Stablility of the L_p-Brunn-Minkowski inequality under hyperplane symmmetry if $0 \leq p < 1$

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Interaction Between PDE and Convex Geometry
Hangzhou, October 18, 2021

Brunn-Minkowski inequality

K, C convex bodies in $\mathbb{R}^n$ ($\text{int}K \neq \emptyset$, $\text{int}C \neq \emptyset$)
Brunn-Minkowski inequality - classical form $\alpha, \beta > 0$

$$V(\alpha K + \beta C)^{\frac{1}{n}} \geq \alpha \cdot V(K)^{\frac{1}{n}} + \beta \cdot V(C)^{\frac{1}{n}}$$

with equality $\Longleftrightarrow$ $C = \gamma K + z$ for $\gamma > 0$, $z \in \mathbb{R}^n$.

Brunn-Minkowski inequality - product form $\lambda \in (0,1)$

$$V((1-\lambda)\, K + \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$$

with equality $\Longleftrightarrow$ $C = K + z$ for $z \in \mathbb{R}^n$

Stability of the Brunn-Minkowski inequality

$\alpha = V(K)^{\frac{-1}{n}}$, $\beta = V(C)^{\frac{-1}{n}}$, $\quad \sigma = \max\left\{\frac{V(C)}{V(K)}, \frac{V(K)}{V(C)}\right\}$

Theorem [Figalli, Maggi, Pratelli $\sim$ 2010]

$$V(K+C)^{\frac{1}{n}} \geq \left[1 + \frac{\gamma(n)}{\sigma^{\frac{1}{n}}} \cdot A(K,C)^2\right] \left(V(K)^{\frac{1}{n}} + V(C)^{\frac{1}{n}}\right)$$

where $A(K,C) = \min_{x \in \mathbb{R}^n} V\Big((\alpha K) \Delta (x + \beta C)\Big)$.

Remark $\gamma(n) = n^{-5-o(1)}$ by Kolesnikov-Milman + Chen

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Theorem [Diskant, Groemer $\sim$ 1973]

$$V(K+C)^{\frac{1}{n}} < (1+\varepsilon)\Big(V(K)^{\frac{1}{n}} + V(C)^{\frac{1}{n}}\Big)$$

for small $\varepsilon > 0 \Longrightarrow \exists \theta > 1$ depending on n and σ such that

$$\left(1 - \theta \cdot \varepsilon^{\frac{1}{n}}\right)(\alpha K - x) \subset \beta C - y \subset \left(1 + \theta \cdot \varepsilon^{\frac{1}{n}}\right)(\alpha K - x)$$

Surface area measure, Minkowski's first inequality

S_K - surface area measure on S^{n-1} of a convex body K in $\mathbb{R}^n$
∂K is $C^2_+ \implies dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}$ ($\kappa(u)$ =Gaussian curvature)

Minkowski problem Monge-Ampere equation on S^{n-1}:

$$\det(\nabla^2 h + h\, I_{n-1}) = \kappa^{-1}$$

where $h(u) = h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ support function.

Idea for given μ on S^{n-1} Minimize $\int_{S^{n-1}} h_C\, d\mu,\ V(C) = 1$

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Idea for given μ on S^{n-1} Minimize $\int_{S^{n-1}} h_C\, d\mu, V(C) = 1$

Minkowski's first inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K.$$

Equality $\iff$ K and C are translates.

Lutwak's L_p Minkowski problem $\sim$ 1990

L_p Minkowski problem $h_K^{1-p} dS_K = \mu$=finite Borel meaure on S^{n-1} where $o \in K$

Monge-Ampere on S^{n-1} for $h = h_K$ if μ has a density function f:

$$h^{1-p} \det(\nabla^2 h + h\, I) = f$$

- $p = 1 \Longrightarrow$ Minkowski problem
- $p = 0 \Longrightarrow$ Logarithmic Minkowski problem
- $p = -n \Longrightarrow$ Determining Centro-affine curvature

State of art

- $p > 1$, $p \neq n$: Chou&Wang, Hug&Lutwak&Yang&Zhang
- $0 < p < 1$: Chen&Li&Zhu "almost complete"
- $p = 0$: positive results by Chen&Li&Zhu
- $p < 0$: ??????????????????

Versions: Huang&Lutwak&Yang&Zhang (L_p dual) Gardner&Hug&Weil&Ye (Orlicz), Hosle&Kolesnikov&Livshyts

Logarithmic (L_0) Minkowski problem

Firey (1974)

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on S^{n-1} if $o \in K$

V_K normalized L_0 surface area measure

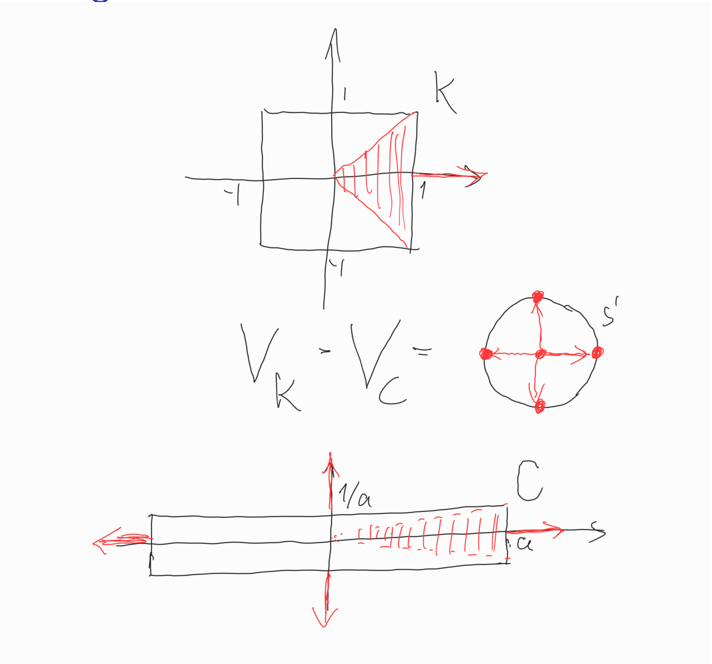
- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$V_K(\{u_i\}) = \frac{h_K(u_i)\mathcal{H}^{n-1}(F_i)}{n} = V(\mathrm{conv}\{o, F_i\}).$$

History

- Earlier characterization by Stancu, Henk&Schürman&Wills, Lutwak&Yang&Zhang, Chou&Wang, He&Leng&Li, Andrews
- B&Lutwak&Yang&Zhang solved in the even case
- Partial result by Chen&Li&Zhu in the general case

Coinciding cone volumes



Uniqueness in the L_p-Minkowski problem

L_p-Minkowski problem on S^{n-1}

$$h^{1-p}\det(\nabla^2 h + h\,I) = f \geq 0$$

- Uniqueness holds if $p > 1$ (Chou&Wang, Hug&Lutwak&Yang&Zhang)
- No uniqueness in general if $p < 1$ (Chen&Li&Zhu)

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For the solution of the Even L_p-Minkowski problem

- Uniqueness Conjectured if $0 < p < 1$
- Uniqueness Conjectured if $p = 0$ and $f > 0$ is C^∞
- No uniqueness if $p < 0$ (Li&Liu&Lu, E. Milman)
- Uniqueness holds if $p_n < p < 1$ and $f > 0$ is C^∞ where $0 < p_n < 1 - \frac{c}{n^{3/2}}$ (Chen&Huang&Li&Liu, Kolesnikov&Milman, Putterman)

L_p Brunn-Minkowski inequality/conjecture

$p > 0$, $\lambda \in (0,1)$, $o \in \mathrm{int}K, \mathrm{int}L$

$\lambda K +_p (1-\lambda)L = \{x \in \mathbb{R}^n : \langle u, x\rangle^p \leq \lambda h_K(u)^p + (1-\lambda)h_L(u)^p \ \forall u\}$

$p \geq 1$ $\quad h_{\lambda K +_p (1-\lambda)L} = \left(\lambda h_K^p + (1-\lambda)h_L^p\right)^{1/p}$

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L_p BM inequality($p \geq 1$)/conjecture($0 < p < 1$, K, C o-symm)

$$V((1-\lambda)K +_p \lambda L)^{\frac{p}{n}} \geq (1-\lambda) V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}$$

with equality $\Longleftrightarrow$ K and L are dilates.

- Proved if $p_n \leq p < 1$ (Chen&Huang&Li&Liu, Putterman)

L_p Minkowski inequality/conjecture for $p > 0$

$$\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p dV_K \geq V(K) \left(\frac{V(L)}{V(K)}\right)^{\frac{p}{n}}$$

with equality in the o-symmetric case $\Longleftrightarrow$ K and L are dilates.

Logarithmic Brunn-Minkowski inequality/conjecture

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Brunn-Minkowski inequality $\lambda \in (0,1)$

$$V((1-\lambda)\,K + \lambda\,C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$

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Logarithmic L_0 sum $o \in K, C$

$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle u, x\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \ \forall u \in S^{n-1}\}$

$\lambda K +_0 (1-\lambda)C \ \subset \ \lambda K + (1-\lambda)C$

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$$\lambda K +_0 (1-\lambda)C \ \subset\ \lambda K + (1-\lambda)C$$

Logarithmic Brunn-Minkowski conjecture
$\lambda \in (0,1)$, K, C are o-symmetric

$$V((1-\lambda)K +_0 \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$$

with equality $\iff$ K and C have dilated direct summands.

Logarithmic Minkowski conjecture

Conjecture (B, Lutwak, Yang, Zhang)

If K and C are convex bodies whose centroid is the origin and $V(K) = V(C)$, then

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K. \tag{1}$$

Assuming K is smooth, equality holds $\iff K = C$.

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Assuming K is smooth, equality holds $\Longleftrightarrow$ $K = C$.

Remark For even C_+^∞ data, uniqueness of the solution of the Log-Minkowski problem $\Longleftrightarrow$ equality holds in (1) only if $K = C$.

Known results

- K is close to some ellipsoid (Colesanti&Livshyts&Marsiglietti, Kolesnikov&Milman, Chen&Huang&Li&Liu)
- K, C have complex symmetry (Rotem)
- K, C - hyperplane symmetry (Saroglou, B&Kalantzopoulos)

Logarithmic sum - approximate estimate

Lemma (Pavlos Kalantzopoulos, K.B.)

If $\lambda \in (0,1)$ *and the centroid of the convex bodies* K *and* L *in* $\mathbb{R}^n$ *is the origin, then*

$$\gamma_1 V(K)^{1-\lambda} V(L)^{\lambda} \leq V\big((1-\lambda)K +_0 \lambda L\big) \leq \gamma_2 V(K)^{1-\lambda} V(L)^{\lambda}$$

where $\gamma_2 > \gamma_1 > 0$ *depend on* n*.*

- Logarithmic Brunn-Minkowski conjecture: $\gamma_1 = 1$
- No reasonable estimate is known for γ_1 and γ_2

Kolesnikov, Emanuel Milman approach (extending Colesanti&Livshyts&Marsiglietti)

$$D^2 h = \nabla^2 h + h\, I_{n-1} \quad \text{for } h \in C^2(S^{n-1})$$

Mixed discriminant For $h_1, \ldots, h_{n-1} \in C^2(S^{n-1})$

$$S(h_1, \ldots, h_{n-1}) = D_{n-1}(D^2 h_1, \ldots, D^2 h_{n-1})$$

Hilbert-Brunn-Minkowski operator ∂K C^2_+, $z \in C^2(S^{n-1})$

$$L_K z = \frac{S(z h_K, h_K, \ldots, h_K)}{S(h_K, \ldots, h_K)} - z$$

Theorem (Hilbert-Kolesnikov-Milman)

$L_K : C^2(S^{n-1}) \to C(S^{n-1})$ *elliptic with self-adjoint extension to* $L^2(dV_K)$

Spectral properties of $-L_K$

Trivial eigenvalues and eigenspaces of $-L_K$

- $\lambda_0(-L_K) = 0$ (corresponding to constant functions)
- linear functions (that are odd) have eigenvalue 1 with multiplicity n

Theorem (Hilbert)

$K \in \mathcal{K}^2_+ \Longrightarrow \lambda_1(-L_K) \geq 1$

Remark: Equivalent with Brunn-Minkowski inequality

Fact $\lambda_{1,e}(-L_K) = \lambda_{n+1}(-L_K)$ for $K \in \mathcal{K}^2_{+,e}$

$\lambda_{1,e} =$first positive eigenvalue when restricted to even functions

Theorem (Kolesnikov&Milman)

$p \in [0, 1)$

local L_p-Brunn-Minkowski conjecture $\Longleftrightarrow$

$\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$ *for* $\forall K \in \mathcal{K}^2_{+,e}$

Some equivalent formulations of the L_p-Brunn-Minkowski conjecture for o-symmetric bodies, $0 \leq p (< p_n) < 1$

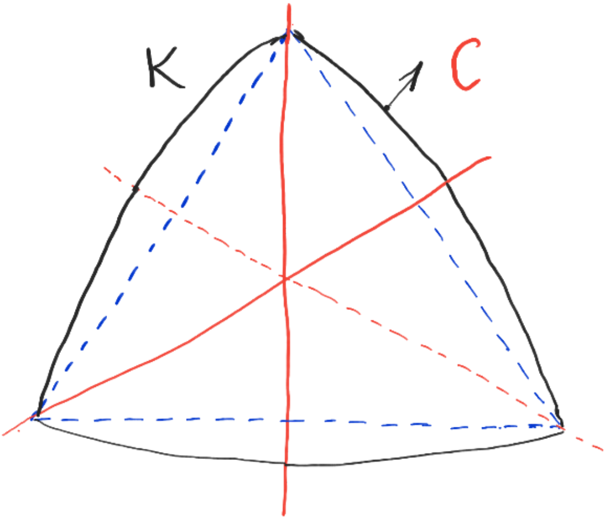
- Monge-Ampere: $h^{1-p} \det(\nabla^2 h + h\, I) = f$ on S^{n-1} has unique even solution if f is even positive and C^∞
- Kolesnikov, E. Milman: $\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$ for $K \in \mathcal{K}^2_{+,e}$ (this proves L_p-B-M locally, Chen&Huang&Li&Liu method yields global result)
- Eli Putterman: $(n-p) \cdot V(L, K[n-1])^2 / V(K) \geq (n-1) V(L[2], K[n-2]) + \frac{1-p}{n} \int_{S^{n-1}} \frac{h_L^2}{h_K}\, dS_K$
- Kolesnikov: Formulation in terms of Optimal Transportation on S^{n-1}

Objects with "hyperplane symmetries"

a function, subset, etc has "hyperplane symmetries" $\iff$
$\exists$ independent $u_1, \ldots, u_n \in S^{n-1}$ s.t. the object is invariant under the reflections through $u_1^\perp, \ldots, u_n^\perp$ $\iff$
invariant under a Coxeter group G of rank n

- Idea comes from
  - Barthe & Fradelizi's work on Mahler's conjecture
  - Barthe & Cordero-Erausquin's work on the Slicing conjecture
- G has a simplicial cone C as fundamental domain, and reflections through the walls of C generate G
- C is mapped into a "coordinate corner" by a linear transform, and results about unconditional bodies are used.

Convex body K with symmetries of a regular simplex



Logarithmic Brunn-Minkowski for bodies with many hyperplane symmetries

$A \in \mathrm{GL}(n,\mathbb{R})$ linear reflection if

- A acts identically on an $(n-1)$-dimensional linear subspace H,
- $\exists u \in \mathbb{R}^n \setminus H$ with $A(u) = -u$

A is an "orthogonal reflection" if $H = u^\perp$.

Theorem (Pavlos Kalantzopoulos, K.B.)

If $\lambda \in (0,1)$ and the convex bodies K and L are invariant under linear reflections $A_1, \ldots, A_n$ are such that $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the associated hyperplanes, then

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}.$$

Equality $\Longleftrightarrow$ $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ where $K_1, \ldots, K_m, L_1, \ldots, L_m$ are invariant under $A_1, \ldots, A_n$, $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are homothetic, $i = 1, \ldots, m$.

Stability of the Log-Brunn Minkowski inequality with hyperplane symmetries

Theorem (B., De)

If K and C in $\mathbb{R}^n$ are invariant under the Coxeter group $G \subset \mathrm{GL}(n)$ generated by n independent linear reflections, and

$$V((1-\lambda)K +_0 \lambda C) \leq (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under G where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that

$$\begin{aligned} K_1 + \ldots + K_m \subset \quad K \quad &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m) \\ C_1 + \ldots + C_m \subset \quad C \quad &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m) \end{aligned}$$

where $c > 1$ is an absolute constant.

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$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

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Coordinatewise product of unconditional convex bodies

K, C unconditional ($(x_1, \dots, x_n) \in K \Longrightarrow (\pm x_1, \dots, \pm x_n) \in K$)

$$K^{1-\lambda} \cdot C^\lambda = \Big\{ \Big(\pm |x_1|^{1-\lambda} |y_1|^\lambda, \dots, \pm |x_n|^{1-\lambda} |y_n|^\lambda \Big) \\ (x_1, \dots, x_n) \in K \ \& \ (y_1, \dots, y_n) \in C \Big\}$$

Theorem (Bollobas&Leader, Uhrin, Saroglou)

If K and C are unconditional convex bodies and $\lambda \in (0, 1)$, then

$$V(K^{1-\lambda} \cdot C^\lambda) \geq V(K)^{1-\lambda} V(C)^\lambda,$$

with equality $\Longleftrightarrow \exists\, \Phi$ *positive definit diagonal matrix s.t.* $K = \Phi C$

Stability of Bollobas-Leader-Uhrin

Theorem (B., De)

If $\tau \in (0, \frac{1}{2}]$, $\lambda \in [\tau, 1-\tau]$ and unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy

$$V(K^{1-\lambda} \cdot C^{\lambda}) \leq (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda},$$

then there exists positive definite diagonal matrix Φ, $\det \Phi = V(K)/V(C)$ such that

$$V(K\Delta(\Phi C)) < c^n n^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}} V(K)$$

where $c > 1$ is an absolute constant.

Stability of the L_p-Brunn Minkowski inequality with hyperplane symmetries if $0 < p < 1$

Theorem (B., De)

If $0 < p < 1$, K and C in $\mathbb{R}^n$ are invariant under the Coxeter group $G \subset \mathrm{GL}(n)$ generated by n independent linear reflections, and

$$V(K +_p C)^{\frac{p}{n}} \leq (1+\varepsilon)\left(V(K)^{\frac{p}{n}} + V(C)^{\frac{p}{n}}\right)$$

for $\varepsilon > 0$, then

$$\left(1 - \gamma \cdot \varepsilon^{\frac{1}{190n}}\right) \cdot C \subset K \subset \left(1 + \gamma \cdot \varepsilon^{\frac{1}{190n}}\right) \cdot C$$

where $\gamma > 1$ depends on n, p and $\sigma = \max\left\{\frac{V(K)}{V(C)}, \frac{V(C)}{V(K)}\right\}$.

Wasserstein distance + Irreducible action

$$\mathrm{Lip}_1 = \left\{ f : S^{n-1} \to \mathbb{R} : \forall a, b \in S^{n-1},\ |f(a) - f(b)| \leq \|a - b\| \right\}.$$

Wasserstein distance $\mu(S^{n-1}) = \nu(S^{n-1}) = 1$

$$d_W(\mu, \nu) = \sup \left\{ \int_{S^{n-1}} f \, d\mu - \int_{S^{n-1}} f \, d\nu : f \in \mathrm{Lip}_1 \right\}.$$

Remark

Convergence w.r.t. the Wasserstein distance $\Longleftrightarrow$ weak convergence.

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Remark

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G is the Coxeter group generated by n independent reflections

- Action of G irreducible if no proper linear subspace invariant
- Action of G reducible otherwise. In this case, $\mathbb{R}^n = \oplus L_i$, $\forall L_i$ invariant proper subspace, $L_i \neq L_j$ orthogonal

Stable determination of V_K - action of G irreducible

Theorem (B.,De)

Let $G \subset O(n)$ be a Coxeter group with irreducible action on $\mathbb{R}^n$. If μ_1 and μ_2 are Borel probability measures on S^{n-1} invariant under G, then the unique G invariant Alexandrov solution h_i of the logarithmic Minkowski problem for $\mu = \mu_i$, $i = 1, 2$, satisfies

$$\|h_1 - h_2\|_\infty \leq \gamma_0 \cdot d_W(\mu_1, \mu_2)^{\frac{1}{95n}}$$

$$r_0 \leq h_1, h_2 \leq R_0;$$

$R_0 = n$, $r_0 = \frac{1}{e}$ and $\gamma_0 = c^n$, $c > 1$ absolute constant.

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$R_0 = n$, $r_0 = \frac{1}{e}$ and $\gamma_0 = c^n$, $c > 1$ absolute constant.

Remark Error $d_W(\mu_1, \mu_2)^{\frac{1}{95n}}$ can't be replaced by less than $d_W(\mu_1, \mu_2)^{\frac{1}{n}}$