LECTURE 1: CHERN CLASSES, EULER CHARACTERISTICS, AND ENUMERATIVE GEOMETRY

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Abstract. We discuss Euler characteristics from various points of view

1. Introduction

Intersection theory has a long and interesting history, and is closely tied to questions of *enumerative geometry*, that is, the counting of solutions to geometric problems in algebraic geometry, or more generally, attaching integer invariants to a given variety or finite collection of varieties.

In this lecture, we look at perhaps the most elementary invariant, the Euler characteristic. A topological space T with the homotopy type of a finite CW complex (say dimension d) has its Euler characteristic

$$\chi^{\mathrm{top}}(T) := \sum_{i=0}^{d} \dim_{\mathbb{Q}} H_i(T, \mathbb{Q})$$

In fact, one can use $\dim_F H_i(T, F)$ for any field F. For an algebraic variety X over $\mathbb{C}$, we have the space $X(\mathbb{C})$, so we have its Euler characteristic

$$\chi^{\mathrm{top}}(X) := \chi^{\mathrm{top}}(X(\mathbb{C}))$$

Over an arbitrary algebraically closed field k, we can use instead étale cohomology with $\mathbb{Q}_\ell$ coefficients for a prime ℓ different from the characteristic.

2. Chow groups and Chern classes

A somewhat more sophisticated definition in the case of a smooth proper scheme X over a field k is to use a version of the *Gauß-Bonnet theorem*

Theorem 2.1 (algebraic Gauß-Bonnet)**.** *Let X be a smooth proper scheme of dimension n over a field k. Then*

$$\chi^{\mathrm{top}}(X_{\bar{k}}) = \deg_k c_n(T_{X/k}) = (-1)^n \deg_k c_n(\Omega_{X/k}).$$

Here $T_{X/k}$ is the tangent bundle of X, $\Omega_{X/k}$ is the sheaf of differentials, c_n is the nth Chern class with values in the Chow group $\mathrm{CH}^n(X)$, and $\deg_k$ is the degree map

$$\deg_k : \mathrm{CH}^n(X) \to \mathrm{CH}^0(k) = \mathbb{Z}.$$

We won't be going into all these objects in detail, but let's just list a few useful objects and their properties.

Chow groups A variety X over a field k has its group of *dimension i algebraic cycles* $Z_i(X)$, the free abelian group on the dimension i subvarieties of X. The

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subgroup $R_i(X) \subset Z_i(X)$ is generated by cycles of the form $\div f$, with f a non-zero rational function on some dimension $i+1$ subvariety of X. The quotient $\mathrm{CH}_i(X) := Z_i(X)/R_i(X)$ is the dimension i Chow group of X. If X has pure dimenison d, we can index by codimension $Z^i(X) := Z_{d-i}(X)$, $\mathrm{CH}^i(X) = \mathrm{CH}_{d-i}(X)$.

Each proper map $f : Y \to X$ induces a functorial pushforward map $f_* : Z_i(Y) \to Z_i(X)$ that passes to $f_* : \mathrm{CH}_i(Y) \to \mathrm{CH}_i(X)$. If $f : Y \to X$ is an arbitrary map with X and Y smooth, we have pullback maps $f^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(Y)$. For X smooth, the graded group $\mathrm{CH}^*(X) := \oplus_i \mathrm{CH}^i(X)$ has a graded-ring structure and f^* is a ring homomorphism. The unit in $\mathrm{CH}^0(X) = \mathrm{CH}_{\dim X}(X)$ is the fundamental class $[X] = 1 \cdot X$.

For f proper, X, Y smooth, we have the projection formula

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y)$$

We have $\mathrm{CH}_0(\mathrm{Spec}\, k) = Z_0(\mathrm{Spec}\, k) = \mathbb{Z}$. For $\pi : X \to \mathrm{Spec}\, k$ proper, we have the *degree map*

$$\deg_k := \pi_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}\, k) = \mathbb{Z}$$

Explicitly, if $p \in X$ is a closed point, $\deg_k(p)$ is the field extension degree $[k(p) : k]$.

Each vector bundle V (locally free coherent sheaf) on a smooth X has Chern classes

$$c_i(V) \in \mathrm{CH}^i(X), i = 1, 2, \ldots$$

with $f^*c_i(V) = c_i(f^*V)$ for $f : Y \to X$ map of smooth varieties. $c_i(V)$ depends only on the isomorphism class of V and $c_i(V) = 0$ for $i > \mathrm{rank}(V)$; we set $c_0(V) = 1 \in \mathrm{CH}^0(X)$. Sending a line bundle L to $c_1(L) \in \mathrm{CH}^1(X)$ defines an isomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$$

For the case $L = \mathcal{O}_X(D)$ for some divisor $D \in Z^1(X)$,

$$c_1(\mathcal{O}_X(D)) = [D] \in \mathrm{CH}^1(X).$$

The top Chern class $c_r(V)$ for $r = \mathrm{rank}(V)$ is also called the *Euler class* and is given by

$$c_r(V) = s_2^* s_{1*}([X])$$

with $s_1, s_2 : X \to V$ any two sections. The canonical choice is $s_1 = s_2 = s_0$, the zero-section, but this is not necessary.

The total Chern class $c(V) := \sum_{i=0}^{\mathrm{rank}(V)} c_i(V)$ satisfies the Whitney formula: If

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence of vector bundles, then $c(V) = c(V')c(V'')$. Also, for the dual bundle $V^\vee$, we have

$$c_i(V^\vee) = (-1)^i c_i(V).$$

3. Intersections, Chern classes and enumerative problems

We give some examples to show how this machinery is useful in solving enumerative problems.

Bézout's theorem. Start with the simplest case: two curves in the plane, C_1, C_2, with no common components. Let C_i have defining equation $F_i(X_0, X_1, X_2)$, a homogeneous polynomial of degree d_i, so the intersection subscheme $C_1 \cap C_2$ is defined by the ideal (F_1, F_2), and is a finite set of points. A each point $p \in C_1 \cap C_2$, we have the *intersection multiplicity*

$$m(C_1, C_2, p) := lng_{\mathcal{O}_{\mathbb{P}^2,p}} \mathcal{O}_{C_1 \cap C_2, p}$$

To explain this, we assume k is algebraically closed and take coordinates so that $p = (1, 0, 0) \in \mathbb{P}^2$. We pass to affine coordinates $x_i = X_i/X_0$ for the open subscheme $U_0 = \mathbb{P}^2 \setminus \{X_0 = 0\} = \operatorname{Spec} k[x_1, x_2]$, so $\mathcal{O}_{\mathbb{P}^2,p}$ is the local ring $k[x_1, x_2]_{(x_1,x_2)}$. Let $f_i = F_i/X_0^{d_i}$, so f_i is the defining equation of $C_i \cap U_0$, and $(f_1, f_2)\mathcal{O}_{\mathbb{P}^2,p}$ is an (x_1, x_2)-primary ideal. Thus $k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2)$ is a $k[x_1, x_2]_{(x_1,x_2)}$-module of finite length ℓ, with $\ell = \dim_k k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2)$, thus

$$m(C_1, C_2, p) = \dim_k k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2)$$

Let

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} m(C_1, C_2, p) \cdot p \in Z^2(\mathbb{P}^2).$$

On the other hand, each F_i is a section s_i of $\mathcal{O}_{\mathbb{P}^2}(d_i)$ and we have

$$s_i^* s_{0*}[\mathbb{P}^2] = [C_i]$$

so

$$c_1(\mathcal{O}_{\mathbb{P}^2}(d_i)) = [C_i]$$

Similarly, we have the section (s_1, s_2) of $\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)$ and

$$(s_1, s_2)^* s_{0*}[\mathbb{P}^2] = [C_1 \cdot C_2] \in \mathrm{CH}^2(\mathbb{P}^2)$$

so

$$c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) = [C_1 \cdot C_2].$$

The Whitney product formula says $c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) = c_1(\mathcal{O}_{\mathbb{P}^2}(d_1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(d_2))$ and since $c_1 : \mathrm{Pic}(\mathbb{P}^2) \to \mathrm{CH}^1(\mathbb{P}^2)$ is a group homomorphism, we have

$$\begin{aligned}
[C_1 \cdot C_2] &= c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) \\
&= c_1(\mathcal{O}_{\mathbb{P}^2}(d_1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(d_2)) \\
&= d_1 d_2 \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1))
\end{aligned}$$

If we now take $d_1 = d_2 = 1$, $F_1 = X_1$, $F_2 = X_2$, we have $C_1 \cdot C_2 = 1 \cdot (1 : 0 : 0)$, so $c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = [1 \cdot (1 : 0 : 0)] \in \mathrm{CH}^2(\mathbb{P}^2)$, and thus

$$[C_1 \cdot C_2] = d_1 d_2 \cdot [(1 : 0 : 0)]$$

Applying the pushforward to the point, $\pi : \mathbb{P}^2 \to \operatorname{Spec} k$, we have $\pi_*(p) = 1$ for all $p \in \mathbb{P}^2(k)$ and so

$$\begin{aligned}
\sum_{p \in C_1 \cap C_2} m(C_1, C_2, p) &= \pi_*(C_1 \cdot C_2) \\
&= \pi_*(d_1 d_2 \cdot [(1 : 0 : 0)]) \\
&= d_1 d_2
\end{aligned}$$

which is exactly Bézout's theorem. The case of n hypersurfaces $H_1, \ldots, H_n$ in $\mathbb{P}^n$ that intersect in finitely many points is exactly the same: if these have degrees $d_1, \ldots, d_n$, then

$$\deg_k H_1 \cdots H_n = d_1 \cdots d_n$$

Lines on a cubic surface Consider a smooth cubic surface $S \subset \mathbb{P}^3$, with defining equation $F \in k[X_0, \ldots, X_3]_3$. We want to count the lines on S. For this, consider the Grassmannian of 2-dimensional subspaces of k^4, $\mathrm{Gr}(2,4)$ (which is the same as lines in $\mathbb{P}^3$), with its tautological subbundle $E_2 \to \mathrm{Gr}(2,4)$ of $\mathrm{Gr}(2,4)\times\mathbb{A}^4$: the fiber of E_2 over a point $x \in \mathrm{Gr}(2,4)$ representing a 2-plane Π in k^4 is $\Pi \subset k^4$. Note that $\mathrm{Gr}(2,4)$ is a smooth proper variety of dimension 4.

The polynomial F determines a degree 3 polynomial function on each fiber Π of E_2, by restricting F to Π, in other words, F gives a section s_F of $\mathrm{Sym}^3 E_2^\vee$ over $\mathrm{Gr}(2,4)$. s_F vanishes at $x \in Gr(2,4)$ exactly when F vanishes on the corresponding plane Π, in other words, when the line $\ell_x := \mathbb{P}(\Pi) \subset \mathbb{P}^3$ is contained in $V(F) = S$. Noting that $\mathrm{Sym}^3 E_2^\vee$ is a vector bundle of rank 4 on $\mathrm{Gr}(2,4)$, we thus have

$$\#\{\text{lines in } S\} = \deg_k s_F^* s_{0*}[\mathrm{Gr}(2,4)] = \deg_k c_4(\mathrm{Sym}^3 E_2^\vee).$$

So, we need to find a way to compute Chern classes of symmetric powers.

This is done via the *splitting principle*, which roughly speaking says that for computing Chern classes of a functor (like Sym^3) applied to a vector bundle, we may assume that the vector bundle is a sum of line bundles. So take $E^\vee = M_1 \oplus M_2$. Let $\xi_i = c_1(M_i)$, then $c_1(E^\vee) = \xi_1 + \xi_2$, $c_2(E^\vee) = \xi_1\xi_2$.

$$\mathrm{Sym}^3 E^\vee = M_1^{\otimes 3} \oplus M_1^{\otimes 2} \otimes M_2 \oplus M_1 \otimes M_2^{\otimes 2} \oplus M_2^{\otimes 3},$$

so

$$\begin{aligned} c_4(\mathrm{Sym}^3 E^\vee) &= c_1(M_1^{\otimes 3}) \cdot c_1(M_1^{\otimes 2} \otimes M_2) \cdot c_1(M_1 \otimes M_2^{\otimes 2}) \cdot c_1(M_2^{\otimes 3}) \\ &= (3\xi_1) \cdot (2\xi_1 + \xi_2) \cdot (\xi_1 + 2\xi_2) \cdot (3\xi_2) \\ &= 9\xi_1\xi_2(2\xi_1^2 + 2\xi_2^2 + 5\xi_1\xi_2) \\ &= 9\xi_1\xi_2(2(\xi_1+\xi_2)^2 + \xi_1\xi_2) \\ &= 9(\xi_1\xi_2)^2 + 18(\xi_1\xi_2)\cdot(\xi_1+\xi_2)^2 \\ &= 9c_2(E^\vee)^2 + 18c_2(E^\vee)\cdot c_1(E^\vee)^2. \end{aligned}$$

The point of the splitting principle is that this identity will hold, even if $E^\vee$ is not a sum of line bundles.

In any case, we now need to compute the degrees of $c_2(E^\vee)^2$ and $c_2(E^\vee)\cdot c_1(E^\vee)^2$. Note that an linear polynomial L in $X_0, \ldots, X_3$ gives a section s_L of $E^\vee$, so $c_2(E^\vee)$ is the class of $V(s_L)$. But $V(s_L)$ is just the variety of lines in $\mathbb{P}^3$ contained in $L = 0$, which is a $\mathbb{P}^2$. Similarly, $c_2(E^\vee)^2$ is the class of $V(s_L)\cdot V(s_{L'})$, in other words, the lines in $V(L)\cap V(L')$, which is just a single line if L and L' are independent. Thus

$$\deg_k c_2(E^\vee)^2 = 1$$

Also $c_2(E^\vee)\cdot c_1(E^\vee)^2$ is just the restriction of $c_1(E^\vee)^2$ to $V(s_L)$, so

$$\deg_k(c_2(E^\vee)\cdot c_1(E^\vee)^2) = \deg_k(c_1(E^\vee_{|\mathbb{P}^2})^2)$$

In general c_1 of a vector bundle V is the same as c_1 of the line bundle $\det V$, so

$$c_1(E^\vee_{|\mathbb{P}^2})^2 = c_1(\det E^\vee_{|\mathbb{P}^2})^2$$

Finally, one shows that $\det E^\vee_{|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$, so using Bézout's therem we have

$$\deg_k(c_1(\det E^\vee_{|\mathbb{P}^2})^2) = \deg_k(c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2) = 1$$

Putting this altogether gives

$$\#\{\text{ lines in } S\} = \deg_k c_4(E^\vee) = 9 + 18 = 27.$$

The Gauß-Bonnet theorem and the Euler characteristic

For X smooth and proper of dimension n, we have $c_n(T_{X/k}) \in \mathrm{CH}^n(X) = \mathrm{CH}_0(X)$ and thus $\deg_k(c_n(T_{X/k})) = (-1)^n \deg_k(c_n(\Omega_{X/k})$ is a well-defined integer. The Gauß-Bonnet theorem says that this is exactly the topological Euler characteristic. On the enumerative side, one can compute $\chi^{\mathrm{top}}(X)$ for X a smooth degree d hypersurface in $\mathbb{P}^{n+1}$ explicitly as follows.

We have the Euler sequence for $T_{\mathbb{P}^{n+1}}$

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{n+2} \to T_{\mathbb{P}^{n+1}} \to 0$$

which gives

$$c(T_{\mathbb{P}^{n+1}}) = c(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{n+2})/c(\mathcal{O}_{\mathbb{P}^{n+1}}) = (1+h)^{n+2}$$

with $h \in \mathrm{CH}^1(\mathbb{P}^{n+1})$ the class of a hyperplane $H \subset \mathbb{P}^{n+1}$. The tangent-normal bundle sequence for $i : X \to \mathbb{P}^{n+1}$ of degree d

$$0 \to T_X \to i^*T_{\mathbb{P}^{n+1}} \to i^*\mathcal{O}_{\mathbb{P}^{n+1}}(d) \to 0$$

gives

$$c(T_X) = i^*\left[c(T_{\mathbb{P}^{n+1}})/c(*\mathcal{O}_{\mathbb{P}^{n+1}}(d))\right] = i^*[(1+h)^{n+2}/(1+dh)]$$

Taking the degree n component gives

$$\deg_k c_n(T_X) = \deg_k i_*c_n(T_X) = \deg_k i_*i^*[h^n \sum_{i+j=n} (-1)^j \binom{n+2}{i} d^j] = \sum_{i+j=n} (-1)^j \binom{n+2}{i} d^{j+1}$$

since

$$i_*i^*h^n = i_*([X] \cdot i^*h^n) = i_*([X]) \cdot h^n = d$$

Here is a table:

n	$\chi^{\mathrm{top}}(X_d)$
1	$-d^2+3d$
2	d^3-4d^2+6d
3	$-d^4+5d^3-10d^2+10d$
4	$d^5-6d^4+15d^3-20d^2+15d$

Another consequence of the Gauß-Bonnet theorem is a version of the Riemann-Hurwitz formula

Theorem 3.1. *Let $f : X \to C$ be a morphism of a smooth proper variety X of dimension n to a smooth projective curve C, giving the differential $df : f^*\omega_C \to \Omega_X$. Suppose that the induced section $df : \mathcal{O}_X \to \Omega_X \otimes f^*\omega_C^{-1}$ has isolated zeros $p_1, \ldots, p_r$, with multiplicities $m_1, \ldots, m_r$. Let X_p be a general (smooth) fiber. Then*

$$\chi^{\mathrm{top}}(X) = \chi^{\mathrm{top}}(X_p) \cdot \chi^{\mathrm{top}}(C) + (-1)^n \cdot \sum_i m_i$$

Proof. Using the splitting principle one shows that for V a rank n bundle and L a lilne bundle, one has

$$c_n(V \otimes L) = \sum_{i=0}^n c_{n-i}(V) \cdot c_1(L)^i$$

Since $c_n(\Omega_X \otimes f^*\omega_C^{-1}) = \sum_i m_i$ and $c_1(f^*\omega_C^{-1})^i = f^*(c_1(\omega_C^{-1})^i) = 0$ (since C has dimension 1), this gives

$$\sum_i m_i = \deg_k(c_n(\Omega_X) + c_{n-1}(\Omega_X) \cdot f^*(c_1(T_C)))$$

Since $\Omega_X = T_X^\vee$, we have

$$\deg_k(c_n(\Omega_X) = (-1)^n\chi^{\rm top}(X).$$

Since the normal bundle to X_p is trivial, we have

$$\Omega_X\otimes\mathcal{O}_{X_p} = \Omega_{X_p}\oplus\mathcal{O}_{X_p}$$

so if $c_1(T_C)) = \sum_i n_ip_i$ with the p_i taken so that X_{p_i} is smooth, we have

$$c_{n-1}(\Omega_X)\cdot f^*(c_1(T_C)) = \sum_i i_{X_{p_i}*}(n_i\cdot c_{n-1}(\Omega_{X_{p_i}}))$$

Each of the fibers X_{p_i} have the same Euler characteristic, so

$$\deg_k c_{n-1}(\Omega_X)\cdot f^*(c_1(T_C)) = (-1)^{n-1}\chi^{\rm top}(X_p)\cdot\chi^{\rm top}(C)$$

Putting this altogether gives the result. □

4. Dualizable objects and abstract Euler characteristics

Let $(\mathcal{C},\otimes,1,\tau)$ be a symmetric monoidal category with symmetry constraint $\tau_{x,y}: x\otimes y\to y\otimes x$.

Definition 4.1. (1) The *dual* of an object x in $\mathcal{C}$ is a triple $(x^\vee,\delta,ev)$ with $x^\vee$ in $\mathcal{C}$, and $\delta: 1\to x\otimes x^\vee$, $ev: x^\vee\otimes x\to 1$ morphisms such that both compositions

$$x\cong 1\otimes x\xrightarrow{\delta\otimes{\rm Id}} x\otimes x^\vee\otimes x\xrightarrow{{\rm Id}\otimes ev} x\otimes 1\cong x$$
$$x^\vee\cong x^\vee\otimes 1\xrightarrow{{\rm Id}\otimes\delta} x^\vee\otimes x\otimes x^\vee\xrightarrow{ev\otimes{\rm Id}} 1\otimes x^\vee\cong x^\vee$$

are identity morphisms.

(2) Suppose x both has dual $(x^\vee,\delta,ev)$ and let $f: x\to x$ be an endomorphism. Define the *trace* ${\rm Tr}_x(f)\in{\rm End}_{\mathcal{C}}(1)$ as the composition

$$1\xrightarrow{\delta} x\otimes x^\vee\xrightarrow{f\otimes{\rm Id}} x\otimes x^\vee\xrightarrow{\tau_{x,x^\vee}} x^\vee\otimes x\xrightarrow{ev} 1$$

The Euler characteristic $\chi_{\mathcal{C}}(x)$ is by definition ${\rm Tr}_{\mathcal{C}}({\rm Id}_x)$.

Examples 4.2. 1. Let $\mathcal{C} = k-\mathbf{Vec}$, the category of k-vector spaces, with $\otimes=\otimes_k$, unit k and $\tau(a\otimes b) = b\otimes a$. Then $V\in k-\mathbf{Vec}$ is dualizable if and only if ${\rm dim}_kV<\infty$, the dual is the usual dual vector space, $ev: V^\vee\otimes_k V\to k$ is the evaluation map $f\otimes v\mapsto f(v)$, and $\delta: k\to V\otimes_k V^\vee$ sends $1\in k$ to $\sum_i e_i\otimes e^i$, where $e_1,\ldots,e_n$ is a basis of V with dual basis $e^1,\ldots,e^n$. The trace is the usual trace and $\chi(V) = {\rm dim}_kV$ as an element of ${\rm End}_k(k)\cong k$.

2. For $\mathcal{C}$ = graded k-vector spaces, we have a similar story, except that $\tau(a\otimes b) = (-1)^{|a||b|}b\otimes a$, for a,b homogeneous of degrees $|a|$, $|b|$. If $V=\oplus_nV_n$, then $\chi(V) = \sum_n(-1)^n{\rm dim}_kV_n$.

3. For $\mathcal{C} = D(k-\mathbf{Vec})$, the derived category, the dualizable objects are the complexes K_* such that the homology $H_*(K_*) = \oplus_nH_n(K_*)$ is finite dimenisonal over k and $\chi(K_*) = \sum_n(-1)^n{\rm dim}_kH_n(K_*)$, again as an element of ${\rm End}(k)\cong k$. Sending a finite CW complex T to its singular chain complex $C_*(T,k)$ we see that

$$\chi(C_*(T,k)) = \chi^{\rm top}(T)$$

in k. We have a similar computation for $\mathcal{C} = D(\mathbf{Ab})$ and for the integral singular chain complex $C_*(T,\mathbb{Z})$, giving $\chi(C_*(T,\mathbb{Z})) = \chi^{\rm top}(T)\in\mathbb{Z} = {\rm End}_{D(\mathbf{Ab})}(\mathbb{Z})$.

4. We may take $\mathcal{C}$ to be the category Sp of spectra, which is symmetric monoidal

with unit the sphere spectrum $\mathbb{S}$. Note that End($\mathbb{S}$) is the 0th stable homotopy group of spheres, which is $\mathbb{Z}$, and that the dualizable objects are the thick subcategory generated by the suspension spectra of finite CW complexes. One recoves the usual topological Euler characteristic

$$\chi_{\mathrm{Sp}}(\Sigma^\infty T_+) = \chi^{\mathrm{top}}(T).$$

5. Morel's theorem and the quadratic Euler characteristic

Morel and Voevodsky have defined a homotopy theory where finite sets in the classical theory get replaced by smooth algebraic varieties over a given field k. The replacement of the stable homotopy category is the *motivic stable homotopy category over* k, SH(k). This is a symmetric monoidal category with unit the *motivic sphere spectrum* $\mathbb{S}_k$. The operation of $\mathbb{P}^1$ suspension, $\Sigma_{\mathbb{P}^1}$ is formally inverted in SH(k).

For each pair of integers a, b one has the associated suspension functor $\Sigma^{a,b}$; for $a \geq b \geq 0$, this is smash product with $S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$ and for arbitrary (a, b), this is defined as

$$\Sigma^{a,b} = \Sigma^{a+2N,b+N}\Sigma_{\mathbb{P}^1}^{-N}; \quad N >> 0.$$

The fact that $S^1 \wedge \mathbb{G}_m \cong \mathbb{P}^1$ implies that this is well-defined, independent of N.

To construct the *Grothendieck-Witt ring over* k, GW(k) one starts with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k (this is the same as non-degenerate quadratic forms over k if $1/2 \in k$. This is a commutative monoid under orthogonal direct sum, and GW(k) is a group completion, that is elements are form differences of non-degenerate symmetric bilinear forms (up to isomorphism). GW(k) is a ring, with product induced by tensor product: for $b : V \times V \to k$, $b' : W \times W \to k$, we have $b \otimes b' : (V \otimes W) \times (V \otimes W) \to k$ with $b \otimes b'(v \otimes w, v' \otimes w') = b(v, v')b'(w, w')$. This makes GW($k$) into a ring.

We will usually work away from characteristic 2, and so will speak mainly of quadratic forms.

A non-degenerate form q has its *rank*, namely, the dimension of the vector space on which it is defined. Sending q to rank$q$ defines a ring homomorphism rank : GW(k) $\to \mathbb{Z}$.

For $u \in k^\times$, we have the rank 1 form $\langle u\rangle$ with $\langle u\rangle(x) = ux^2$, more generally, we have the rank n form $\sum_{i=1}^n \langle u_i\rangle$ with $\sum_{i=1}^n \langle u_i\rangle(x_1, \ldots, x_n) = \sum_{i=1}^n u_i x_i^2$. Away from characteristic 2, every quadratic form is isomorphic to such a "diagonal" form. The hyperbolic form is $H(x, y) = x^2 - y^2 = \langle 1\rangle + \langle -1\rangle$. For a form q, we have $q \cdot H = \mathrm{rank}(q) \cdot H$. The Witt ring $W(k)$ is defined by

$$W(k) := \mathrm{GW}(k)/(H).$$

For k algebraically closed, the rank homomorphism is an isomorphism GW(k) $\cong \mathbb{Z}$. For $k = \mathbb{R}$, Sylverster's theorem of inertia says that each $q \in \mathrm{GW}(\mathbb{R})$ is uniquely of the form $q = a \cdot \langle 1\rangle + b \cdot \langle -1\rangle$, $a, b \in \mathbb{Z}$, and the signature homomorphism

$$\mathrm{sig} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$$

is given by $\mathrm{sig}(a \cdot \langle 1\rangle + b \cdot \langle -1\rangle) = a - b$.

Theorem 5.1 (Morel)**.** *There is a natural isomorphism*

$$\mathrm{GW}(k) \cong \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$$

Each smooth proper variety over k, X, defines a dualizable object $\Sigma^\infty_{\mathbb{P}^1}X_+$ in $\mathrm{SH}(k)$, so one has the associated Euler characteristic

$$\chi(X/k) := \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}X_+) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) = \mathrm{GW}(k)$$

If we assume that k has characteristic zero, or if we invert p if k has characteristic $p > 0$, $\Sigma^\infty_{\mathbb{P}^1}U_+$ is dualizable for all smooth U, so the definition of $\chi(X/k)$ extends to arbitrary smooth U over k. Under the same assumptions, $\chi(X/k)$ extends to the Euler characteristic with compact support, $\chi_c(Z/k)$ for arbitrary finite type k-schemes, with $\chi(X/k) = \chi_c(X/k)$ for X smooth and proper.

The formal properties of categorical Euler characteristics and additional structural properties of $\mathrm{SH}(k)$ yield a number of properties of these Euler characteristics: For $u \in k^\times$, let $\langle u\rangle$ denote the rank one form $\langle u\rangle(x, y) = uxy$.

- $\chi(\Sigma^{a,b}X/k) = (-1)^a(\langle -1\rangle)^b \cdot \chi(X/k)$
- If Z contains a closed subscheme W with open complement U, then
$$\chi_c(Z/k) = \chi_c(U/k) + \chi_c(W/k)$$
If Z and W are smooth, and W has codimension c in Z, then
$$\chi_c(Z/k) = \chi(U/k) + \langle -1\rangle^c\chi(W/k)$$
- If $E \to B$ is a fiber bundle with fiber F, locally trivial in the Nisnevich topology, and E, B and F are smooth, then
$$\chi(E/k) = \chi(B/k) \cdot \chi(F/k)$$
- For X a smooth k-scheme, we have $\mathrm{rank}\chi(X/k) = \chi^{\mathrm{top}}(X)$. If $k = \mathbb{C}$, this says $\mathrm{rank}\chi(X/\mathbb{C}) = \chi^{\mathrm{top}}(X(\mathbb{C}))$. If $k = \mathbb{R}$, we have $\mathrm{sig}\chi(X/\mathbb{R}) = \chi^{\mathrm{top}}(X(\mathbb{R}))$.
- Suppose X is *cellular*: there is a stratification $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = X$ with $X_i \subset X$ closed of dimension i, such that $X_i \setminus X_{i-1}$ is a disjoint union of affine spaces $\mathbb{A}^i_k$. Then $\mathrm{CH}^j(X)$ is a free abelian group of finite rank for each j, and letting $r_+ = \sum_{j \text{ even}} \mathrm{rank}\mathrm{CH}^j(X)$, $r_- = \sum_{j \text{ odd}} \mathrm{rank}\mathrm{CH}^j(X)$, we have
$$\chi(X/k) = r_+ \cdot \langle 1\rangle + r_- \cdot \langle -1\rangle.$$
For example
$$\chi(\mathbb{P}^n/k) = \sum_{i=0}^n \langle -1\rangle^i$$
- Let $Z \subset X$ be a smooth closed subscheme of a smooth k-scheme X, of codimension c and let $\tilde{X}$ be the blow-up of X along Z. Then
$$\chi(\tilde{X}/k) = \chi(X/k) + (\sum_{i=1}^{c-1}\langle -1\rangle^i) \cdot \chi(Z/k).$$

Since the rank n form $\sum_{i=0}^{n-1}\langle -1\rangle^i$ comes up alot, we denote this by n_ϵ.

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