\Phi_j(TK) = \Phi_j(K)$ , for all  $T \in \text{SL}(n)$ . See Grinberg [15]. For more information, we refer to Gardner [13] and Dafnis and Paouris [7].

Hadwiger [18, p. 268] and Lutwak [20] established the following Brunn-Minkowski type inequalities for harmonic quermassintegrals and affine quermassintegrals, respectively.

**Proposition 4.6.** *Suppose  $K, L \in \mathcal{K}^n$  and  $j \in \{1, \dots, n-1\}$ . Then*

$$\hat{W}_j(K+L)^{\frac{1}{n-j}} \geq \hat{W}_j(K)^{\frac{1}{n-j}} + \hat{W}_j(L)^{\frac{1}{n-j}}$$

and

$$\Phi_j(K+L)^{\frac{1}{n-j}} \geq \Phi_j(K)^{\frac{1}{n-j}} + \Phi_j(L)^{\frac{1}{n-j}}.$$

*If  $j = n-1$ , equality holds in each inequality if and only if  $w_K = \lambda w_L$  for some constant  $\lambda > 0$ . If  $1 \leq j < n-1$ , equality holds in each inequality if and only if  $K$  and  $L$  are homothetic.*

From Proposition 4.6 and the  $L_p$  transference principle, we obtain the following result.

**Theorem 4.7.** *Suppose  $K, L \in \mathcal{K}_o^n$  and  $1 < p < \infty$ . Then*

$$\hat{W}_j(K +_p L)^{\frac{p}{n-j}} \geq \hat{W}_j(K)^{\frac{p}{n-j}} + \hat{W}_j(L)^{\frac{p}{n-j}}$$

and

$$\Phi_j(K +_p L)^{\frac{p}{n-j}} \geq \Phi_j(K)^{\frac{p}{n-j}} + \Phi_j(L)^{\frac{p}{n-j}}.$$

*Equality holds in each inequality if and only if  $K$  and  $L$  are dilates.*

*Proof.* We prove this theorem for affine quermassintegrals. The proof for harmonic quermassintegrals is similar. For  $K \in \mathcal{K}^n$ , define

$$F(K) = \Phi_j(K)^{\frac{1}{n-j}}.$$

Then  $F$  is positively homogeneous. From the definition of  $\Phi_j$  and Lemma 2.3,  $F$  is strictly increasing. From Proposition 4.6,  $F$  is concave. Hence, from the  $L_p$  transference principle and Theorem 3.4, we obtain the theorem.  $\square$

#### 4.4. An application to projection bodies

For a convex body  $K \in \mathcal{K}^n$ , its mixed projection bodies  $\Pi_0 K, \Pi_1 K, \dots, \Pi_{n-1} K$  are defined by

$$h_{\Pi_i K}(u) = W_i^{(n-1)}(K|u^\perp),$$

for  $u \in S^{n-1}$  and  $i \in \{0, 1, \dots, n-1\}$ , where  $W_i^{(n-1)}(K|u^\perp)$  denotes the  $i$ th quermassintegral of  $K|u^\perp$  defined in the subspace  $u^\perp$ . For more information about mixed projection bodies, we refer to Gardner [11, p. 185], Lutwak [21, 23], Parapatits and Schuster [33], and Schneider [36, p. 578].

In [23], Lutwak established the following Brunn-Minkowski type inequality for projection bodies.

**Proposition 4.8.** *Suppose  $K, L \in \mathcal{K}^n$ ,  $j \in \{1, \dots, n\}$ , and  $k \in \{1, \dots, n-1\}$ . Then*

$$W_{n-j}(\Pi_{n-1-k}(K + L))^{\frac{1}{jk}} \geq W_{n-j}(\Pi_{n-1-k}K)^{\frac{1}{jk}} + W_{n-j}(\Pi_{n-1-k}L)^{\frac{1}{jk}},$$

*with equality if and only if  $K$  and  $L$  are homothetic.*

From Proposition 4.8 and the  $L_p$  transference principle, we obtain the following result.

**Theorem 4.9.** *Suppose  $K, L \in \mathcal{K}_o^n$ ,  $j \in \{1, \dots, n\}$ ,  $k \in \{1, \dots, n - 1\}$ , and  $p \in (1, \infty)$ . Then*

$$(4.4) \quad W_{n-j}(\Pi_{n-1-k}(K +_p L))^{\frac{p}{jk}} \geq W_{n-j}(\Pi_{n-1-k}K)^{\frac{p}{jk}} + W_{n-j}(\Pi_{n-1-k}L)^{\frac{p}{jk}},$$

with equality if and only if  $K$  and  $L$  are dilates.

*Proof.* Some facts about mixed projection bodies are in order.

First, for each  $K \in \mathcal{K}^n$ , the compact convex set  $\Pi_{n-1-k}K$  is a convex body. Indeed, for all  $u \in S^{n-1}$ , since  $K|u^\perp$  is  $(n - 1)$ -dimensional, it follows that  $W_{n-1-k}^{(n-1)}(K|u^\perp) > 0$ , i.e.,  $h_{\Pi_{n-1-k}K}(u) > 0$ .

Second,  $\Pi_{n-1-k}(\lambda K) = \lambda^k \Pi_{n-1-k}K$ , for all  $\lambda > 0$ . Indeed, for all  $u \in S^{n-1}$ ,

$$\begin{aligned} h_{\Pi_{n-1-k}(\lambda K)}(u) &= W_{n-1-k}^{(n-1)}\left((\lambda K)|u^\perp\right) \\ &= \lambda^k W_{n-1-k}^{(n-1)}\left(K|u^\perp\right) \\ &= \lambda^k h_{\Pi_{n-1-k}K}(u). \end{aligned}$$

Third, if  $K, L \in \mathcal{K}^n$  and  $K \subseteq L$ , then  $\Pi_{n-1-k}K \subseteq \Pi_{n-1-k}L$ . Indeed, for all  $u \in S^{n-1}$ , since  $K|u^\perp \subseteq L|u^\perp$ , it follows that  $W_{n-1-k}^{(n-1)}(K|u^\perp) \leq W_{n-1-k}^{(n-1)}(L|u^\perp)$ , i.e.,  $h_{\Pi_{n-1-k}K}(u) \leq h_{\Pi_{n-1-k}L}(u)$ .

Fourth,  $\Pi_{n-1-k}(K + x) = \Pi_{n-1-k}K$ , for all  $x \in \mathbb{R}^n$ . Indeed, for all  $u \in S^{n-1}$ ,

$$\begin{aligned} h_{\Pi_{n-1-k}(K+x)}(u) &= W_{n-1-k}^{(n-1)}\left((K + x)|u^\perp\right) \\ &= W_{n-1-k}^{(n-1)}\left(K|u^\perp + x|u^\perp\right) \\ &= W_{n-1-k}^{(n-1)}\left(K|u^\perp\right) \\ &= h_{\Pi_{n-1-k}K}(u). \end{aligned}$$

Hence, the functional  $F = W_{n-j}(\Pi_{n-1-k}(\cdot))^{1/jk}$  over  $\mathcal{K}^n$  is strictly positive, positively homogeneous, increasing, and translation invariant. Meanwhile, Proposition 4.8 implies that  $F$  is concave.

Hence, from the  $L_p$  transference principle and Proposition 4.8 together with Theorem 3.5, we obtain the theorem.  $\square$

**4.5. An application to capacities**

The  $q$ -capacity of a convex body  $K$  in  $\mathbb{R}^n$ , for  $1 \leq q < n$ , is

$$\text{Cap}_q(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^q dx \right\},$$

where the infimum is taken over all nonnegative functions  $f$  such that  $f \in L^{\frac{nq}{n-q}}(\mathbb{R}^n)$ ,  $\nabla f \in L^q(\mathbb{R}^n; \mathbb{R}^n)$ , and  $K$  is contained in the interior of  $\{x : f(x) \geq 1\}$ .

The following is the remarkable capacity Brunn-Minkowski inequality.

**Proposition 4.10.** *Suppose  $K, L \in \mathcal{K}^n$ , and  $1 \leq q < n$ . Then*

$$(4.5) \quad \text{Cap}_q(K + L)^{\frac{1}{n-q}} \geq \text{Cap}_q(K)^{\frac{1}{n-q}} + \text{Cap}_q(L)^{\frac{1}{n-q}},$$

*with equality if and only if  $K$  and  $L$  are homothetic.*

Borell [3] first established (4.5) for the case  $q = 2$  (the Newtonian capacity), and the equality condition was proved by Caffarelli, Jerison, and Lieb [5]. When  $1 < q < n$ , the inequality was proved by Colesanti and Salani [6]. The case  $q = 1$  is just the Brunn-Minkowski inequality for surface area of convex bodies:

$$W_1(K + L)^{\frac{1}{n-1}} \geq W_1(K)^{\frac{1}{n-1}} + W_1(L)^{\frac{1}{n-1}}, \quad K, L \in \mathcal{K}^n,$$

due to the fact  $\text{Cap}_1(K) = \mathcal{H}^{n-1}(\partial K) = nW_1(K)$ , for  $K \in \mathcal{K}^n$ .

For more information on the role of capacity in the Brunn-Minkowski theory and its dual, we refer to Gardner and Hartenstine [12] and the references within.

From Proposition 4.10 and the  $L_p$  transference principle, we obtain the following result.

**Theorem 4.11.** *Suppose  $K, L \in \mathcal{K}_o^n$ ,  $1 \leq q < n$ , and  $1 < p < \infty$ . Then*

$$\text{Cap}_q(K +_p L)^{\frac{p}{n-q}} \geq \text{Cap}_q(K)^{\frac{p}{n-q}} + \text{Cap}_q(L)^{\frac{p}{n-q}},$$

*with equality if and only if  $K$  and  $L$  are dilates.*

*Proof.* From Evans and Gariepy [8, pp. 150–151], the functional

$$F = \text{Cap}_q(\cdot)^{\frac{1}{n-q}} : \mathcal{K}^n \rightarrow (0, \infty)$$

is positively homogeneous, increasing, and translation invariant. Meanwhile, Proposition 4.10 implies that  $F$  is concave.

Hence, from the  $L_p$  transference principle and Proposition 4.10 together with Theorem 3.5, Theorem 4.11 is obtained.  $\square$

### Acknowledgement

We thank the referee very much for the extremely thorough and helpful report on our manuscript.

### References

- [1] F. Barthe, *The Brunn-Minkowski theorem and related geometric and functional inequalities*, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006, 1529–1546.
- [2] S. G. Bobkov and M. Ledoux, *From Brunn-Minkowski to sharp Sobolev inequalities*, Annali di Matematica **187** (2008), 369–384.
- [3] C. Borell, *Capacitary inequalities of the Brunn-Minkowski type*, Math. Ann. **263** (1983), 179–184.
- [4] H. Brascamp and E. Lieb, *On extension of the Brunn-Minkowski and Prékopa-Leindler inequality, including inequalities for log concave functions, and with an application to diffusion equation*, J. Funct. Anal. **22** (1976), 366–389.
- [5] L. Caffarelli, D. Jerison, and E. Lieb, *On the case of equality in the Brunn-Minkowski inequality for capacity*, Adv. Math. **117** (1996), 193–207.
- [6] A. Colesanti and P. Salani, *The Brunn-Minkowski inequality for  $p$ -capacity of convex bodies*, Math. Ann. **327** (2003), 459–479.
- [7] N. Dafnis and G. Paouris, *Estimates for the affine and dual affine quermassintegrals of convex bodies*, Illinois J. Math. **56** (2012), 1005–1021.
- [8] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton (1992).
- [9] W. J. Firey,  *$p$ -means of convex bodies*, Math. Scand. **10** (1962), 17–24.

- [10] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.
- [11] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, second edition, New York (2006).
- [12] R. J. Gardner and D. Hartenstine, *Capacities, surface area, and radial sums*, Adv. Math. **221** (2006), 601–626.
- [13] R. J. Gardner, *The dual Brunn-Minkowski theory for bounded Borel sets: Dual affine quermassintegrals and inequalities*, Adv. Math. **216** (2007), 358–386.
- [14] R. J. Gardner, D. Hug, and W. Weil, *Operations between sets in geometry*, J. Eur. Math. Soc. **15** (2013), 2297–2352.
- [15] E. L. Grinberg, *Isoperimetric inequalities and identities for  $k$ -dimensional cross-sections of convex bodies*, Math. Ann. **291** (1991), 75–86.
- [16] P. M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin, 2007.
- [17] H. Hadwiger, *Konkave Eikörperfunktionale und höhere Trägheitsmomente*, Comment. Math. Helv. **30** (1956), 285–296.
- [18] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie*, Springer-Verlag, Berlin (1957).
- [19] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge (1959).
- [20] E. Lutwak, *A general isoperimetric inequality*, Proc. Amer. Math. Soc. **90** (1984), 415–421.
- [21] E. Lutwak, *Mixed projection inequalities*, Trans. Amer. Math. Soc. **287** (1985), 91–106.
- [22] E. Lutwak, *Inequalities for Hadwiger's harmonic quermassintegrals*, Math. Ann. **280** (1988), 165–175.
- [23] E. Lutwak, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), 901–916.
- [24] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.
- [25] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.



- [26] E. Lutwak, D. Yang, and G. Zhang, *A new ellipsoid associated with convex bodies*, Duke Math. J. **104** (2000), 375–390.
- [27] E. Lutwak, D. Yang, and G. Zhang,  *$L_p$  affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.
- [28] E. Lutwak, D. Yang, and G. Zhang, *On the  $L_p$ -Minkowski problem*, Trans. Amer. Math. Soc. **356** (2004), 4359–4370.
- [29] E. Lutwak, D. Yang, and G. Zhang,  *$L_p$  John ellipsoids*, Proc. Lond. Math. Soc. **90** (2005), 497–520.
- [30] E. Lutwak, D. Yang, and G. Zhang, *The Brunn-Minkowski-Firey inequality for non-convex sets*, Adv. in Appl. Math. **48** (2012), 407–413.
- [31] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed  $n$ -dimensional space*, in: Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. D. Milman, eds.), Lecture Notes in Mathematics, Vol. 1376, Springer-Verlag, Berlin, 1989, 64–104.
- [32] G. Paouris and E. Werner, *Relative entropy of cone-volumes and  $L_p$  centroid bodies*, Proc. Lond. Math. Soc. **104** (2012), 253–286.
- [33] L. Parapatits and Franz E. Schuster, *The Steiner formula for Minkowski valuations*, Adv. Math. **230** (2012), 978–994.
- [34] D. Ryabogin and A. Zvavitch, *The Fourier transform and Firey projections of convex bodies*, Indiana Univ. Math. J. **53** (2004), 667–682.
- [35] L. A. Santaló, *Integral Geometry and Geometric Probability*, Cambridge University Press, Cambridge (2004).
- [36] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, second edition, Cambridge University Press, Cambridge (2014).
- [37] C. Schütt and E. Werner, *Surface bodies and  $p$ -affine surface area*, Adv. Math. **187** (2004), 98–145.
- [38] E. M. Werner and D. Ye, *New  $L_p$  affine isoperimetric inequalities*, Adv. Math. **218** (2008), 762–780.
- [39] D. Ye, *Inequalities for general mixed affine surface areas*, J. Lond. Math. Soc. **85** (2012), 101–120.

DEPARTMENT OF MATHEMATICS  
WUHAN UNIVERSITY OF SCIENCE AND TECHNOLOGY  
WUHAN, 430081, P. R. CHINA  
*E-mail address:* zoudu@wust.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, TONGJI UNIVERSITY  
SHANGHAI, 200092, P. R. CHINA  
*E-mail address:* xiongge@tongji.edu.cn

RECEIVED SEPTEMBER 22, 2015