Sumset estimates in convex geometry.

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BIRS Workshop "Interaction Between Partial Differential Equations and Convex Geometry"

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- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.
- We will denote by $|K|$ - volume of $K \subset \mathbb{R}^n$.
- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.

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- Fradelizi, Lángi, A.Z. The conjecture is true for a compact star-shaped set $A \subset \mathbb{R}^n$ when $n=2,3$ for any m and $n \geq 4$ and m large enough (i.e. $m \geq (n-1)(n-2)$).

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$$\#(X+B_1+\ldots+B_k) \leq \alpha_1 \ldots \alpha_k \cdot \#(X).$$

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$$\log|A+B_1+B_2| + \log|A| \le n\log 3 + \log|A+B_1| + \log|A+B_2|.$$

The above would represent log-submodularity of volume, if we could remove the constant!

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$c(B_2^n, B, C) \leq 1$, for any (convex) B, C and Euclidean ball $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

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Note: the lower bound for c_n and some improvements of upper bound 3^n was also done by P. Nayar and T. Tkocz,

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Consider convex compact sets $A, B, C \subset \mathbb{R}^n$, $n \geq 2$, then

$$|A||A+B+C| \leq 2^{n-2}|A+B||A+C|.$$

Idea of the proof: use mixed volumes!

$K_1, K_2, \dots, K_r$ be convex bodies in $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_r \geq 0$

Then $|\lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_r K_r|$ is a homogeneous polynomial (in $\lambda_1, \dots, \lambda_r$) of degree n and

$$|\lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_r K_r| = \sum_{i_1, i_2, \dots, i_r = 1}^{r} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n}.$$

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Main Definitions: Mixed Volume

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We would need the following formula

$$|A + B| = \sum_{i=0}^{n} \binom{n}{i} V(A[i], B[n-i]),$$

Here we use the notation

$$V(A[i], B[n-i]) = V(\underbrace{A, \dots, A}_{i-\text{times}}, \underbrace{B \dots, B}_{(n-i)-\text{times}}).$$

3 body inequalities

Find c_n such that for any convex compact sets $A, B, C \subset \mathbb{R}^n$, $n \geq 2$: $|A||A+B+C| \leq c_n|A+B||A+C|$.

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Taking the product of the last two equations we will compare it with the first sum. We will do it term by term comparing terms $i+j = n+k$, i.e. the terms for which A has homogeneity $n+k$:

$$\binom{n}{k} |A| V(A[k], (B+C)[n-k]) \leq c_n \sum_{j=k}^{n} \binom{n}{j} \binom{n}{n+k-j} V(A[j], B[n-j]) V(A[n+k-j], C[j-k])$$

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Again we can not do to much better than "term by term" comparison! I.e. comparing terms with fixed m. After simplifications we need to find c_n such that for $m, j \geq 0$ and $m+j \leq n$:

$$\frac{|A|}{n!} \frac{V(A[n-m-j], B[m], C[j])}{(n-m-j)!} \leq c_n \frac{V(A[n-m], B[m])}{(n-m)!} \frac{V(A[n-j], C[j])}{(n-j)!}.$$

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Lower bound

Find c_n such that for any convex compact sets $A, B, C \subset \mathbb{R}^n$, $n \geq 2$:

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Also Note if the inequality is true in a class of convex bodies closed by linear transformations, then

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and Fradelizi, Giannopoulos & Meyer observed that the inequality is optimal in the class of convex bodies. Thus $c_n \geq 2 - \frac{2}{n}$ and, in particular, $c_3 \geq \frac{4}{3}$.

Consider a collection $\mathcal{K}'$ of compact convex sets in $\mathbb{R}^n$ stable by sums and dilation. Then the following statements are equivalent:

1. $|A||A+B_1+B_2| \leq |A+B_1||A+B_2|$, for every A, B_1, B_2 in $\mathcal{K}'$.

M. Fradelizi, M. Madiman, M. Meyer, A. Z. 2021+

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Let $\mathcal{L}$ be a class of a compact convex sets in $\mathbb{R}^n$ preserved under a liner transformations. The following are equivalent.

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4. $|A+[0,u]+[0,v]||A| \leq |A+[0,u]||A+[0,v]|$ for any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$.

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3. $|A||P_{[u,v]^\perp} A|_{n-2}\sqrt{1-\langle u,v\rangle^2} \leq |P_{u^\perp} A|_{n-1}|P_{v^\perp} A|_{n-1}$, for any $A \in \mathcal{L}$ and any $u, v \in S^{n-1}$.
4. $|A+[0,u]+[0,v]||A| \leq |A+[0,u]||A+[0,v]|$ for any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$.
5. $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1} V(A[n-1], Z_1)V(A[n-1], Z_2)$, for any $A \in \mathcal{L}$ and any Z_1, Z_2 zonoids.

Consider a collection $\mathcal{K}'$ of compact convex sets in $\mathbb{R}^n$ stable by sums and dilation. Then the following statements are equivalent:

1. $|A||A+B_1+B_2| \leq |A+B_1||A+B_2|$, for every A, B_1, B_2 in $\mathcal{K}'$.
2. For every $A, B_1, B_2 \in \mathcal{K}'$

$$|A|V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1)V(A[n-1], B_2).$$

Let $\mathcal{L}$ be a class of a compact convex sets in $\mathbb{R}^n$ preserved under a liner transformations. The following are equivalent.

1. $|A||\partial(A+[0,u])| \leq |\partial A||A+[0,u]|$ for any $A \in \mathcal{L}$ and any $u \in \mathbb{R}^n$.
2. $\frac{|A|}{|P_{u^\perp} A|_{n-1}} \leq \frac{|\partial A|}{|\partial(P_{u^\perp} A)|}$, for any $A \in \mathcal{L}$ and any $u \in S^{n-1}$.
3. $|A||P_{[u,v]^\perp} A|_{n-2} \sqrt{1-\langle u,v\rangle^2} \leq |P_{u^\perp} A|_{n-1} |P_{v^\perp} A|_{n-1}$, for any $A \in \mathcal{L}$ and any $u, v \in S^{n-1}$.
4. $|A+[0,u]+[0,v]||A| \leq |A+[0,u]||A+[0,v]|$ for any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$.
5. $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1} V(A[n-1], Z_1)V(A[n-1], Z_2)$, for any $A \in \mathcal{L}$ and any Z_1, Z_2 zonoids.
6. Define $P(t) = |A + t([0,u]+[0,v])|$, then $P(t)$, as a polynomial on $\mathbb{R}$, has only real roots, for any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$.

Let $\mathcal{Z}$ be the class of zonoids in $\mathbb{R}^n$, then

1. For every A, B_1, B_2 in $\mathcal{Z}$
$$|A+B_1+B_2||A| \leq |A+B_1||A+B_2|.$$
2. For every $A, B_1, B_2 \in \mathcal{Z}$
$$|A|V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1) V(A[n-1], B_2).$$
3. $|A||\partial(A+[0,u])| \leq |\partial A||A+[0,u]|$ for any $A \in \mathcal{Z}$ and any $u \in \mathbb{R}^n$.
4. $\frac{|A|}{|P_{u^\perp} A|_{n-1}} \leq \frac{|\partial A|}{|\partial(P_{u^\perp} A)|}$, for any $A \in \mathcal{Z}$ and any $u \in S^{n-1}$.
5. $|A||P_{[u,v]^\perp} A|_{n-2}\sqrt{1-\langle u,v\rangle^2} \leq |P_{u^\perp} A|_{n-1}|P_{v^\perp} A|_{n-1}$, for any $A \in \mathcal{Z}$ and any $u, v \in S^{n-1}$.
6. Define $P(t) = |A + t([0,u]+[0,v])|$, then $P(t)$, as a polynomial on $\mathbb{R}$, has only real roots, for any $A \in \mathcal{Z}$ and any $u, v \in \mathbb{R}^n$.

M. Fradelizi, M. Madiman, M. Meyer, A.Z. (2021):

Let $\mathcal{Z}$ be the class of zonoids in $\mathbb{R}^n$, then

1. For every A, B_1, B_2 in $\mathcal{Z}$
$$|A+B_1+B_2||A| \leq |A+B_1||A+B_2|.$$
2. For every $A, B_1, B_2 \in \mathcal{Z}$
$$|A|V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1) V(A[n-1], B_2).$$
3. $|A||\partial(A+[0,u])| \leq |\partial A||A+[0,u]|$ for any $A \in \mathcal{Z}$ and any $u \in \mathbb{R}^n$.
4. $\frac{|A|}{|P_{u^\perp} A|_{n-1}} \leq \frac{|\partial A|}{|\partial(P_{u^\perp} A)|}$, for any $A \in \mathcal{Z}$ and any $u \in S^{n-1}$.
5. $|A||P_{[u,v]^\perp} A|_{n-2} \sqrt{1-\langle u,v\rangle^2} \leq |P_{u^\perp} A|_{n-1} |P_{v^\perp} A|_{n-1}$, for any $A \in \mathcal{Z}$ and any $u, v \in S^{n-1}$.
6. Define $P(t) = |A + t([0,u]+[0,v])|$, then $P(t)$, as a polynomial on $\mathbb{R}$, has only real roots, for any $A \in \mathcal{Z}$ and any $u, v \in \mathbb{R}^n$.
7. For every zonoids A and B in $\mathbb{R}^n$ and every $u \in S^{n-1}$, one has
$$\frac{|A+B|}{|P_{u^\perp}(A+B)|_{n-1}} \geq \frac{|A|}{|P_{u^\perp} A|_{n-1}} + \frac{|B|}{|P_{u^\perp} B|_{n-1}}.$$