Knots, Surfaces, and 3-manifolds (23w5031)

Mario Eudave Muoz (National Autonomous University of Mexico),
Puttipong Pongtanapaisan (Arizona State University),
Jennifer Schultens (University of California, Davis)

June 11 - 16, 2023

1 Overview of the Field

This workshop focused on various facets of knots, surfaces, and 3-manifolds, which are essential objects in low-dimensional topology. The fundamental problem in knot theory is to decide if one knot can be continuously deformed into the other without creating any self-intersections in the process. In general, the task of distinguishing knots can be difficult as there are many diagrams representing the same knot. To make progress on this problem, it has been proven useful to analyze knot invariants: quantities that one can associate to a knot with the property that equivalent knots have the same invariant. Despite significant progress, many basic invariants are not well-understood. For instance, it is not known how the crossing number of knots behaves under an elementary operation of connect summing.

More interesting topics for investigation arise as we move up one dimension to the realm of surfaces. Many researchers are trying to understand the mapping class groups, which capture the idea of symmetries for surfaces. Some researchers try to exploit the fact that a manifold of dimension greater than three can be represented as curves on surfaces as new directions to prove difficult conjectures. Unsurprisingly, analyzing how surfaces in 3-manifolds intersect also allows us to draw conclusions about knots. This is of great interest to biologists as knots are used to model polymers whose functions (protein folding, transcriptions, etc.) can depend on the shapes they take in space.

2 Recent Developments and Open Problems

2.1 Circular generalized Heegaard splittings

The *handle number* of a knot K in S^3 is the minimum number of critical points among regular Morse functions $f : S^3 \backslash K \to S^1$. Recall that a knot is *fibered* if there exists a fibration $f : S^3 \backslash K \to S^1$ such that f is well-behaved near K. That is, its complement can be filled nicely by copies of an oriented surface. Thus, the handle number is a measure of how far a given knot is from being fibered. Manjarrez-Gutirrez proved that the handle number is additive under connected sum for small knots [11]. In 2021, Baker generalized the result so that the handle number is additive for all knots using the machinery of circular generalized Heegaard splittings [4]. A consequence of Baker's technique is that the handle number is realized on an incompressible Seifert surface. However, it is not known whether the statement is true when the word "incompressible" is replaced by other natural adjectives.

Problem: Is the handle number of a knot always realized by a minimal genus Seifert surface?

The handle number can also be defined in the relative setting. Given a knot in a solid torus, one can discuss a notion of handle number that measures how far a given knot is from being a braid. More formally, there is a projection $f : S^1 \times D^2 \to S^1$ and one counts the number of local extrema of f restricted to the knot (minimized over all embeddings). Thus, one is keeping track of instances of backtracking as one traverses along the knot. The relative version of Problem 1 is also an open question. Here, the minimum wrapping number is the analog of being minimum genus in the previous case.

Problem: Does there exist a representative $K \subset S^1 \times D^2$ with minimum wrapping number realizing the handle number?

Another related complexity measure coming from regular Morse functions $f : S^3 \backslash K \to S^1$ is the *circular width*. In this case, one keeps track of the genera of regular level surfaces (called the *thick surfaces* and the *thin surfaces*) instead of counting the number of critical points, and arrange such quantities using the dictionary order to yield a multi-set valued invariant. Examples of knots that Araceli Guzmn Tristan presented have circular width $\{3, 3\}$, and Scott Taylor asked whether other multi-sets of length at least two arise as circular width of some knots.

Problem: Find examples of knots in a circular width minimizing position containing multiple high genus thin surfaces.

2.2 Generalized bridge numbers of satellite knots

Schubert and Schultens showed that the bridge number behaves as expected under satellite operations. Scott Taylor exploited his way of "thinning" 3-manifolds that he developed with Tomova combined with the technique of searching for crushable handles (for the more precise statement of the definition, please see the recording) to investigate the behavior of generalized bridge numbers. More specifically, he showed that the genus one bridge number of a satellite knot $b_1(T)$ in S^3 or lens spaces that are not $S^1 \times S^2$ is at least the wrapping number times $b_1(C)$, where C is the companion knot. Lively discussions followed Taylor's presentation and some problems were proposed.

Problem: The usual satellite operation involves knotting up a solid torus containing a knot into the shape of the companion. Investigate the generalized bridge number under the operation of knotting up a higher genus handlebody containing a knot. This operation is interesting because one may get a hyperbolic knot as a result.

Problem: When the companion is a torus, what is the behavior of genus one bridge number under satellite operation if the ambient 3-manifold is not a lens space?

2.3 Link concordance

Recall that two knots are isotopic if one knot can be continuously deformed into the other without creating any self-intersections at any time during the deformation. Rather than requiring such an equivalence relation, one can consider a less stringent concept. Under this less restrictive relation of concordant, two links are equivalent if they can be connected by a certain kind of cylinders living in a 4D space. More precisely, a link $L \subset S^3$ of m-component is smoothly (resp. topologically) *concordant* to a link $L' \subset S^3$ of m-component if there exists m disjoint annuli $A_1 \cup A_2 \cup \cdots \cup A_m$ in $S^3 \times [0, 1]$ such that $\partial A_i = L_i \times \{0\} \cup -L'_i \times \{1\}$.

A link $L = L_1 \cup \cdots \cup L_m$ is *slice* if it is concordant to the m-component unlink. A related concept is ribbonness, which can be defined in multiple ways. From the 3-dimensional perspective, a link is *ribbon* if it bounds m disjointly immersed disks with only ribbon singularities. Alternately, a link is ribbon if it bounds m disjoint smoothly embedded disks in the 4-ball with no maxima with respect to the radial height function.

A difficult open question is whether every slice knot is ribbon. To try to solve this long standing conjecture, some researchers considered versions of the concept called homotopy ribbon. A knot is *homotopy ribbon* if it bounds a properly embedded disk whose exterior can be built with only 0,1, and 2 handles. A weaker version of homotopy ribbon also exists, where one just asks that induced map $i_* : \pi_1(E(L)) \to \pi_1(E(D))$ to be surjective. Harvey mentioned the following question in her presentation.

Question: There is a 3-dimensional characterization of ribbon links. Is there a 3-dimensional characterization of homotopy ribbon links?

The idea is that ribbon implies homotopy ribbon. Therefore, it is useful to study homotopy ribbon obstructions.

It turns out that if K is a slice genus one knot, and suppose there is a surgery curve a that is itself slice as a knot in S^3. One can then perform surgery on F by cutting along a and using two copies of a slice disk for a. This implies that K is a slice knot. It turns out that the converse is not true [6]. Harvey suggested in her lecture that there should exists a counterexample in the setting of multiple-component links as well.

Question [Generalized Kauffman Conjecture]: Find a slice boundary link with disjoint Seifert surfaces of genus one such that any derivative has non-vanishing Milnor's invariant $\overline{\mu}(123)$.

The conjecture in the knot setting was originally due to Kauffman. Here, the Milnor's invariant is a well-known slice obstruction.

In the setting of knots, it is true that a slice knot is algebraically slice. That is, given a slice knot, its Seifert form is metabolic. Harvey also posed a question for links.

Question: If a link is slice, then what can be said about its Seifert form?

2.4 Hyperbolic 3-manifolds

There has been some progress made on the following question.

Question: What is the smallest volume n-cusped hyperbolic 3-manifold?

Namely, Cao-Mayerhoff showed in 2001 that the figure-eight knot complement and the (5,1)-surgery on the Whitehead double complements are the smallest volume 1-cusped hyperbolic 3-manifold. Agol and Yoshida also answered the question for the case where $n = 2$ or 4. For other values of n not mentioned, the question is still open (there have been some candidates proposed by Agol coming from a covering construction).

2.5 Complexes of curves and surfaces

2.5.1 Pants complex

A *pants decomposition* of a surface is a collection of curves cutting the surface into pairs of pants. Consider a graph where each vertex of it is a pants decomposition. Two vertices are connected by an edge if the two pants decomposition differ by an elementary move: replacing a curve in the first pants decomposition with a new curve that intersects the original curve once or twice. This graph, which is called the *pants graph* receives a lot of interests by geometric group theorists and has connections with Teichmller space. Interestingly, it can also be used to study three and four manifolds.

A collection of curves in a pants decomposition determines a 3-dimensional handlebody. Since a Heegaard splitting is a decomposition of a 3-manifold into two handlebodies, two vertices in the pants graph gives a 3-manifold. The minimum number of edges needed to travel between the two vertices gives some information about the Dehn surgery number of the 3-manifold. Analogously, Islambouli et al. showed that essentially a loop in the pants graph represents a 4-manifold. Thus, the minimum length over all the loops representing the same 4-manifold measures how complicated the manifold is. Aranda et al. characterized the shapes of the loops representing 4-manifolds with simple topology.

Problem: Classify 4-manifolds that can be constructed as loops in the pants complex of length at most n, for some fixed n.

2.5.2 Kakimizu complex

Instead of letting a vertex be a collection of curves, one can let a vertex be a minimal genus Seifert surface of a knot. Two vertices are connected by an edge if their representatives can be realized disjointly. Finally, one asks for the complex to be flagged. Such a complex is called the *Kakimizu complex*. Neel computed the complex for alternating knots up to 11 crossings. Saul Schleimer was interested in whether there is a software whose output is a Kakimizu complex.

Problem: Find an algorithm to compute the Kakimizu complex of all alternating knots.

3 Presentation Highlights

Our workshop was diverse in many aspects. We had participants from the United States, Mexico, Canada, Spain, England, Australia, France, Germany, Israel, Japan, China, and Korea. Roughly, 50% of the speakers were women.

Our workshop was in hybrid mode consisting of both online talks and in-person talks. Each presentation on the first two days of the workshop lasted an hour. These talks were expository lectures designed to introduce early-career researchers to the overview, recent developments, and open problems in the field. The last three days of the workshop consisted of 25-minute research talks.

The first talk on hyperbolic 3-manifolds was given by David Gabai. He described the standard models of hyperbolic spaces. He also mentioned the utility of the software SnapPy and its influence in the proofs of diverse results. Many illustrations from his presentation were created using the this software.

There were several exciting talks regarding novel techniques to estimate knot invariants. Alfonsn defined the notion of ball number and discussed an upper bound in terms of a function of the crossing number [1]. His presentation included an elegant proof using techniques from Lorenz geometry and circle packing. Baker and Manjarrez-Gutirrez described their work, which explores connections between the concept of *nearly fibered* knots arising in Heegaard Floer theories and handle numbers: they were able to determine the handle numbers in terms of the properties of the guts [5]. Araceli Guzmn Tristan presented infinitely many examples of knots for which she and her collaborators were able to determine the exact value of the *circular width* [7]. This is a multi-set valued invariant that is difficult to calculate.

For algebraic techniques, Silvero gave an effective criterion devised with her collaborators to obstruct positivity coming from Khovanov homology. Emily Hamilton also gave applications to 3-manifolds via subgroup separability. Parlak investigated the correspondence between Anosov flows and certain types of triangulations. This led to examples of interest to dynamicists. In particular, her examples pointed out a gap in a proof of another result in dynamical systems.

Some talks related to more peripheral topics were presented by Rieck and Hinojosa. For the former, the author considered tools for potential future use on bad orbifolds. For the latter, the author studied wild knots, which is a fascinating subject. In particular, Montesinos showed that every closed orientable 3-manifold is a 3- fold branched covering of S^3 with branched set a wild knot. Hinojosa described an example of branched covering along a wild knot, where the total space is not a manifold.

One day of the workshop was dedicated to a celebration of Misha Kapovich on the occasion of his 60th birthday. Most speakers on this day were researchers who had worked with Kapovich as graduate students or postdoctoral fellows. Influential mathematicians such as Curtis McMullen were given the opportunity to share favorite memories of M. Kapovich and revolutionary mathematical discoveries inspired by his ideas. For instance, Liu discussed her work with Wang on full rank cusps in a complete hyperbolic manifold [10], which is related to Misha's contribution of a finitely generated free Kleinian group that has infinitely many rank one cusps [8]. Amazingly, Moon Duchin applied topological concepts to the issue of redistricting voting precincts. By modeling voting districts using an adjacency graph, and the dual to the adjacency graph she analyzes the fairness of the division into districts. She mentioned the inspiration coming from studying Teichmuller space by looking at a discrete object called the pants complex. Her work has been mentioned by politicians, and even used in court cases pertaining to redistricting.

4 Scientific Progress Made and Outcome of the Meeting

Many participants used the computer room and the blackboards provided by CMO to make progress in existing projects. Many new groups are also formed with the goal of solving interesting problems that arose during the workshop. Speakers on June 15 brought the mathematics of Misha Kapovich to life, and the participants were exposed to his legacy. Some preprints were produced as a consequence of time spent at the CMO, for instance [3, 2].

Participants spent time gaining a better understanding of certain problem relating to talks. For instance, it was believed that the proof of additivity of handle number in the relative setting (knots in $S^1 \times D^2$) should be straightforward. However, participants found some specific situations where the statement is false. We think that the statement might be true with some extra assumptions added. Other conversations that happened outside of the recordings include a survey of results related to contact branched covering by Rodriguez-Viorato, and generalizing classical results in linear Heegaard splitting settings to circular Heegaard splittings. A result along this line was obtained in [9], and we expect that similar techniques can be used.

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