Yamabe flow on some singular spaces.

Joint work with :
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Plane of the talk

- Yamabe flow on the smooth setting

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- A singular setting
- Existence of Yamabe flow
- Asymptotic behavior of Yamabe flow

Yamabe flow

On closed manifold, the Yamabe flow [R. Hamilton,1989] is the gradient flow of the Hilbert functional

$$g \mapsto \int_M \mathrm{Scal}_g \mathrm{dv}_g$$

$$\text{on } \mathcal{C}(g_0) := \{g = e^{2f} g_0, \mathrm{vol}_g(M) = \int_M \mathrm{dv}_g = 1\}.$$

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The question was to solve by parabolic method the Yamabe problem

$$\text{Given } g_0, \text{ find } g \in e^{2f} g_0 \text{ with } \mathrm{Scal}_g = \mathrm{C} \text{ and } \mathrm{vol}_g(M) = 1.$$

Existence of Yamabe metric on closed smooth manifold

Introducing $Y(M, g_0) = \inf_{g \in \mathcal{C}(g_0)} \int_M \mathrm{Scal}_g \mathrm{dv}_g$, we know that there is $g = e^{2f} g_0$ such that

$$\mathrm{Scal}_g = Y(M, g_0) \text{ and } \mathrm{vol}_g(M) = 1.$$

This problem has been solved by Yamabe, Trudinger, Aubin and Schoen. As an abstract of the story, we have

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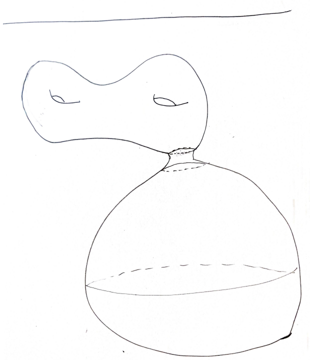
- (Aubin, 1974, Schoen 1984) : if $Y(M, g_0) = Y(\mathbb{S}^n, [\mathrm{rounded}])$ then $M = \mathbb{S}^n$ and $g_0 = e^{-2f} [\mathrm{rounded}]$.

Convergence of the Yamabe flow on the smooth setting : a brief summary

- The Yamabe flow exists of all times,
- There is either convergence or concentration (formation of bubbles).
- Positive mass theorem prevents the formation of bubble.

Mostly from [Schwetlick- Struwe, 2003] and [Brendle, 2005 & 2007].



Some stratified space : the surface of a cube

We are looking for the geometry of the surface of a cube :

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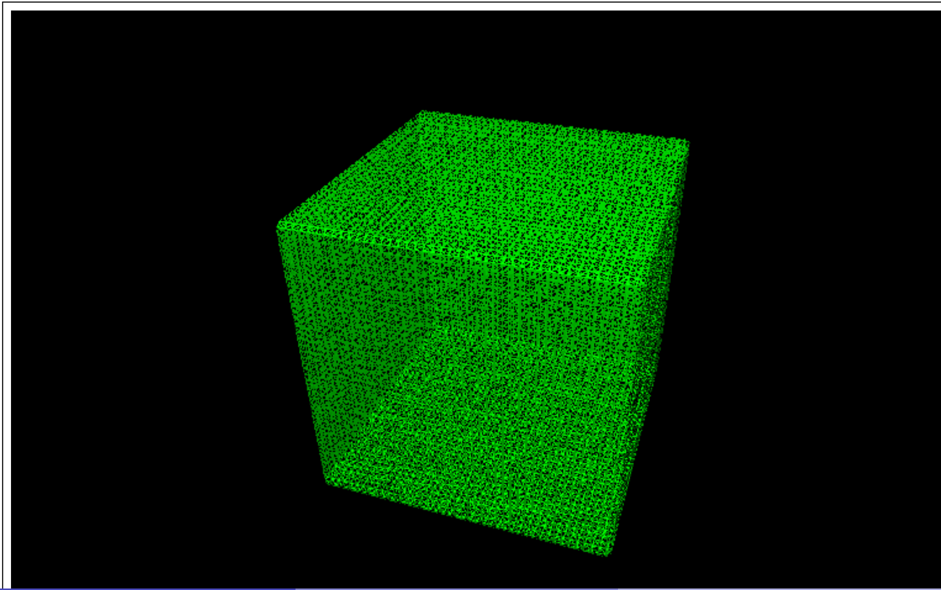

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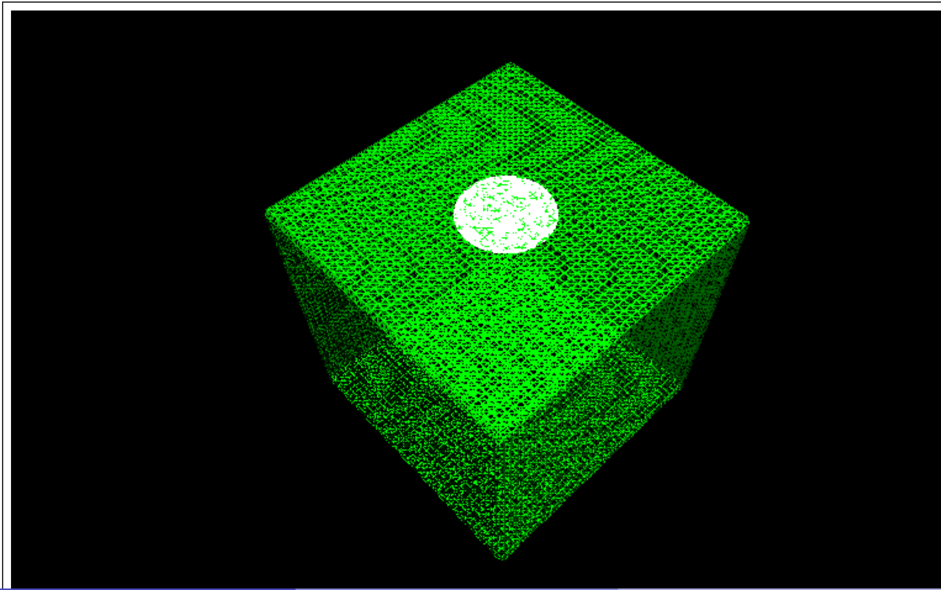

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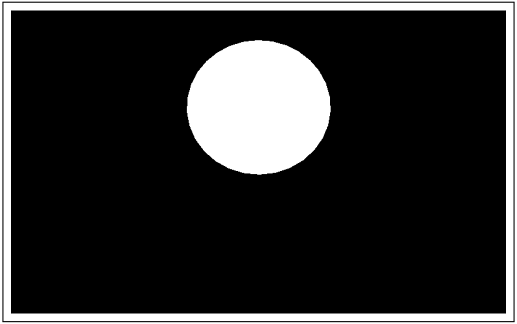

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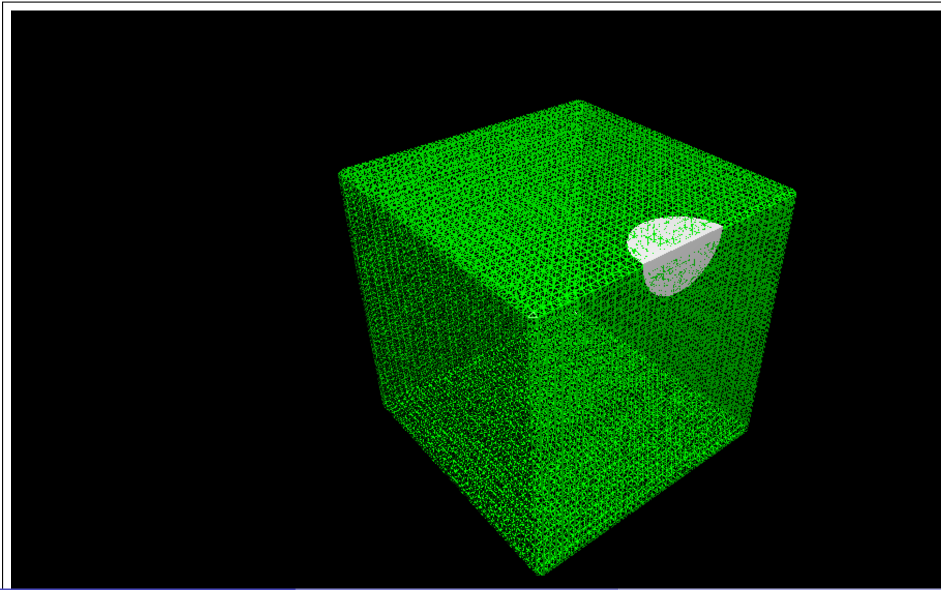

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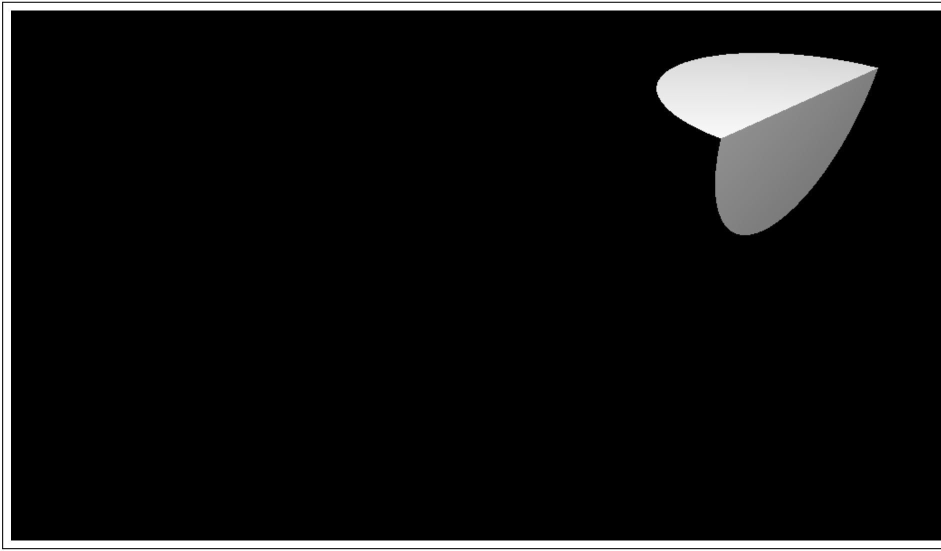

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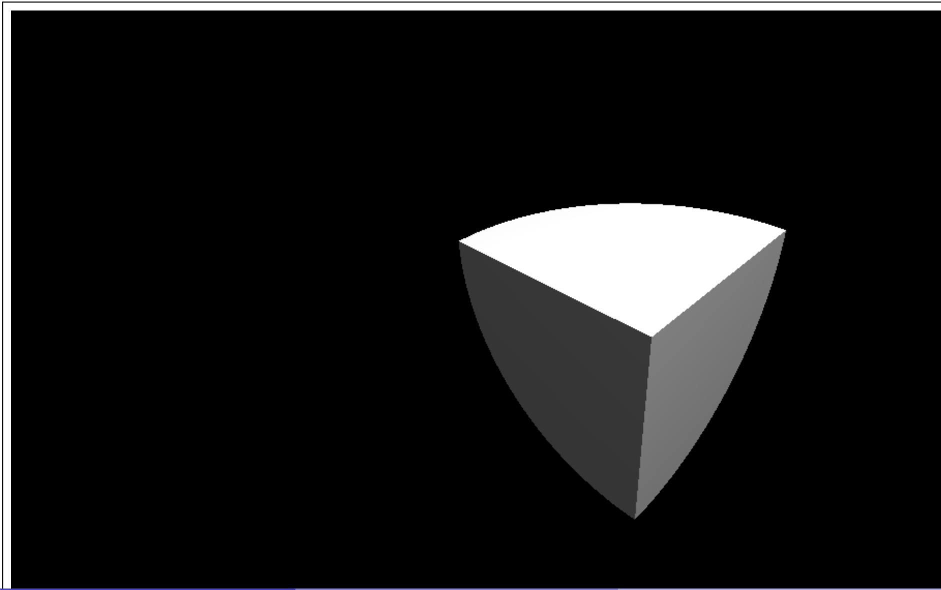

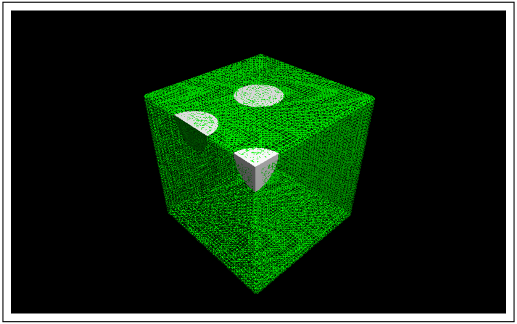

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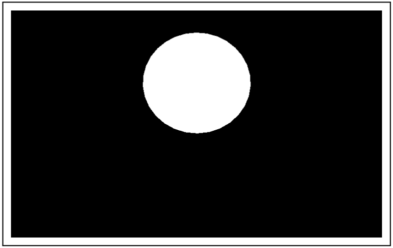
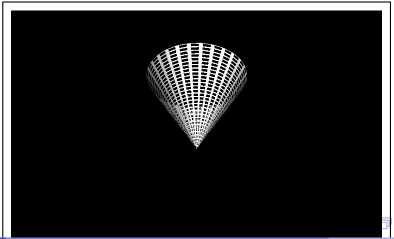
Summary : the surface of a cube has a decomposition $X \supset X_0$, where

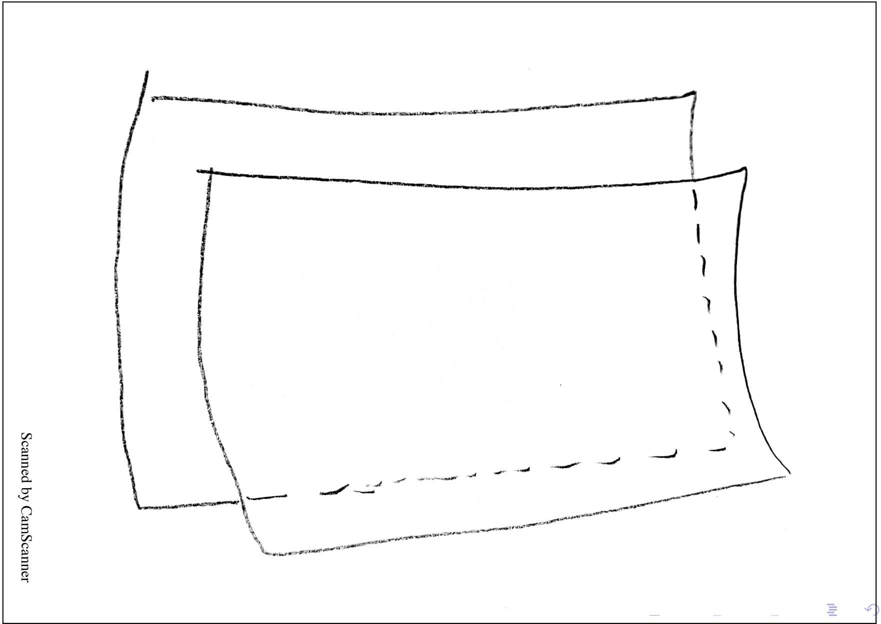
- near each point of $X \setminus X_0$, the geometry is Euclidean



- X_0 is the collection of 8 vertex and near each of these point the geometry is a cone over a circle of length $3\pi/2$



Some stratified space : the surface of a pillow



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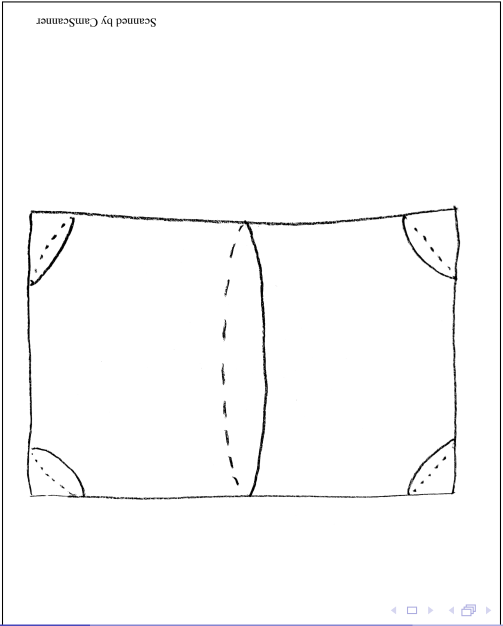

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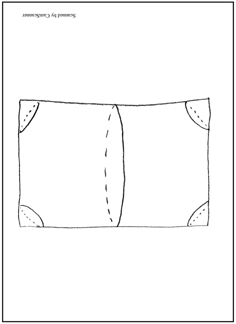

Summary : the surface of a pillow has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Euclidean
- X_0 is the collection of 4 vertex and near each of these point the geometry is a cone over a circle of length π

Some stratified space : the surface of a *berlingot*

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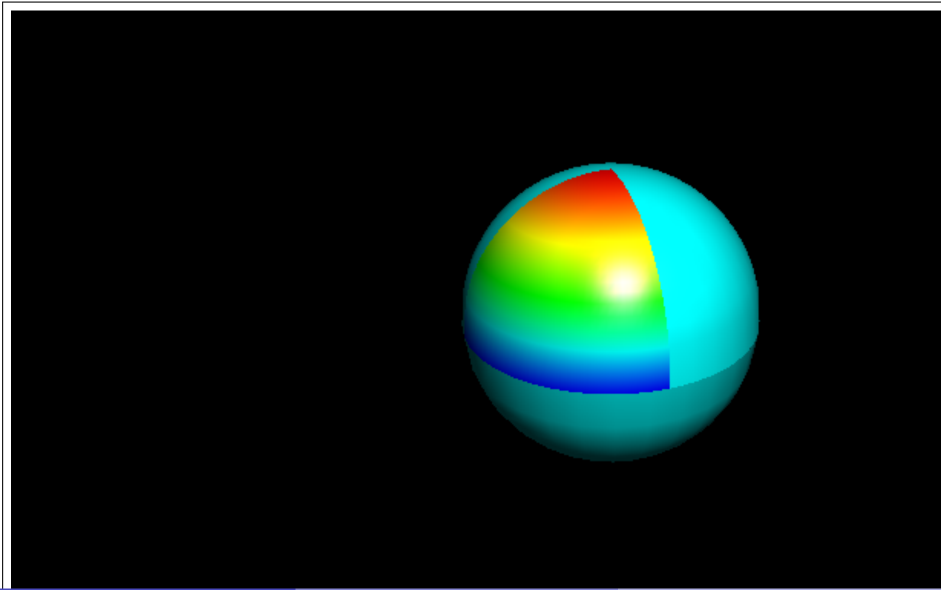

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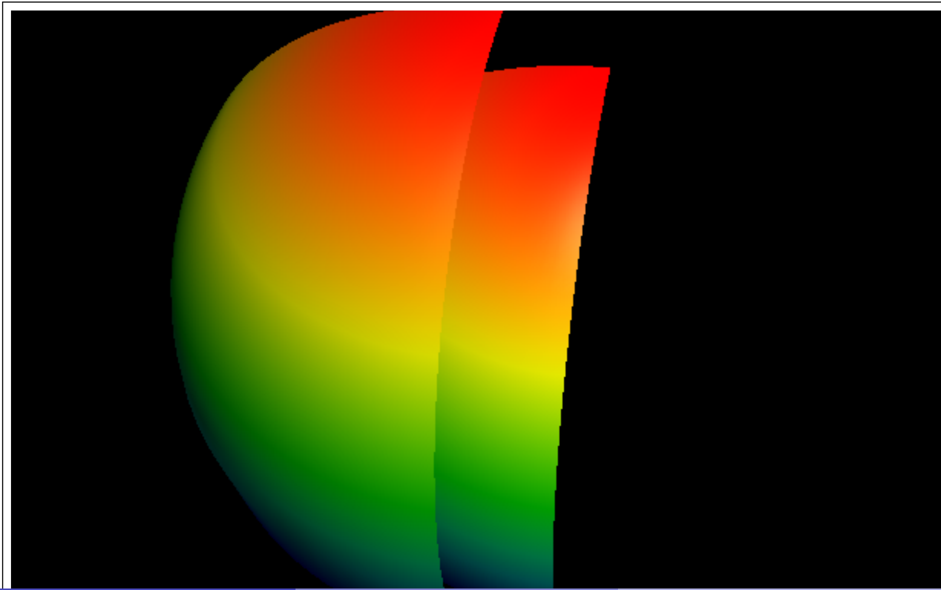

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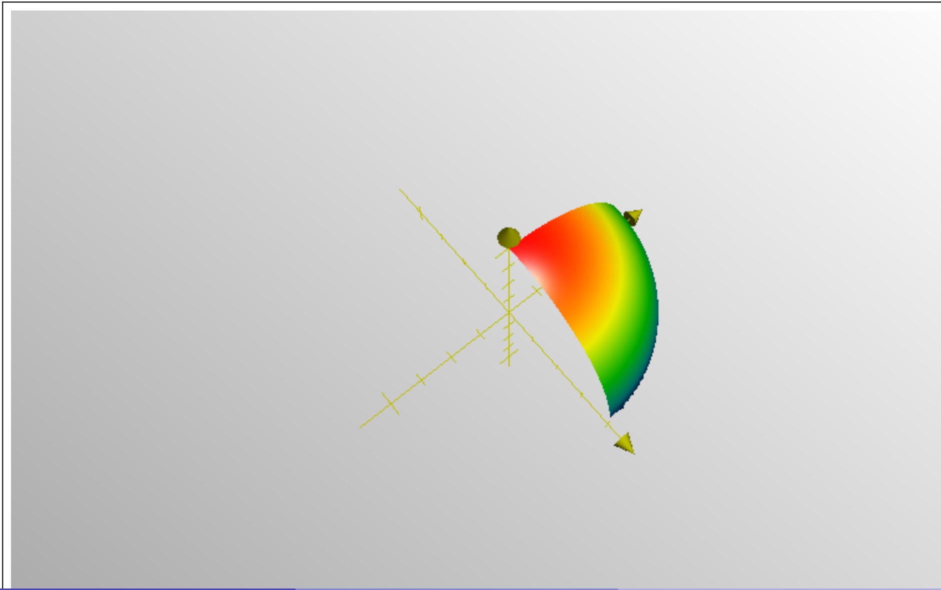

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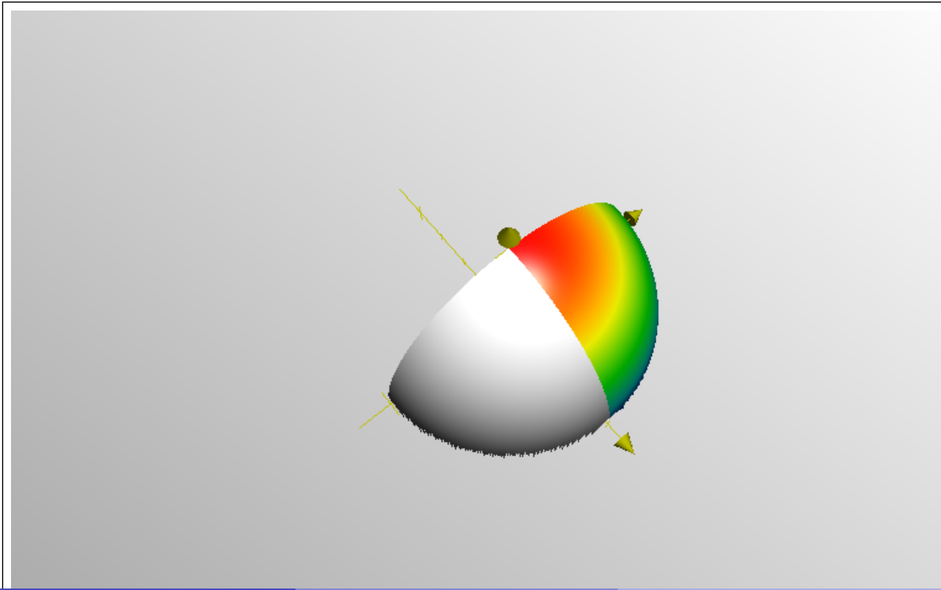

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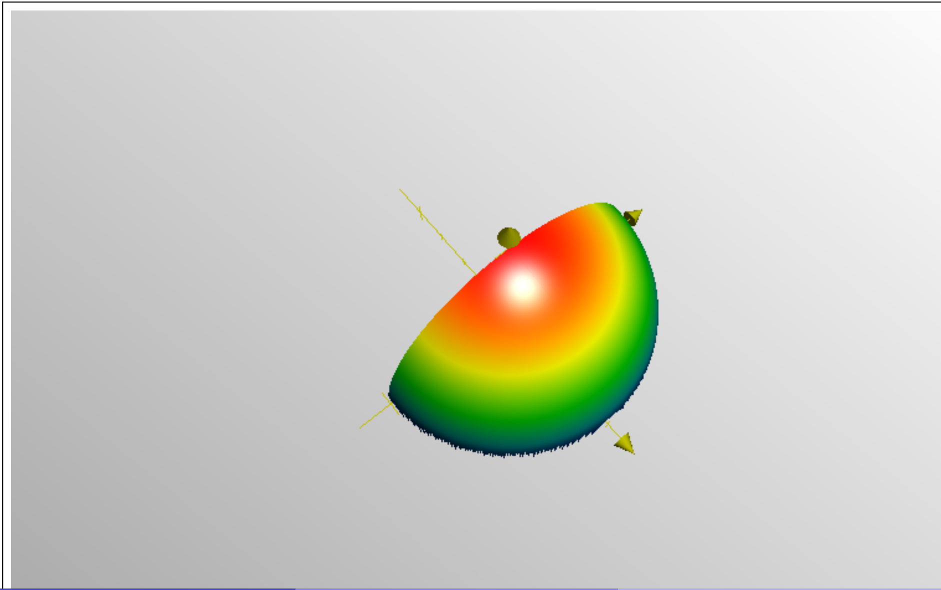

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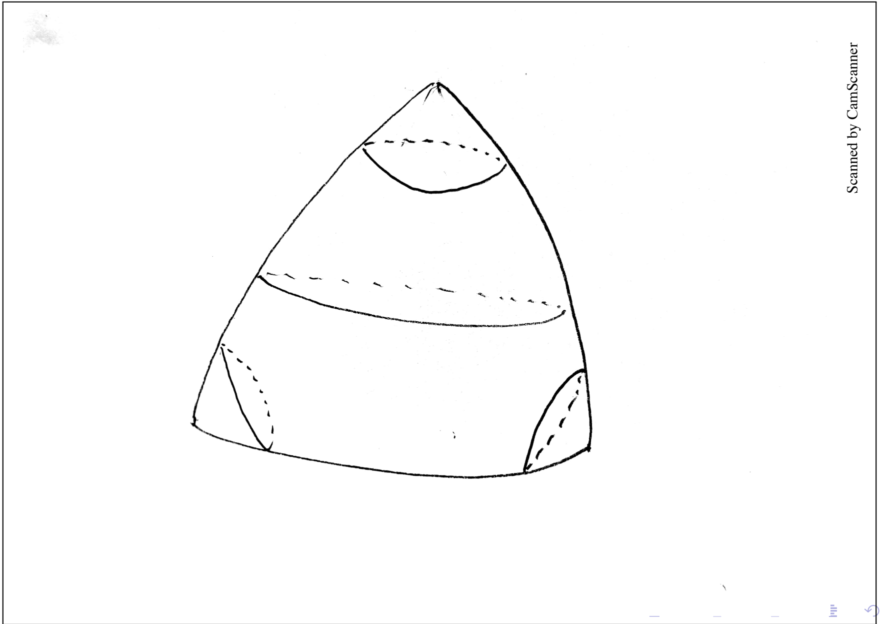

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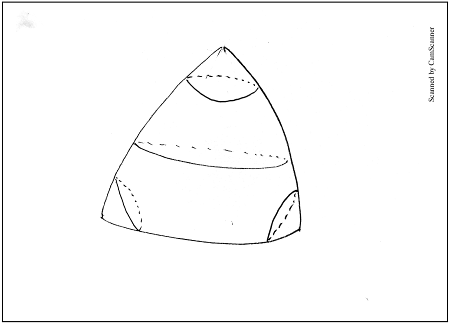
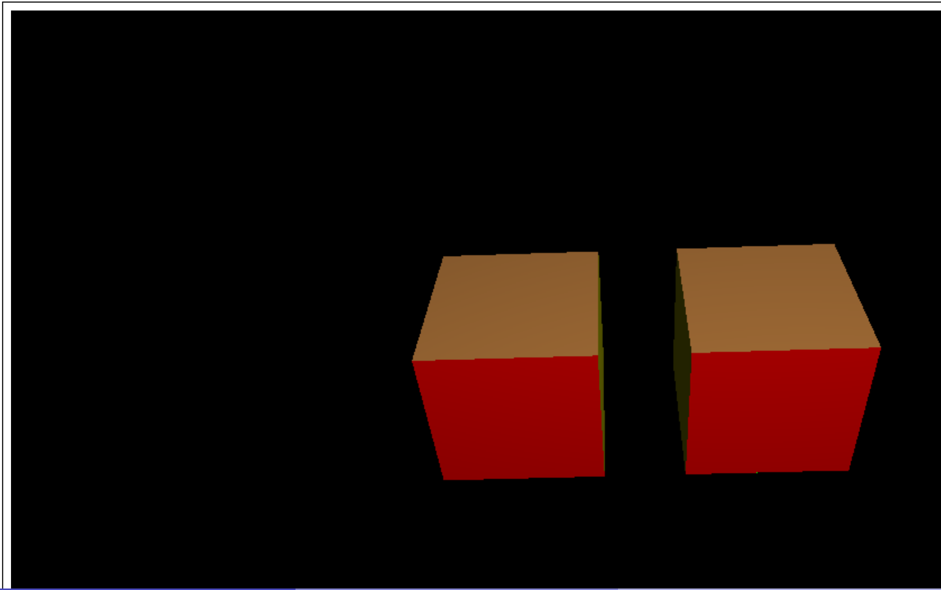


Summary : the surface of a berlingot has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Spherical (Riemannian)
- X_0 is the collection of 3 vertex and near each point the geometry is a cone over a circle of length π

Some stratified space : the double solid cube



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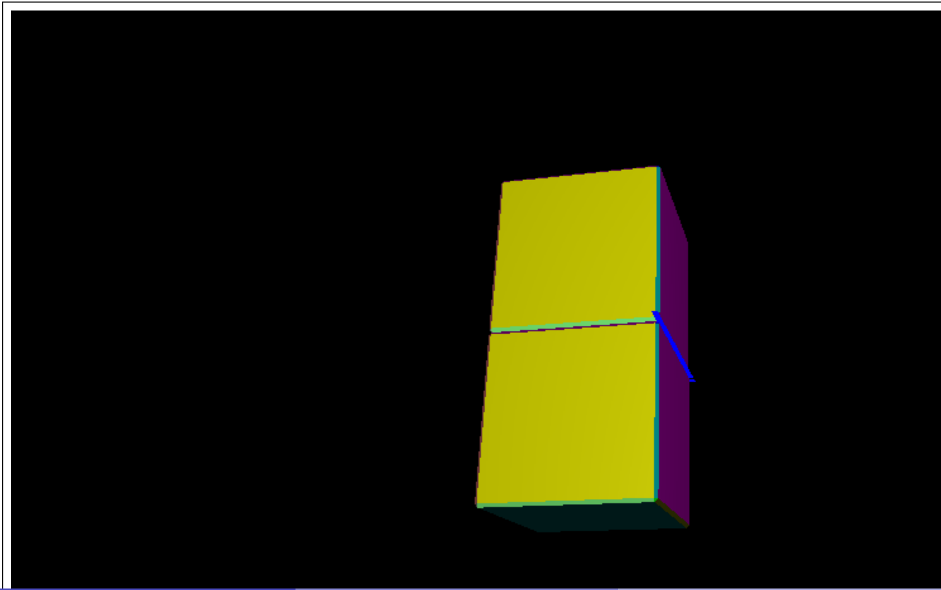

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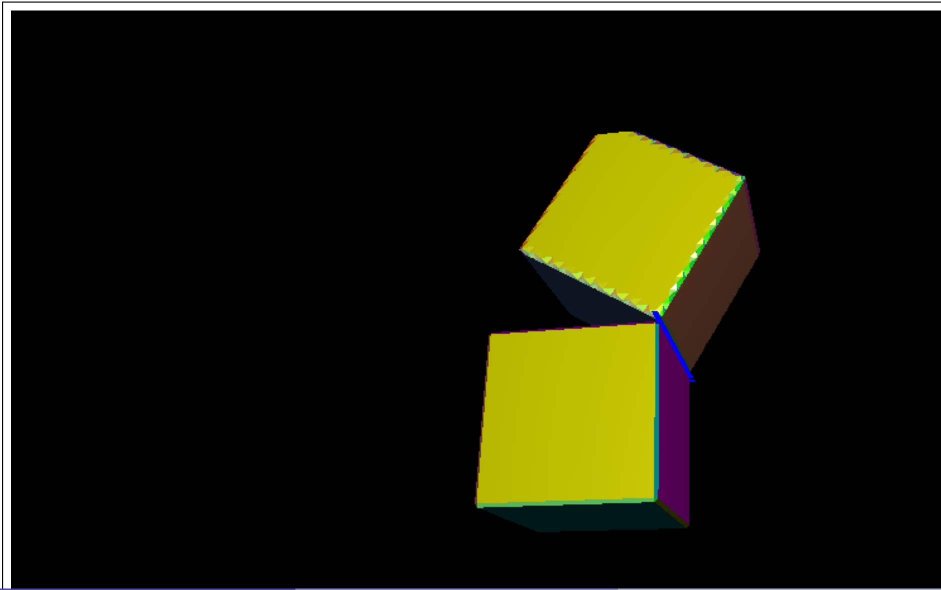

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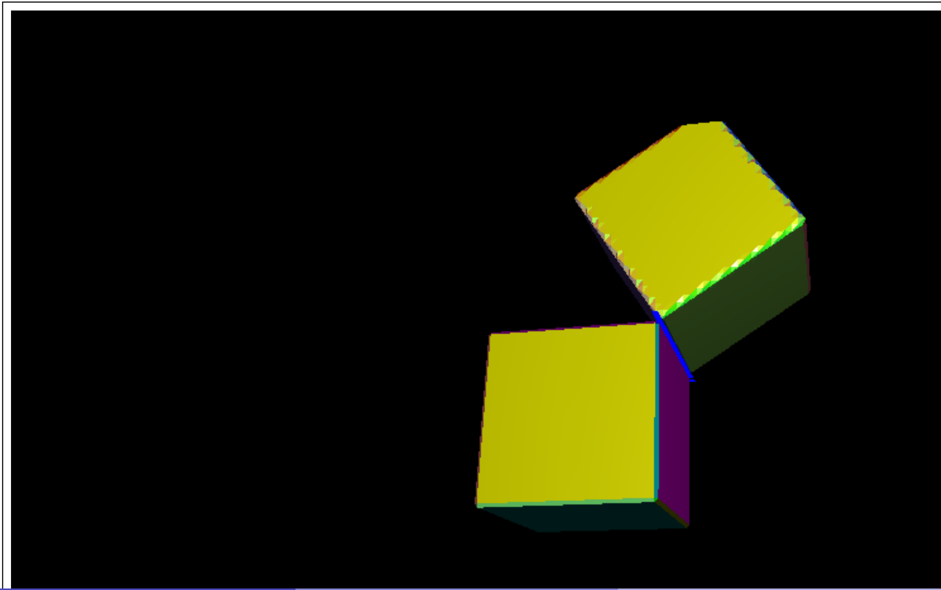

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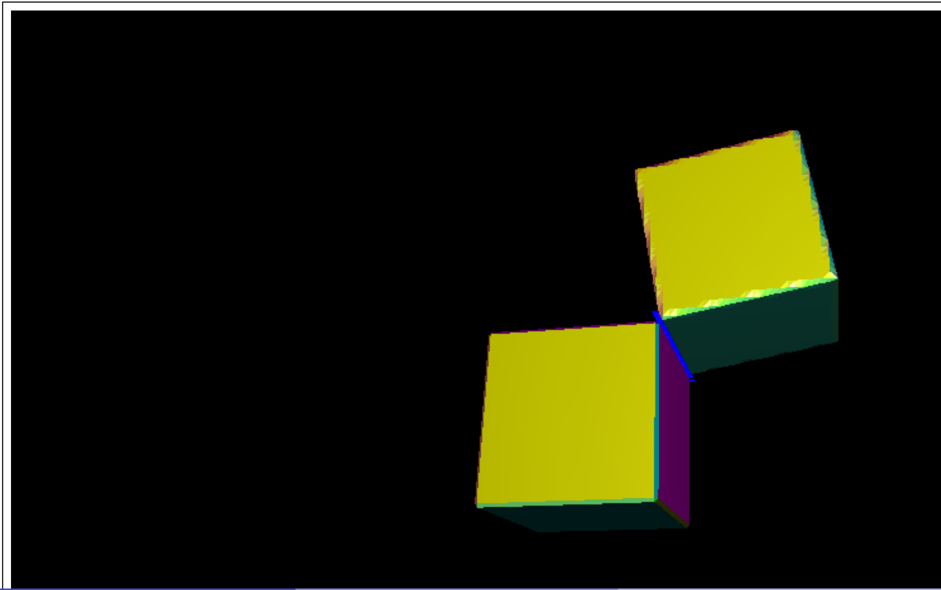

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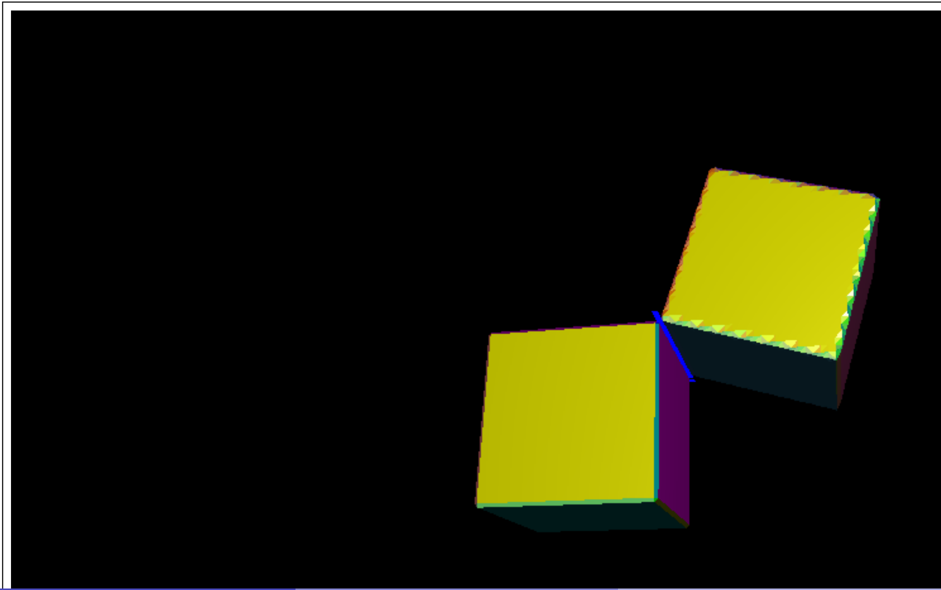

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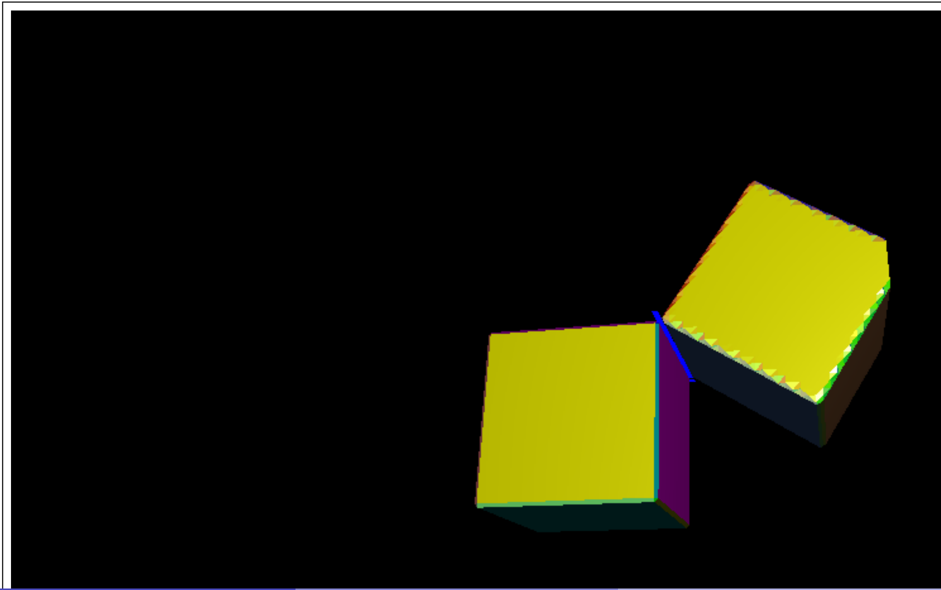

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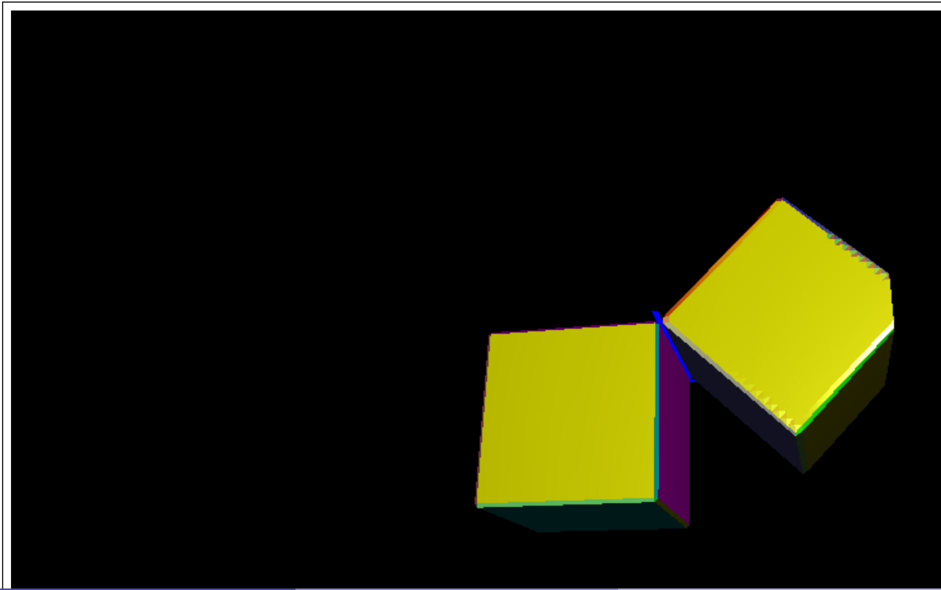

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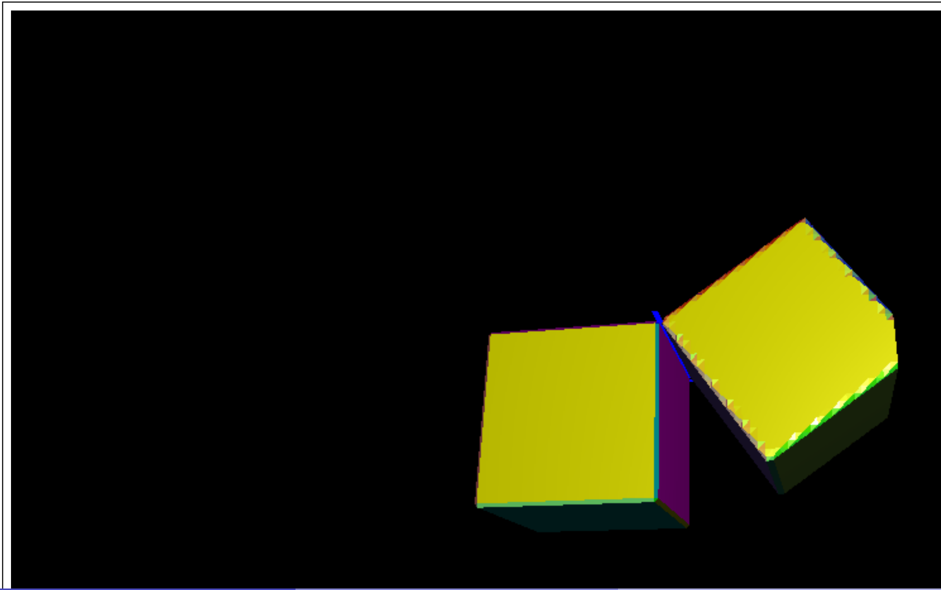

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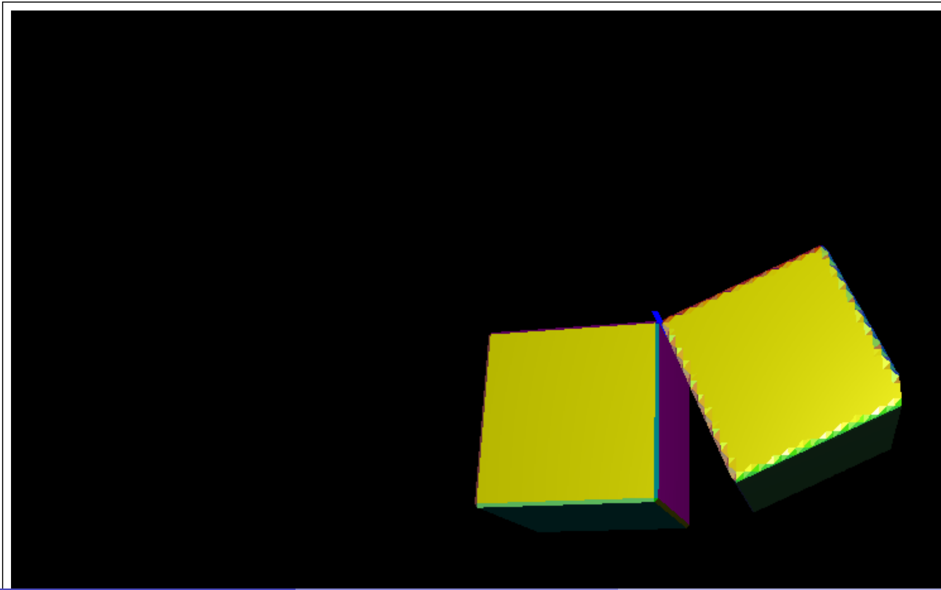

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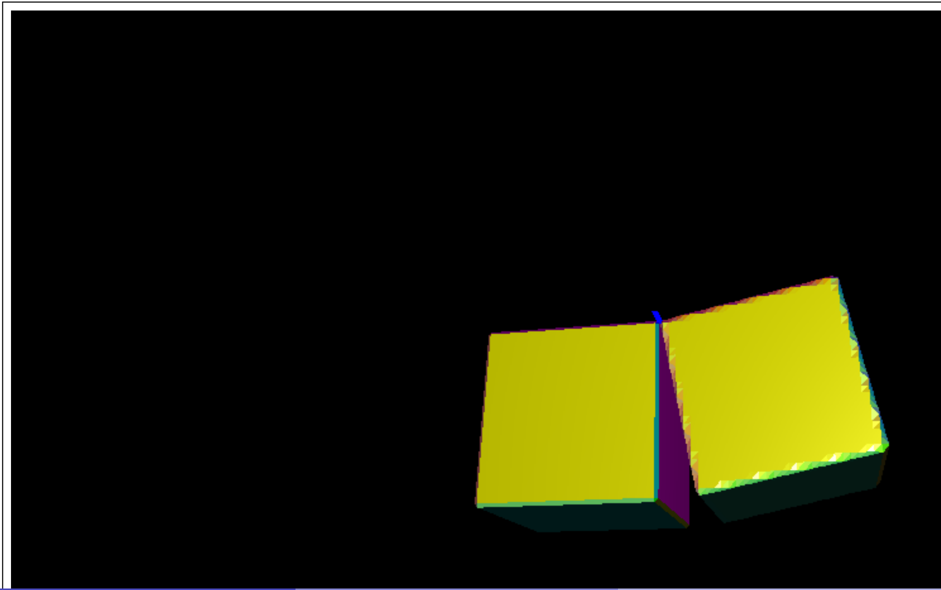

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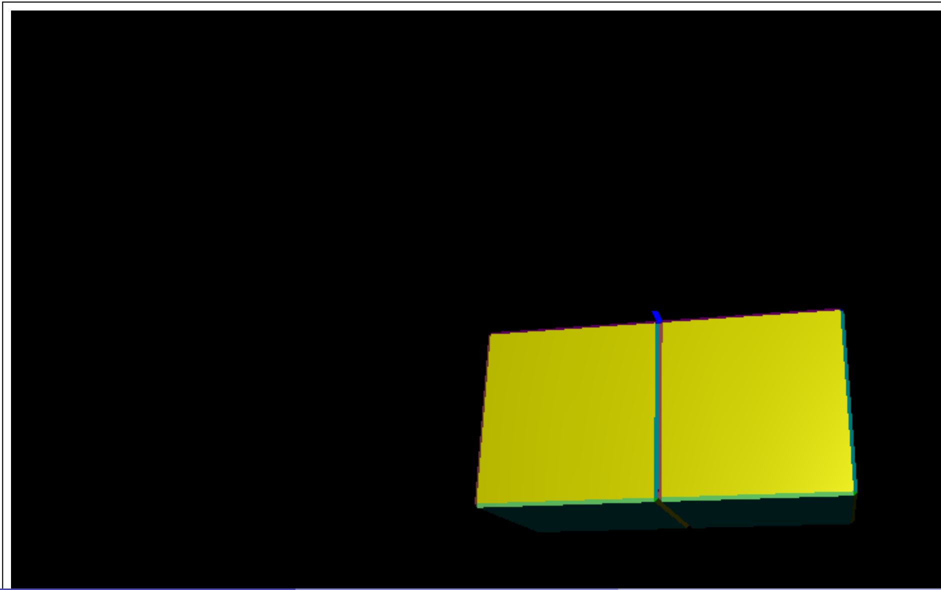

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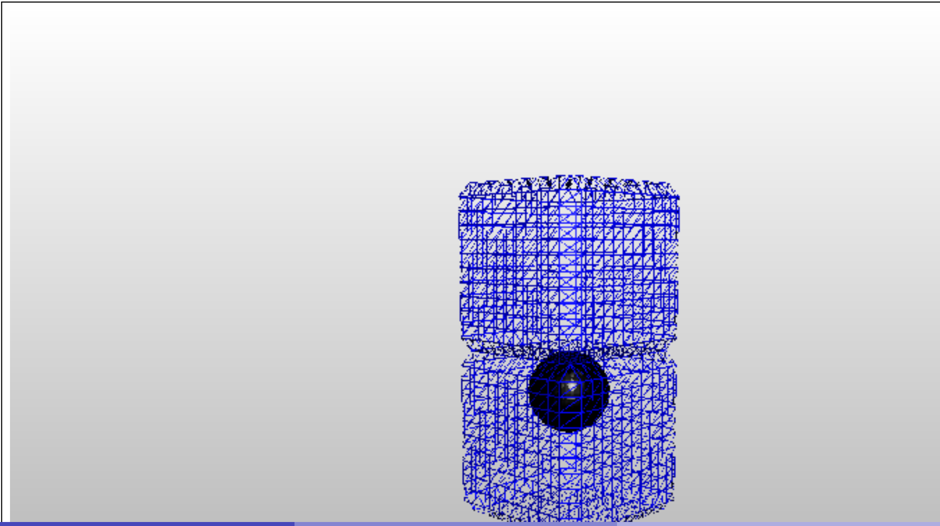
The geometry of the double solid cube is the following : at a point interior or on a face of the cube, the geometry is Euclidean.



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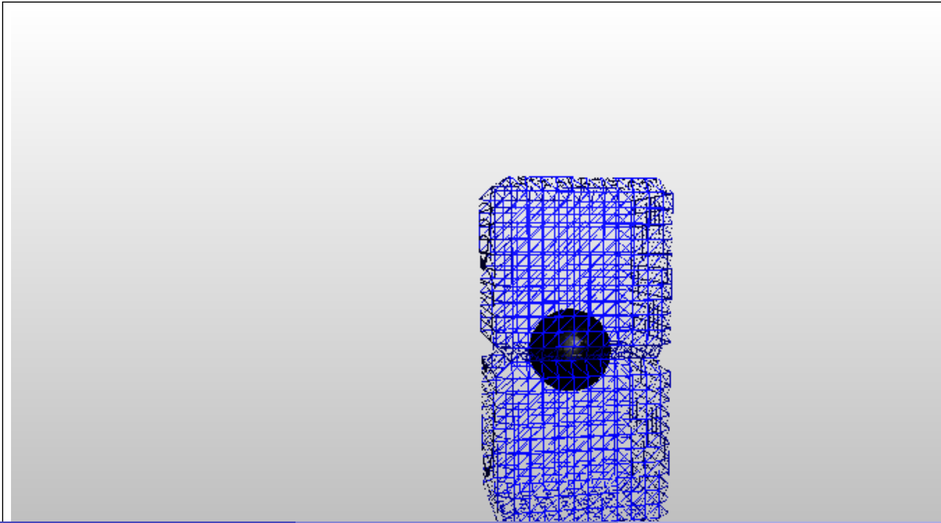

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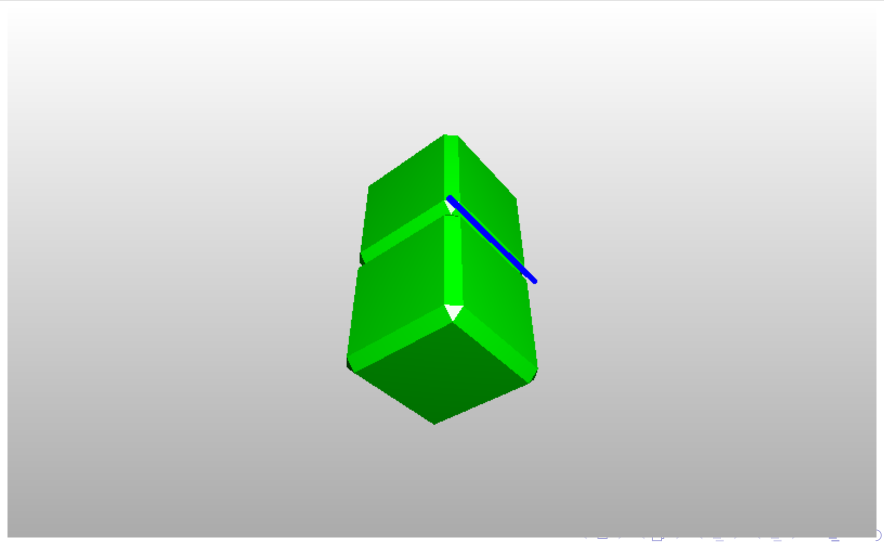

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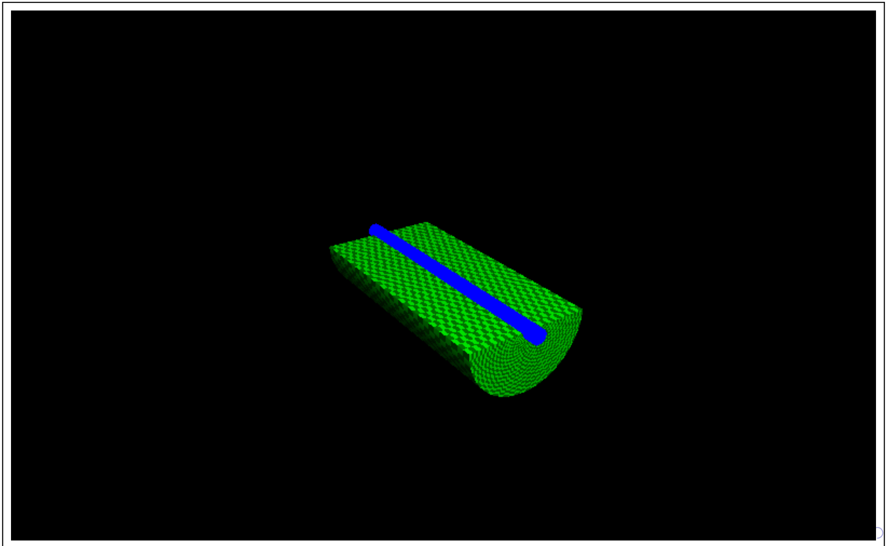

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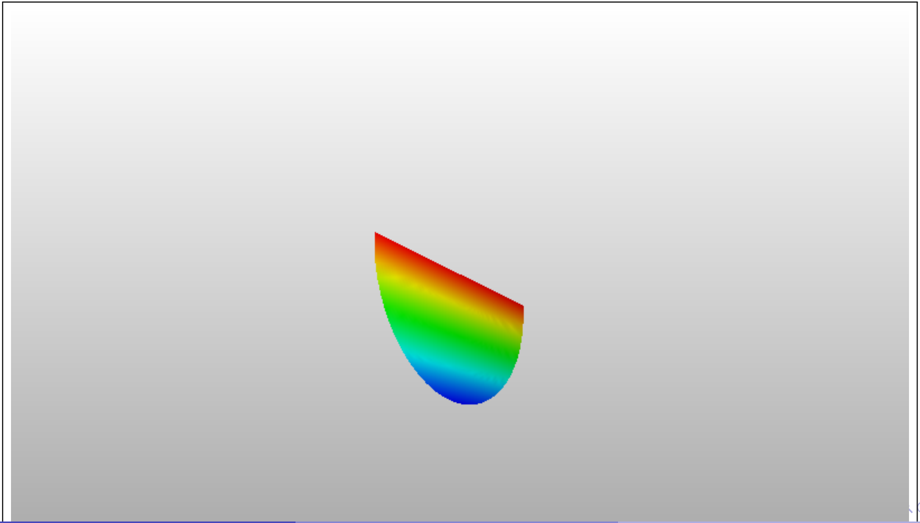

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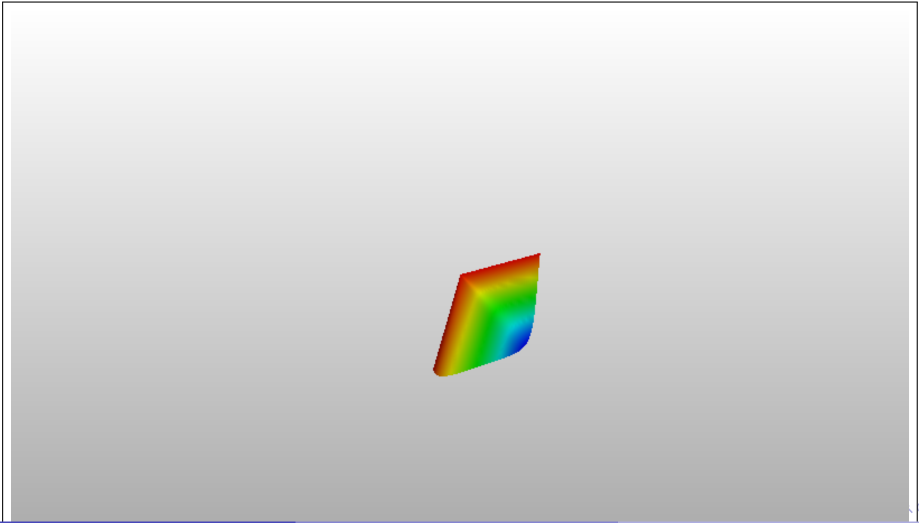

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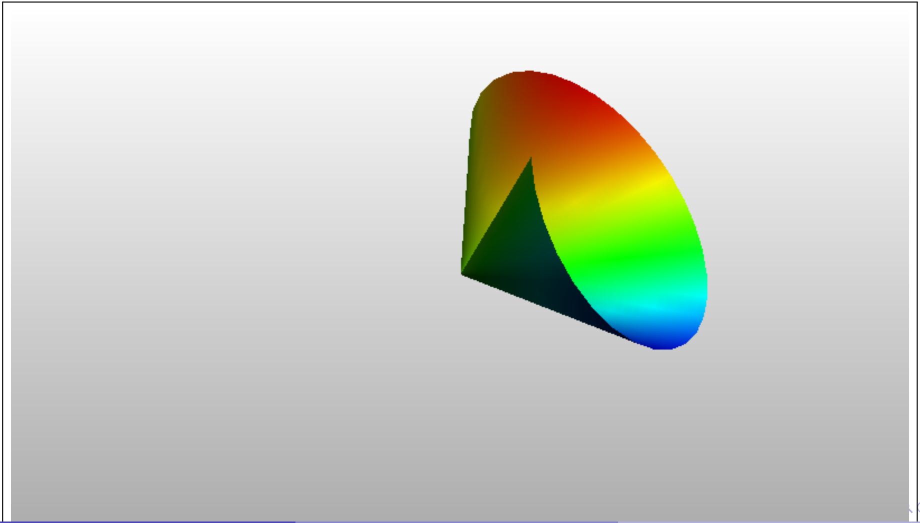

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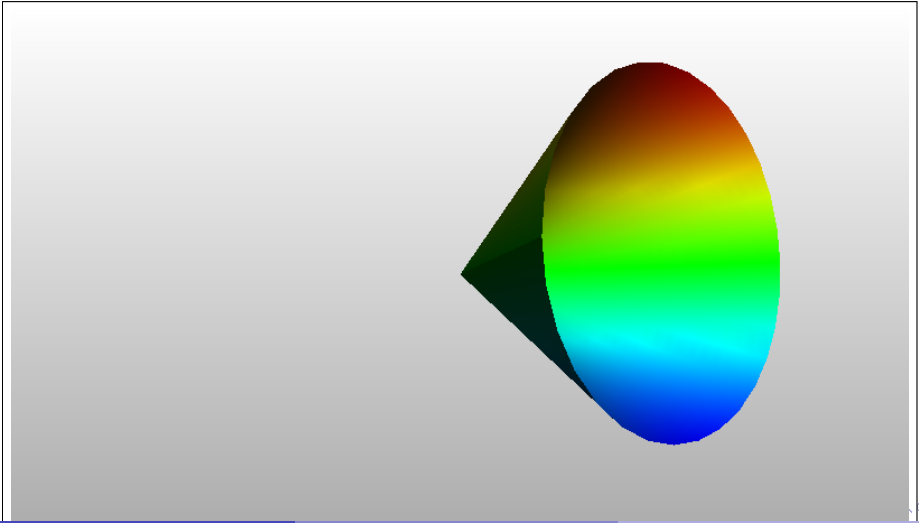

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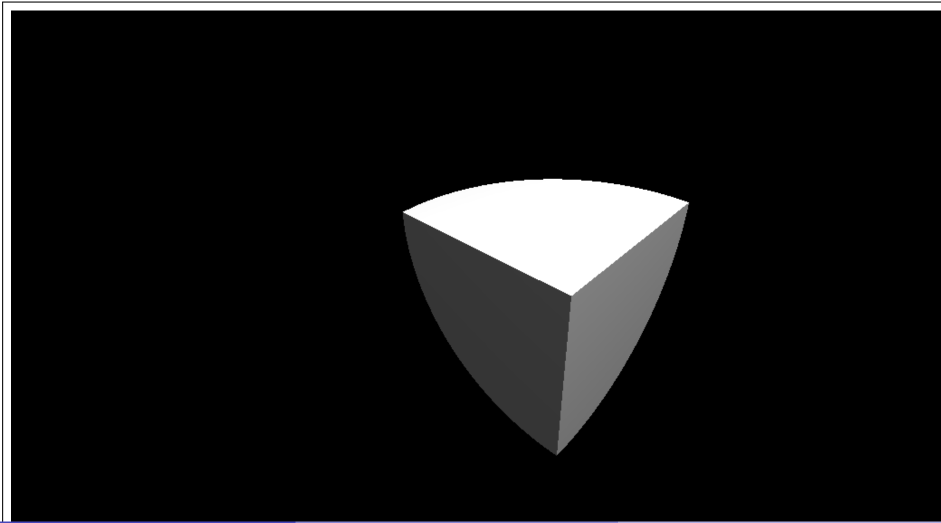
The geometry of the double solid cube is the following : at a vertex of the cube, we have to glue two cones over $1/8$ of sphere.



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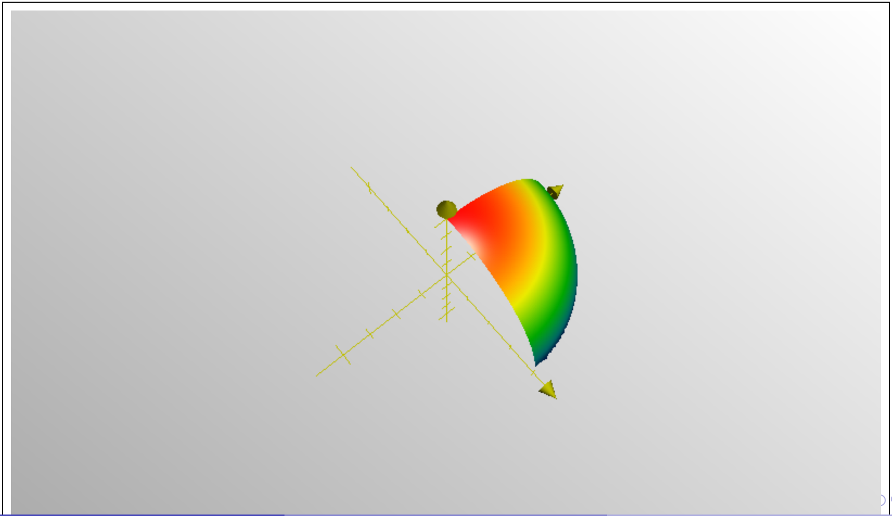

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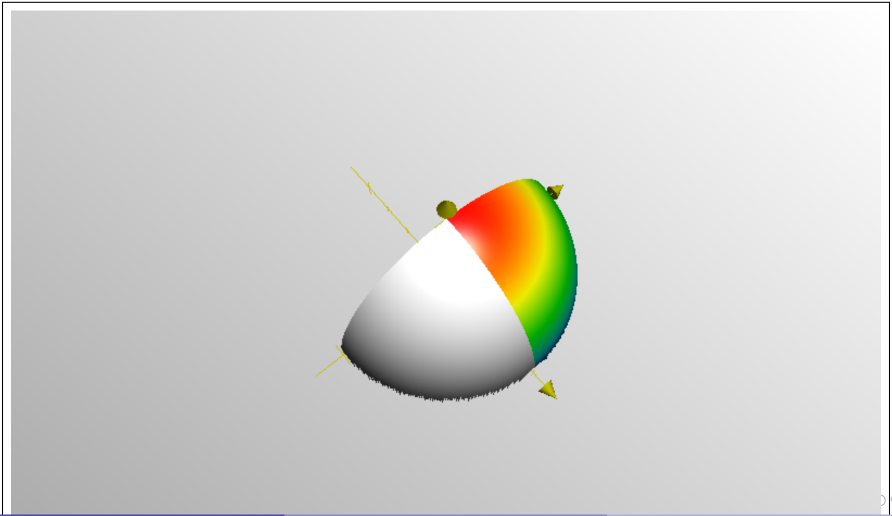

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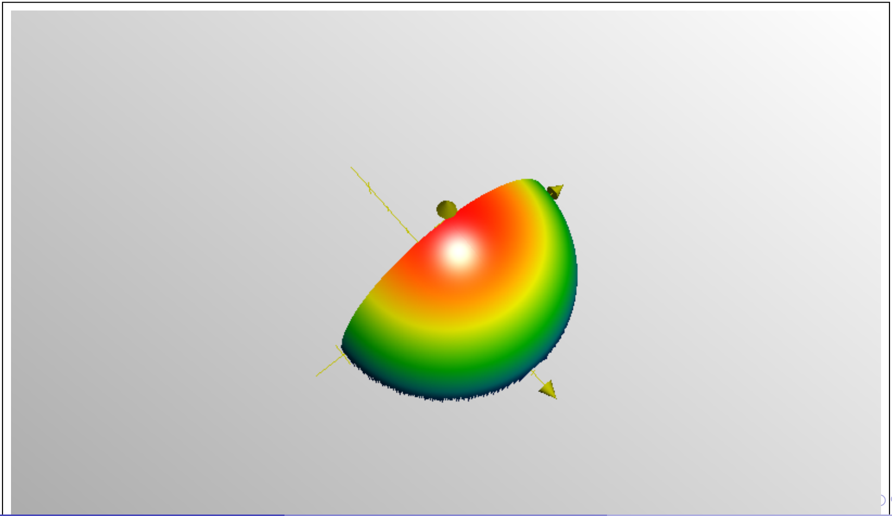

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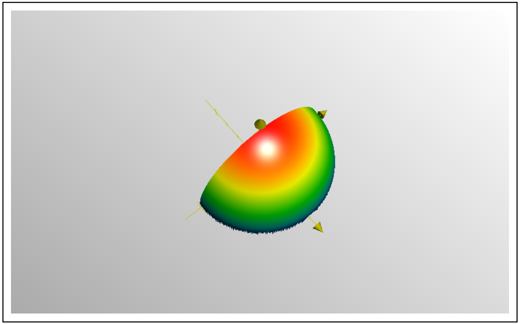

This is a cone over the berlingot !

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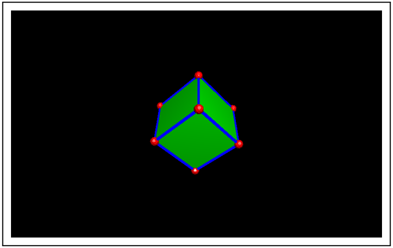
Summary : The double solid cube has a decomposition $X \supset X_1 \subset X_0$:



- On $X \setminus X_1$ the geometry is Euclidean
- $X_1 \setminus X_0$ is the union of 12 unit segments and at a point on $X_1 \setminus X_0$, the geometry is the product of an interval with a cone whose link has length π.
- X_0 consists of 8 points and the geometry near these points looks like a cone over a berlingot.

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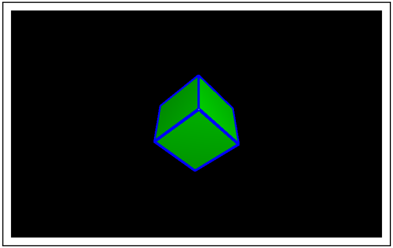

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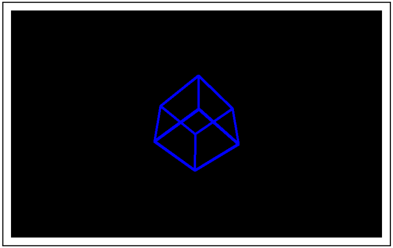

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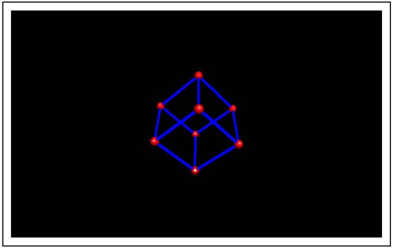

What are stratified spaces ?

The basics object are cone over metric space : if Σ is a complete metric space with distance d_Σ, the cone $C(\Sigma)$ over Σ is the completion of the product $(0, \infty) \times \Sigma$ with the distance for
$p = (t, x),\ q = (s, y) \in (0, \infty) \times \Sigma$

$$d(p,q) = \begin{cases} t + s & \text{if} \quad d_Y(x,y) \geq \pi \\ \sqrt{t^2 + s^2 - 2ts\cos d_Y(x,y)} & \text{if} \quad d_Y(x,y) \leq \pi \end{cases}$$

We have only to blown down $\{0\} \times \Sigma$ to a point (the vertex of the cone) from $[0 + \infty) \times \Sigma$.

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$$X \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$$

such that

- near each point $x \in X \setminus X_{n-2} = X_{\mathrm{reg}}$, the geometry is Riemannian.

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where Σ_x is a $(n-k-1)$- dimensional stratified space.

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Some remarks :

- It is an inductive definition.
- Near $x \in X_k \setminus X_{k-1}$, the $\mathbb{R}^k$-*directions* are tangent to the stratum X_k.

Yamabe flow on stratified spaces

The Yamabe flow is a parabolic flow $g = u^{\frac{4}{n-2}} g_0$:

$$\frac{4}{n-2}\frac{\partial}{\partial t}u = \sigma(t)u - u^{-\frac{4}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_{g_0}u + \mathrm{Scal}_{g_0}u\right),$$

where $\sigma(t) = \int_M \mathrm{Scal}_g \mathrm{dv}_g = \int_M \left(\frac{4(n-1)}{n-2}|du|_0^2 + \mathrm{Scal}_{g_0}u^2\right)\mathrm{dv}_{g_0}$.

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Our convention for Δ_g is that

$$\int_M |du|_0^2 \mathrm{dv}_{g_0} = \int u\,\Delta_{g_0}u\ \mathrm{dv}_{g_0}.$$

Yamabe flow on stratified spaces, existence

Theorem (Carron, Olsen Lye & Vertman, 2021)

Assume that X is a stratified space of dimension $n > 2$ and that g_0 is a Riemannian metric on X_{reg} such that

$$\mathrm{vol}_{g_0} X_{reg} = 1 \text{ and } \mathrm{Scal}_{g_0} \in L^{p > \frac{n}{2}}$$

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This a parabolic counterpart of some existence result for the Yamabe problem on stratified spaces [Akutagawa-C-Mazzeo 2014].

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We used functional analytic method to get the existence of solution of equation of the type

$$\frac{\partial}{\partial t} u = u^{\beta} \left(-Lu + Vu\right)$$

on Dirichlet space $(X, d, \mu, \mathcal{E})$ where

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$$\forall\, w \ \in \mathcal{D}(\mathcal{E}) \colon \ \|w\|^2_{L^{\frac{2n}{n-2}}} \leq A \left(\mathcal{E}(w) + \|w\|^2_{L^2}\right).$$

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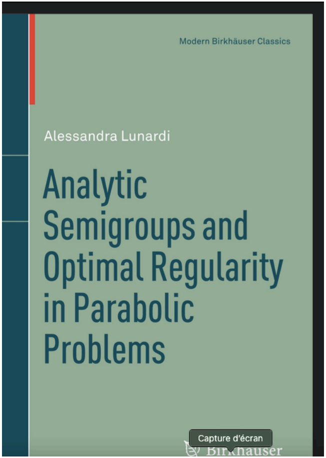
This abstract framework generalizes $\mathcal{E}(u) = \int_M |du|^2_g \mathrm{dv}_g = \int_M \Delta_g u u \mathrm{dv}_g$.

Yamabe flow on stratified spaces, existence

Together with the suitable general theorem that can be found in the very good book :



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This general theorem also provides useful information about the regularity of the Scalar curvature.

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- We do not know wether $\Delta_{g_0} \mathrm{Scal}_g \in L^p$
- But we know that $\Delta_g \mathrm{Scal}_g \in L^p$
- And we can deduce Scalar curvature estimate from the equations :

$$\frac{\partial}{\partial t} \mathrm{Scal}_{g(t)} + (n-1)\Delta_{g(t)} \mathrm{Scal}_{g(t)} = -\mathrm{Scal}_{g(t)} \left(\mathrm{Scal}_{g(t)} - \sigma(t) \right).$$

Yamabe flow on stratified spaces, asymptotic behavior

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Yamabe flow on stratified spaces, asymptotic behavior

- We also have a dichotomy compactness vs concentration (formation of bubbles).
- We do not have a positive mass theorem.
- There are examples where only concentration occurs.

Yamabe flow on stratified spaces,bubble

Under the hypothesis that

$$\|\mathrm{Ricci}_{g_0}\|_{L^\infty} < \infty$$

and that cone angles of the tangent spaces along $X_{n-2} \setminus X_{n-3}$ are always less than 2π, we get a very description of the blow-up profile.

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For $g(t) = u^{\frac{4}{n-2}}(t)g_0$ a solution of the Yamabe flow. We can extract subsequence $t_k \to \infty$ and find

- $u_\infty \in H^1(X)$ solving the equation

$$c_n \Delta_{g_0} u_\infty + \mathrm{Scal}_{g_0} u_\infty = \sigma_\infty u_\infty^{\frac{n+2}{n-2}}$$

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  such that $u(t_k)$ converges weakly to u_∞.
- a finite number of bubbles $B_k^1, \ldots, B_k^L$ such that

  $$\lim_{k\to\infty} \left\| u(t_k) - u_\infty - \sum_{j=1}^{L} B_k^j \right\|_{H^1} = 0.$$

Yamabe flow on stratified spaces,bubble

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and that cone angles of the tangent spaces along $X_{n-2} \setminus X_{n-3}$ are always less than 2π, we get a very description of the blow-up profile.
We have

$$B_k^\ell(x) = \left(\frac{c\lambda_\ell \epsilon_k(\ell)}{(\lambda_\ell \epsilon_k(\ell))^2 + d(x, p_k(\ell)))^2} \right)^{\frac{n-2}{2}}$$

and the bubble are well separated if $i \neq j$ then

$$\lim_{k\to+\infty} \frac{\epsilon_k(j)}{\epsilon_k(i)} + \frac{\epsilon_k(i)}{\epsilon_k(j)} + \frac{d^2(p_k(j), p_k(i))}{\epsilon_k(i)\epsilon_k(j)} = +\infty.$$

Yamabe flow on stratified spaces,bubble

We can adapt the classical blow-up analysis (Struwe) in this general setting and a key point is provided by a result of I. Mondello (2016).

Theorem, (I. Mondello, 2016)

Assume that $\|\mathrm{Ricci}_{g_0}\|_{L^\infty} < \infty$ and that cone angles of the tangent spaces along $X_{n-2} \setminus X_{n-3}$ are always less than 2π. If X_p is a blow-up/tangent space of X at $p \in X$ and $u\colon X_p \to \mathbb{R}_+$ a solution of the equation

$$c_n \Delta u = \sigma u^{\frac{n+2}{n-2}}.$$

Then there is a constant $\lambda > 0$ such that

$$u(x) = \left(\frac{c\lambda}{\lambda^2 + d_{X_p}(x,p)^2}\right)^{\frac{n-2}{2}},$$

with

$$c = \sqrt{\frac{n(n-1)}{\sigma}}.$$