Interplay between Geometric Analysis and Discrete Geometry

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1 Overview of the Field

Convexity, as a branch of classical mathematics, is located at the confluence of geometry, analysis, topology and combinatorics. Although its origins can be traced back to Archimedes and its systematic study started at the end of the nineteenth century, it was not until the mid-twentieth century that convexity became a well established branch of mathematics. Nowadays, it is an important area that is attracting young researchers and students because of its intrinsic beauty, its fascinating and intriguing open questions that have an instant intuitive appeal, but also, because of its many relations with other areas of mathematics and its multiple applications, such as in economics, engineering and data science.

The purpose of this workshop was to bring together key people working in this area, in order to explore recent progress and to help focus on future research directions. In particular, we wanted to invite many young researchers, so that they can network among each other and with established researchers.

2 Workshop Structure

We were able to have 17 in-person participants at CMO, together with about 13 more online participants. This turned it into a smaller workshop than originally anticipated, but it led to an environment where there was more time available for informal working in groups. This was done in the afternoons, with talks usually scheduled in the mornings. There was a nice balance between in-person and online talks, and together with the technology made available by CMO made this a successful hybrid workshop.

3 Recent Developments Addressed by the Workshop

3.1 Meissner and Ball Polytopes

Interestingly and contrary to common belief, the *Reuleaux tetrahedron*, constructed in a similar way as the classical Reuleaux triangle is not a body of constant width. In 1911, Meissner introduced special convex *bodies of constant width*, based on a regular tetrahedron, which are now named after him. It is a well-known and still unsolved conjecture that these *Meissner bodies* have the smallest volume among all three-dimensional convex bodies of the same constant width, see [14] for a survey. In [13] it is asked whether it is possible to confirm this conjecture with the help of computers. In 2016, L. Montejano and E. Roldán [24] found a natural connection between the constructions of Meissner bodies and auto-dual polyhedra or auto-dual ball polyhedra. In this case, its one skeleton is a planar involutive self-dual graph metrically embedded in the three dimensional Euclidean space in such a way that the distance between two of its vertices is less or equal than 1 and it is exactly one if and only if one point is in the dual cell of the other. Can it be generalized in dimension four? All explicit higher dimensional constant width bodies presented until now have some symmetry properties, like rotational or tetrahedral symmetry, see [17] for constructions and various characteristic properties. It is natural to ask for symmetry properties of bodies of constant width in higher dimensions.

3.2 Approximation of convex bodies by polytopes

The general question in the area is to find a convex polytope that is close (with respect to some notion of distance) to a given convex body, and, at the same time, has low complexity (eg. few vertices, or few facets), see [2, 8, 12]. Many results naturally combine analytical and combinatorial methods. Recent works of Barvinok [5] and of Naszódi, Nazarov and Ryabogin [30] provide sharp bounds for the problem of approximation in the Banach-Mazur distance, in the case of fine approximation, that is, where the polytope has a large number of vertices. Rough approximation, on the other hand, remains poorly understood (a computational geometric approach is presented in [29]). Is there a constant $c > 0$ such that for any d, any convex body K in $\mathbb{R}^d$, and for any $0 < t < 1/2$, there is a convex polytope P with e^{cdt^2} vertices for which $tK' \subseteq P \subseteq K'$, for a properly translated image K' of K.

3.3 The Boltyanskii–Hadwiger Illumination Problem

This problem asks whether any convex body in d-space may be covered by 2^d translates of its interior, see [6, 7]. According to a 60-years old classical result of Rogers, roughly 4^d translates suffice. On the other hand, the conjecture has been proved for a variety of special convex bodies, see [6, 7]. Recently, a combinatorial approach to the problem was presented in [27]. Moreover, Livshyts and Tikhomirov [19] showed that any convex body close to the cube in the Banach-Mazur metric and different from a parallelotope can be covered by at most $2^d - 1$ translates of its interior. The following question is related, and possibly, within reach (see also [28]). Does the conjecture hold for bodies that are close to the Euclidean ball B, say $B \subseteq K \subseteq 5B$? Another question in the theory of geometric coverings is the *Translative Plank Conjecture*, a converse to *Tarski's classical plank problem*, first formulated by Makai and Pach [22] in connection with the problem of approximating functions. It states that for a given sequence of slabs in Euclidean d-space, there are translation vectors such that the translated slabs cover the whole space if, and only if their total width is infinite. Here, a slab means a region bounded by two parallel hyperplanes. Recent progress has been seen in [15, 16].

4 Pre-workshop lectures

Two pre-workshop lectures were presented by Alina Stancu and Vlad Yaskin. These lectures were given live on Zoom, and recordings were made available to workshop participants.

4.1 Alina Stancu (Concordia University): An introduction to affine invariants of convex bodies

Convex bodies considered here are at least C^2, strictly convex, and have positive Gauss curvature. Affine invariants of convex bodies are quantities or measures that are invariant under transformations from $SL(d)$. For example the volume of K but not the surface area of K. Affine surface area is an affine invariant, introduced by Blaschke, defined as $\Omega(K) = \int_{\partial K} k^{1/(d+1)} dS_{\partial K}$, where k is the Gauss curvature. This quantity vanishes for polytopes, and in 1991 was shown to be upper semicontinuous by Lutwak [20]. The affine isoperimetric inequality states that $\Omega(K)^{d+1}/|K|^{d-1}$ is maximized by ellipsoids. By rewriting the defining integral for affine surface area, a certain local affine invariant is derived, which can sometimes be used to show that a convex body is an ellipsoid. Modifying this quantity gives the p-affine surface area.

Some applications of affine surface area were then discussed.

Barany and Prodromou [4] found an unexpected application of affine perimeter in the plane. The maximum number of vertices of a convex $\frac{1}{t}\mathbb{Z}^2$-polygon contained in a convex body K in $\mathbb{R}^2$ can be determined asymptotically in terms of the affine perimeter of convex subsets of K. Specifically, if $m(K, \frac{1}{t}\mathbb{Z}^2)$ denotes the largest n such that there exists an n-gon with vertices in $K \cap \frac{1}{t}\mathbb{Z}^2$, and $A(K)$ denotes $\sup\{\Omega(S) \mid S \subset K\}$, then $\lim_{t\to\infty} t^{-2/3} m(K, \frac{1}{t}\mathbb{Z}^2) = \frac{3}{(2\pi)^{2/3}} A(K)$.

Also, for any convex body K in $\mathbb{R}^2$ there exists a unique convex body K_a such that $\Omega(K_a) = A(K)$, and if K is *elliptic* ($k^{1/2} : \mathbb{S}^1 \to \mathbb{R}$ is the support function of a convex body), $K_a = K$.

Schneider [33] generalised this to $\mathbb{R}^d$, where existence of K_a is known, but not uniqueness.

Winternitz [37] showed that if K is sufficiently smooth and contained in an ellipsoid E, then $\Omega(K) \leq \Omega(E)$. Leichtweiss [18] and Lutwak [20] independently proved that if L is elliptic and $K \subseteq L$, then $\Omega(K) \leq \Omega(L)$.

Another application is on expectations and floating bodies. Consider a convex body with volume 1, and let K_n be the convex hull of n random points from K. Rényi and Sulanke [32] showed that the expected number of $(d-1)$-dimensional faces of K_n is asymptotically $c_d \Omega(K) n^{\frac{d-1}{d+1}}(1 + o(1))$. Barany [1] generalized this to k-dimensional faces.

The floating body of K of index δ, denoted $K_{(\delta)}$, is the body that remains after cutting off all hyperplane sections of volume δ. Barany and Larman [3] showed that the expected volume of $K \setminus K_n$ is $\Theta(|K_{(1/n)})$.

An intuitive way to see the affine invariance of $\Omega(K)$ wheree K is a convex body in the plane: the affine perimeter can be expressed in terms of areas of triangular caps of polygons circumscribing K, which can immediately be seen to be affine invariant. In higher dimensions, Werner showed that $\Omega(K) = \lim_{\delta\searrow 0} \frac{|K| - |K_{(\delta)}|}{\delta^{2/(d+1)}}$ if K is sufficiently smooth. This also immediately gives that $\Omega(K)$ is affine invariant.

Bárány and Larman (1988) showed that for all convex bodies of volume 1, and with $t \leq (2d)^{-2d}$, $t \log^{d-1} \frac{1}{t} \ll |K_{(t)}| \ll t^{d/(d+1)})$. The right-hand side occurs for smooth convex bodies, and can be used to deduce the affine isoperimetric inequality via Werner's characterization of affine surface area in terms of the floating body. The left-hand side occurs for polytopes.

An open question: If $K(\delta)$ is homothetic to K for some $0 < \delta|K|/2$, then K is an ellipsoid. It is known [34] that if δ is sufficiently small, and K is C^{2+} with positive Gaussian curvature everywhere, then the answer is positive.

4.2 Vlad Yaskin (University of Alberta): Some extremal problems involving centroids of convex bodies

Grünbaum-type inequalities

Grünbaum's inequality states that for any n-dimensional convex body K with volume $|K|$, the volume cut off by any hyperplane through its centroid is at least $\left(\frac{n}{n+1}\right)^n |K| > e^{-1}|K|$. This is sharp, as shown by a cone.

He asks the following question: Does there exist an absolute constant $c > 0$ such that for any hyperplane section $H \cap K$ of K through its centroid, any hyperplane of H through the (original) centroid cuts off an $(n-1)$-dimensional volume of at least $c|K \cap H|_{n-1}$? Grünbaum's result cannot be used, because the centroid of $K\cap$ does not coincide with the centroid of K. A more general version of this result (for any k-dimensional subspace instead of $n-1$-dimensional) was shown by Fradelizi, Meyer and Yaskin [11], although with non-optimal constants.

The following version for projections of constant bodies onto arbitrary subspaces was shown by Stephen and Zhang [36]: For any k-dimensional subspace E, if K is projected onto E, then any hyperplane through the projection of the centroid of K cuts off a k-dimensional volume of the projection $K|E$ of at least $\left(\frac{k}{n+1}\right)^k |(K|E)|_k$. This inequality is optimal.

Meyer, Nazarov, Ryabogin and Yaskin [23] showed the following functional version of Grünbaum's inequality: For any log-concave $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} x f(x)dx = 0$, then $\int_0^\infty f(tu)dt \geq e^{-n} \int_{-\infty}^\infty f(tu)dt$

for all $u \in \mathbb{S}^{n-1}$. Again, the constant is sharp.

This was further generalized to γ-concave functions by Myroshnychenko, Stephen and Zhang [25], from which they got as a corollary that if a convex body K is intersected by a k-dimensional subspace E through its centroid, then hyperplane sections of the intersection $K \cap E$ through the original centroid cuts off a k-dimensional volume of at least $\left(\frac{k}{n+1}\right)^k |(K \cap E)|_k$, again with a sharp constant.

Makai and Martini [21] and Fradelizi [10] showed that for any convex body and any k-dimensional subspace K through the centroid of K, $|K \cap E|_k \geq \left(\frac{k+1}{n+1}\right)^k \max_{x \in K} |(K-x) \cap E|_k$, also with a sharp constant.

M. Stephen and Yaskin [35] generalized these results to intrinsic volumes:

$$V_i(K \cap E) \geq \left(\frac{i+1}{n+1}\right)^i \max_{x \in K} V_i((K-x) \cap E), \quad i = 1, \ldots, k.$$

Problems on distances between centroids

The original motivation for these problems come from Grünbaum-type inequalities. In general, how far apart can $P_H(c(K))$, the projection of the centroid of K onto a hyperplane H and $c(P_H(K))$, the centroid of the projection of K onto H be, relative to some linear measure of K? The measure that will be used is $w_u(K)$, the width of K in the direction u parallel to $P_H(c(K)) - c(P_H(K))$.

In dimension 2, Stephen showed $|P_H(c(K)) - c(P_H(K))| \leq \frac{1}{6} w_K(u)$, with $1/6$ being the best constant.

Let $D_n > 0$ be the smallest number such that $|P_H(c(K)) - c(P_H(K))| \leq D_n \cdot w_K(u)$ for all convex bodies K in $\mathbb{R}^n$ and all hyperplanes H in $\mathbb{R}^n$, with u the unit vector parallel to $P_H(c(K)) - c(P_H(K))$. Myroshnychenko, Tatarko and Yaskin [26] showed that $D_3 = 1 - \sqrt{2/3}$, for each fixed n, D_n is the maximum of a certain rational function of one variable, the sequence (D_n) is increasin with $\lim_{n \to \infty} D_n \approx 0.2016$. The extremizers are completely known.

Croft, Falconer and Guy [9] ask: How far apart can the centroids of K and its boundary ∂K be relative to some linear measure of K?

Nazarov, Ryabogin and Yaskin [31] show that in the plane, $|c(\partial K) - c(K)| \leq \frac{1}{6} \operatorname{diam}(K) \leq \frac{1}{12} \operatorname{perim}(\partial K)$. More generally, for any direction u, $|\langle c(\partial K) - c(K), u \rangle| \leq \frac{1}{6} w_K(u)$. The constants are all best possible.

In higher dimensions, they suspect that the same constants D_n from before will be the optimal constants in this problem.

5 Research Talks

5.1 Károly Bezdek: From the Kneser–Poulsen conjecture to r-ball-bodies

Starting the workshop, this online talk introduced the original Kneser–Poulsen conjecture originally proposed independently by Kneser (1955) and Poulsen (1954), which states that the volume of a union of a finite number of balls decreased after there centers are moved closer to each other. As a second topic, *ball-polyhedra*, the intersections of finitely many congruent balls, were considered. Both topics have been studied from the point of view of convex and discrete geometry. The talk aimed to bridge the two topics by discussing a selection of old and new results, many of them by the speaker.

5.2 Andriy Prymak: Convex bodies of constant width with exponential illumination number

Borsuk's number $f(n)$ is the smallest integer such that any set of diameter 1 in the n-dimensional Euclidean space can be covered by $f(n)$ sets of smaller diameter. The currently best known asymptotic upper bound $f(n) \leq (\sqrt{3/2} + o(1))^n$ was obtained by Shramm (1988) and by Bourgain and Lindenstrauss (1989) using different approaches. Bourgain and Lindenstrauss estimated the minimal number $g(n)$ of open balls of diameter 1 needed to cover a set of diameter 1 and showed $1.0645^n \leq g(n) \leq (\sqrt{3/2} + o(1))^n$. On the other hand, Schramm used the connection $f(n) \leq h(n)$, where $h(n)$ is the illumination number of n-dimensional convex bodies of constant width, and showed $h(n) \leq (\sqrt{3/2} + o(1))^n$. The best known asymptotic lower bound on

$h(n)$ is subexponential and is the same as for $f(n)$, namely $h(n) \geq f(n) \geq c^{\sqrt{n}}$ for large n established by Kahn and Kalai with $c \approx 1.203$ (1993) and by Raigorodskii with $c \approx 1.2255$ (1999). In 2015 Kalai asked if an exponential lower bound on $h(n)$ can be proved. The speaker, in joint work with Andrii Arman and Andriy Bondarenko, showed that $h(n) \geq (\cos(\pi/14) + o(1))^{-n}$ by constructing the corresponding n-dimensional bodies of constant width, which answers Kalai's question in the affirmative. The construction is based on a geometric argument combined with a probabilistic lemma establishing the existence of a suitable covering of the unit sphere by equal spherical caps having sufficiently separated centers. The lemma also allows to improve the lower bound of Bourgain and Lindenstrauss to $g(n) \geq (2/\sqrt{3} + o(1))^n \approx 1.1547^n$. Improving work of Naszódi (2016), they show that for all $D \in (1, 2/\sqrt{3})$ there exists a convex body K that is D-close to a Euclidean ball and has illumination number $> c\sqrt{n}D^n$.

5.3 Ferenc Fodor: Central limit theorems and floating bodies

There have been several results recently concerning central limit theorems for random polytopes in various geometric settings that were proved via Stein's method. Some of the key ingredients of these arguments are floating bodies and their visibility regions. In the last few years, there has also been quite much work done on random polytope models in which the notion of convex hull was modified. The combination of these two topics raises questions about generalizations of floating bodies and their use in proving limit theorems and more. As an application, the speaker, in joint work with Dániel Papvári (Szeged), showed a quantitative central limit theorem for the area of random disc-polygons, and raise several open questions.

5.4 Cameron Strachan: The boundary structure of C-polygons

Given a convex domain C, a C-polygon is an intersection of $n > 1$ homothets of C. In this talk the speaker explored the boundary structure of C-polygons and demonstrated how different properties on the boundary of C, such as smoothness or strict convexity, imply bounds on the complexity of the boundary of a C-polygon.

5.5 Grigory Ivanov: Coarse approximations of polytopes

Different quantitative versions of classical convexity results have recently gained attention. In this talk, the speaker focused on coarse approximation of polytopes. He, in joint work with Márton Naszódi, showed that the convex hull of a carefully selected set of $2d$ vertices from a well-centered polytope $P \subset \mathbb{R}^d$ contains a homothet $c(d)P$ of the original polytope P. Although the proof of this result requires only a basic understanding of linear algebra, its implications extend beyond its simplicity, shedding light on quantitative Helly-type and Carathéodory-type results.

5.6 Ádám Sagmeister: Reduced convex bodies in spaces of constant curvature and Pál's isominwidth inequality

We call a convex body K reduced, if for any different convex body contained in K has a smaller minimal width. Reduced bodies are extremizers to some inequalities in convex geometry, and they also give a different perspective to the broadly studied family of bodies of constant width. There are multiple recent studies about reduced bodies in Minkowski spaces and spherical reduced bodies, and we also present a hyperbolic approach after introducing an extended version of Leichtweiss' width function. In this online talk, the speaker, in joint work with Károly J. Böröczky, András Csépai and Ansgar Freyer, proved the hyperbolic version of Pál's inequality, stating that the regular triangle has the smallest area among convex bodies of minimal width, and we will also discuss a stability version of the theorem.

5.7 Vladyslav Yaskin: An analogue of polynomially integrable bodies in even-dimensional spaces

For a convex body K in $\mathbb{R}^n$, its parallel section function is given by $A_{K,\xi}(t) = |K \cap (\xi^\perp + t\xi)|$, where $\xi \in S^{n-1}$ and $t \in \mathbb{R}$. We say that K is *polynomially integrable* if $A_{K,\xi}(t)$ is a polynomial of t on its support. It was shown by Koldobsky, Merkurjev, and Yaskin that the only polynomially integrable bodies are ellipsoids in odd dimensions. In even dimensions such bodies do not exist. In this talk the speaker discussed an analogue of polynomially integrable bodies in even dimensions: these are the bodies for which the Hilbert transform of $A_{K,\xi}(t)$ is a polynomial of t (on an appropriate interval). In joint work with M. Agranovsky, A. Koldobsky, and D. Ryabogin, he showed that ellipsoids in even dimensions are the only convex bodies satisfying this property.

5.8 Illya Ivanov: Facet-lean polyhedra: maybe the hardest polyhedra to illuminate

Convex polyhedron in $\mathbb{E}^d$ is a bounded intersection of a finite set of halfspaces. A polyhedron is facet-lean if omitting any halfspace from the intersecting set makes the intersection unbounded. In this talk the speaker classified the facet-lean polyhedra, proved that any facet-lean polyhedron in $\mathbb{E}^d$ can be illuminated by 2^d light sources, and outlined possible directions that might lead to proving the illumination conjecture for convex polyhedra in general case.

5.9 Sean Dewar: Introducing genericity to convex body packings

Given n homothetic copies of a convex body with disjoint interiors, it is natural to ask how many possible contacts can occur between them. For example, Oded Schramm proved in his thesis and other early research that (with some special exceptions) every planar graph can be realised as the contact graph of such a (2-dimensional) packing. However, this requires that the scaling for each homothetic copy of the convex body is very carefully selected; for example, while a disc can be chosen to lie in the interior of three pairwise-touching discs, the radius of such a disc is essentially fixed. Connelly, Gortler and Theran proved in 2019 that if the radii of a disc packing are randomly selected, then the packing can have at most $2n-3$ contacts, a significant departure from the maximum achievable count of $3n-6$. In this talk, the speaker discussed his own analogous results on the topic when dealing with a variety of types of convex bodies in 2-dimensions and higher, including: smooth and strictly convex centrally symmetric convex bodies, squares, convex bodies with positive curvature boundaries, spheres and cubes.

5.10 Attila Pór: Orientation Preserving Map of the Grid and Projective Rigidity

In this online talk, the speaker introduced a function on order types that measures their rigidity with respect to projective transformations and give some examples. He showed that for large n the $n \times n$ grid is *rigid*, in the sense that every orientation-preserving map of the $n \times n$ grid is $O(\frac{1}{n})$ close to a projective transformation. Other examples include the order types on at most five points, some convex n-gons and the square grid on 9 and 16 points.

5.11 Alexander Litvak: Volume ratio between projections of convex bodies

In this online talk, the speaker discussed volume ratios between convex bodies and their projections. In joint work with D. Galicer, M. Merzbacher, and D. Pinasco, he showed that for every n-dimensional convex body K, there exists a centrally-symmetric convex body L such that for any two projections P, Q of rank $k \leq n$ the volume ratio between PK and QL is large. This result is sharp (up to logarithmic factors) when $k \geq n^{2/3}$.

5.12 Zsolt Langi: On a strengthened version of a problem of Conway and Guy on convex polyhedra

A boundary point q is an equilibrium point of a 3-dimensional convex body K with center of mass c if the plane through q and orthogonal to the segment $[q, c]$ supports K. In this case, in particular, K can be balanced on a horizontal plane in a position touching it at q. Assuming nondegeneracy of the body, three types of equilibrium points are distinguished: stable, unstable and saddle-type. A consequence of the Poincaré-Hopf theorem is that the numbers S, U and H of the stable, unstable and saddle-type points of a 3-dimensional convex body, respectively, satisfy the equation $S - H + U = 2$. If a convex body has a unique stable or unstable point, it is called *monostable* and *mono-unstable*, respectively, and a body which is simultaneously monostable and mono-unstable is called *mono-monostatic*. A famous example of a mono-monostatic convex body is the so-called *Gömböc*, constructed by Domokos and Várkonyi. A 1969 question of Conway and Guy, also asked independently by Shephard in 1968, asks if there is a monostable convex polyhedron with a k-fold rotational symmetry, where $k > 2$. The answer to this problem was given by the speaker in 2022, who proved the existence of such a polyhedron for all positive integers $k > 2$. In this talk, he, in joint work with G. Domokos and P. Várkonyi, strengthened this result of his by characterizing the possible symmetry groups of monostable, mono-unstable and mono-monostatic convex polyhedra and convex bodies.

5.13 Gyivan E López: On Borsuk's number and the Vázsonyi problem in $\mathbb{R}^3$

In this talk the speaker introduced some connections between the Borsuk partition problem and the Vázsonyi problem, two attractive and famous problems in discrete and combinatorial geometry, both involving the diameter of a bounded set $S \subset \mathbb{R}^3$.

Borsuk's problem asks whether every set $S \subset \mathbb{R}^d$ with finite diameter $\mathrm{diam}(S)$ is the union of $d+1$ sets of diameter less than $\mathrm{diam}(S)$. For $\mathbb{R}^2$ and $\mathbb{R}^3$ this holds, but for dimensions greater than 63 the statement is false.

Vázsonyi's problem on the other hand asks for the maximum number of diameters over all sets of n points in $\mathbb{R}^d$. In $\mathbb{R}^3$, the answer is $2n-2$ and the configurations attaining this number are already known.

In joint work with Déborah Oliveros and Jorge Ramírez Alfonsín, the speaker presented an equivalence between the critical sets with Borsuk number 4 in $\mathbb{R}^3$ and the minimal structures for the Vázsonyi problem.

5.14 Luisa Fernanda Higueras Monaño: A topological insight into the polar involution of convex sets

Denote by $\mathcal{K}_0^n$ the class of closed convex sets $A \subseteq \mathbb{R}^n$ containing the origin $0 \in A$, and recall that the polar duality (or polarity) is the map on $\mathcal{K}_0^n$ sending $A \in \mathcal{K}_0^n$ to its polar set A°. It is well-known that polarity is an involution on $\mathcal{K}_0^n$ with a unique fixed point, and that it reverses inclusions In this talk, the speaker, in joint work with Natalia Jonard-Pérez, exhibited a topological characterization of the polar duality, and described its relation with Anderson's Conjecture, which is an open problem regarding the characterization of all continuous involutions with a unique fixed point on the Hilbert cube $Q = \prod_{i=1}^{\infty}[-1,1]$. To this end, she showed that $\mathcal{K}_0^n$, endowed with the Attouch-Wets metric, is homeomorphic with Q and the polar duality is topologically conjugate with the standard involution $\sigma : Q \to Q$ given by $\sigma(x) = -x$. On the geometric side, she also proved that among all involutions on $\mathcal{K}_0^n$ reversing the inclusion relation, those and only those with a unique fixed point are topologically conjugate with the polar duality.

5.15 Luis Montejano: Complex Ellipsoids

The complex isometric Banach conjecture states that if any two n-dimensional subspaces of a complex Banach space are isometric, then the space is a Hilbert space. The solution of this conjecture relies heavily on a characterization of the complex ellipsoid.

An ellipsoid is the image of a ball under an affine transformation. If this affine transformation is over the complex numbers, we refer to this affine image of a ball as a *complex ellipsoid*. Characterizations of real ellipsoids have received much attention over the years. On the other hand, characterizations of complex ellipsoids have scarcely been considered. In this talk, the speaker gave an overview of joint work with Jorge Arocha and Javier Bracho, which is the study of complex ellipsoids. This is naturally related to the study of complex symmetry. So, characterizing and understanding complex symmetry is vital to characterizing complex ellipsoids. A subset K of $\mathbb{C}^n$ is *complex symmetric* if $K = zK$ for all $z \in \mathbb{C}$ such that $|z| = 1$. The main result of this talk is an unexpected characterization of complex ellipsoids. This result has no analogue over the real numbers, and states that if every complex line intersects K in some disc, then K is a complex ellipsoid. The proof is topological, and it is an open question if there is a proof of this result that avoids topology.

5.16 Bushra Basit: Geometric extremum problems in spaces of constant curvature

Böröczky and Peyerimhoff proved that among simplices inscribed in a ball in spherical and hyperbolic space, respectively, the regular simplices have maximal volume. In this online talk, the speaker explained her result that among simplices circumscribed about a ball in hyperbolic space, the regular simplices have minimal volume. She also considered analogous questions for d-dimensional spherical and hyperbolic polytopes with $d+2$ vertices. These results are joint work with Zsolt Lángi.

5.17 Boaz Slomka: Vertex generated polytopes

In this online talk, which is joint work with Shiri Artstein-Avidan and Tomer Falah, the speaker discussed some curious Brunn-Minkowski type theorems involving boundaries of convex bodies and restricted Minkowski sums. For example, if K and T are compact sets with connected boundary in $\mathbb{R}^n$, then $\mathrm{vol}\left(\frac{\partial K+\partial T}{2}\right) \geq \sqrt{\mathrm{vol}(K)\,\mathrm{vol}(T)}$, with equality in the convex case if and only if they are translates, or, in dimension 2, homothets. Another result is the following restricted Brunn-Minkowski type inequality: If K and T are in $\mathbb{R}^n$ with $\left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(T)}\right)^{1/n} \in [c/\sqrt{n}, \sqrt{n}/c]$, then $\mathrm{vol}(\partial(K)+\partial(T))^{2/n} \geq \mathrm{vol}(K)^{2/n} + \mathrm{vol}(T)^{2/n}$.

These results led them to the study of classes of polytopes with special covering properties. Starting off, the speaker observed that if K is compact with connected boundary, then $\partial K + \partial K = 2K$ (also observed by Fradlizi, Langi, Zvavitch). If K is a polytope P, we can ask about lower-dimensional boundaries. If the k-skeleton of P is denoted by $\partial^k P$, they showed that $\partial^{\lceil n/2 \rceil} P + \partial^{\lfloor n/2 \rfloor} P = 2P$. For a simplex, the value $k = \lfloor n/2$ is the smallest k such that $\partial^k P + \partial^{n-k} P = 2P$. If we denote the smallest such k by $k(P)$, then as just mentioned, $k(P) \leq \lfloor n/2 \rfloor$ for all polytopes P and $k(\text{simplex}) = \lfloor n/2 \rfloor$. Also, $k(B_1^n) = \lfloor (n-1)/2 \rfloor$. A polytope P is called *vertex generated* if $k(P) = 0$. It is shown that P is vertex generated if and only if $P = \mathrm{cl} \sum_{j=1}^{\infty} s^{-j} V(P)$, where $V(P)$ is the vertex set of P. Some more properties of this notion shown are: All zonotopes are vertex generated, if P is vertex generated, then $P+I$ is also, where I is a segment, and for each P there exists a zonotope Z such that $P+Z$ is vertex generated. If P is vertex generated, then it has at least 2^n vertices, with equality only for the affine cube.

In the plane, the collection of vertex-generated polygons is dense with respect to the Hausdorff distance. Another metric that dominates the Hausdorff metric is introduced, and it is shown that the collection of vertex-generated polytopes is closed in this metric.

5.18 Roman Prosanov: On hyperbolic 3-manifolds with polyhedral boundary

It is known that convex bodies in Euclidean 3-space are rigid with respect to the induced intrinsic metric on the boundary. Classically, there has been two approaches: the rigidity of convex polyhedra and the rigidity of smooth convex bodies, though there is also a common generalization obtained by Pogorelov. Thurston's work from the 1970s highlighted the ubiquity and the diversity of hyperbolic manifolds in the 3-dimensional case. Hyperbolic 3-manifolds with convex boundary constitute a large and interesting class to study from various perspectives. In the 1990s to 2000s, an analogue of the Weyl problem for hyperbolic 3-manifolds with smooth convex boundary was resolved in the works of Labourie and Schlenker. Curiously enough, a polyhedral counterpart was not known until recently. One of the reasons is that some metrics on the boundary of such 3-manifolds that are "intrinsically polyhedral" admit realizations that are not so polyhedral and are somewhat more difficult to handle. In this online talk, the speaker described the state of art around these and related problems, and presented a recent proof of the respective polyhedral result in a generic case. Another outcome is a rigidity result for a family of so-called convex cocompact hyperbolic 3-manifolds, important in the theory of Kleinian groups. This is a step towards a resolution of conjectures of Thurston.

6 Problem Session

We held a problem session one evening, in which the floor was open to anyone to state an open research problem. The problems that were presented are given here with their authors.

Luis Montejano: Complex homotheties

Let K_1, K_2 be convex bodies in $\mathbb{R}^{2d} = \mathbb{C}^d$ such that for any complex linear hyperplane H in $\mathbb{C}^d$, we have that $H \cap K_1$ and $H \cap K_2$ are complex homothets, that is, $H \cap K_1 = z(H \cap K_2)$ for a $z \in \mathbb{C}$ depending on H. We may even assume that $|z| = 1$. Does it follow that there is a $z \in \mathbb{C}$ such that $K_1 = zK_2$?

Grigory Ivanov: The Gram matrix and the ball

The *Gram matrix* of a sequence $v_1, \dots, v_n$ of vectors in $\mathbb{R}^d$ is the matrix $G \in \mathbb{R}^{n \times n}$ with entries $G_{ij} = \langle v_i, v_j \rangle$. It is well known that if the Gram matrices of two sequences of vectors are equal, then the two sequences of vectors are congruent.

Question: Given the Gram matrix, can we decide quickly without reconstructing the vectors whether the Euclidean unit ball centered at the origin is contained in the convex hull of the vectors?

Ilya Ivanov: A Kakeya type problem

Let C denote the set of 8 vertices of the inscribed cube of the unit sphere $\mathbb{S}^2$.

What is the minimum (infimum) area of a measurable subset S of $\mathbb{S}^2$ that has the property that any rotation UC of C (where $U \in SO(3)$) intersects S?

It is clear that a hemisphere works (with measure 2π), and that $1/8$ is a lower bound.

Instead of the vertex set of a cube, the same question can be asked for other sets, such as the vertex set of an equilateral triangle inscribed in a great circle, or the vertex set of a regular inscribed tetrahedron.

Edgardo Roldán–Pensado: Covering the Banach–Mazur compactum

Give a bound on $r > 0$ such that the balls

$$\{K \in \mathcal{K}^2 \;:\; d_{\text{BM}}(K, \ell_2^2) \le r\}$$

and

$$\{K \in \mathcal{K}^2 \;:\; d_{\text{BM}}(K, \ell_\infty^2) \le r\}$$

cover the two dimensional Banach–Mazur compactum $\mathcal{K}^2$. More generally, how can we cover the Banach–Mazur compactum with a small number of small-radius balls?

Leonardo Martinez: Realization of orderings

Consider points $p_1, \dots, p_n, q_1, q_2, \dots, q_m$ in $\mathbb{R}^d$. Suppose that the distances $d(p_i, q_j)$ are pairwise distinct. Then, these distances induce a linear order on $[n] \times [m]$ given by $(i_1, j_1) < (i_2, j_2)$ if and only if $d(p_{i_1}, q_{j_1}) < d(p_{i_2}, q_{j_2})$.

If d is fixed, not all possible linear orders on $[n] \times [m]$ can be obtained in this way. Which is the minimal d such that all possible linear orders on $[n] \times [m]$ are representable? The question is mostly solved, since we know the following:

- Every linear order on $[n] \times [m]$ can be obtained via points in $\mathbb{R}^{\min(m,n)}$ (Almendra-Herández and Martínez-Sandoval).
- There are linear orders on $[n] \times [n+1]$ that cannot be obtained via points in $\mathbb{R}^{n-1}$. (Maldonado, Raggi, Roldán-Pensado)
- Every linear order on $[3] \times [3]$ can be obtained via points in $\mathbb{R}^2$ (computational verification).

However, we do not know whether all possible linear orders on $[n] \times [n]$ can always be obtained via points in $\mathbb{R}^{n-1}$. The first open instance is the following:

Problem. Can any possible linear order on $[4] \times [4]$ be induced as the linear order of the distances $p_i q_j$ for points $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$ in $\mathbb{R}^3$?

Andrei Gavriliuk: Realizing a tight fan as a fan over a convex polytope

By a *polytopal fan*, we understand a normal (face-to-face) polytopal complex $\mathcal{F}$ in $\mathbb{R}^d$ which consists of cones (not necessarily pointed cones) which tile $\mathbb{R}^d$ (all of them sharing a common face which is minimal by inclusion). A face F of a polytopal complex $\mathcal{F}$ is called *standard* if it can be represented in a form $C_1 \bigcap C_2$ where C_1 and C_2 are two full-dimensional cones from $\mathcal{F}$. A standard face S of a polytopal fan $\mathcal{F}$ is called *standard–symmetric* if any two full dimensional cones C_1 and C_2 of $\mathcal{F}$ such that $S = C_1 \bigcap C_2$ are locally centrally symmetric to each other about some interior point p of the face F.

Here, local symmetry means that there is an $\varepsilon > 0$ such that $C_1 \bigcap \mathbf{B}(p, \varepsilon)$ is symmetric to $C_2 \bigcap \mathbf{B}(p, \varepsilon)$ about p.

Note. If the face S in the above definition is the common 0-dimensional apex of the cones in the fan $\mathcal{F}$, then we consider S itself as its only interior point p.

A polytopal fan $\mathcal{F}$ is called a *tight fan*, if each of its standard faces are also a standard-symmetric face.

For a d-dimensional convex polytope $P \subset \mathbb{R}^d$ and an arbitrary point p inside P, the family of cones $\{cone(p, G)|\ G \text{ is a face of } P\}$ form a pointed polytopal fan $\mathcal{F}_{p,P}$ (where by $cone(p, G)$ we understand a cone with the apex p over G). This is called a *polytopal fan* (or, a *fan over the polytope P with apex p*).

Problem. Is it true that any pointed tight fan is a polytopal fan (ie., it can be realized as a fan over some convex polytope)?

Note. Any not pointed tight fan can be reduced to a pointed tight fan by a transversal section through an internal point of the common face of all cones of the fan. So the problem may be formulated for arbitrary tight fans.

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