Moduli and ednomorphisms of vector bundles

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CIMAT

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- The results for rank 3 are part of the Ph. D. Thesis of my student Rocío Ríos Siérra.
- The results for HN-length > 2 are in progress and the article will be submitted soon in arXiv.

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- (For higher dimensional varieties, fix a polarization H, Then $deg_H := c_1(E) \cdot [H]^{\dim X - 1}$ $\mu_H(E) := \frac{deg(E)_H}{rk(E)}$)

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- E is unstable (no semistable) if $\mu(F) > \mu(E)$ for some subbundle F.

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such that for $1 \leq i \leq m$,

1. E_i/E_{i-1} is semistable and
2. $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_m/E_{m-1})$.

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4. E is semistable iff the $HN - lenght = 1$

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- If (n, d) are coprime $\tilde{M}(n, d) = M(n.d)$

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- New construction of ”good moduli spaces” by Jarod Alper, Daniel Halpern-Leistner, Jochen Heinloth

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- We want to construct the moduli using the extension.
- The first step is when the indecomposable bundle E is an extension

$$\rho : 0 \to E_1 \to E \to E_2 \to 0$$

of two semistable vector bundles.

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  where $T = H^1(C, E_1 \otimes E_2^*)$parameterize the isomorphic classes of indecomposable vector bundles that are extensions of E_2 by E_1.
- Moreover, there is a universal extension and a universal family $\mathcal{G}(F, G)$ parameterized by $\mathbb{P}(T)$.

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- However, $End(E) \neq End(F)$.
- The S-equivalence identifies bundles with different algebra of endomorphisms.

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- $$\begin{array}{ccccc} & & p_{12}^*\mathcal{U}_1 \otimes p_{13}^*\mathcal{U}_2^* & & \mathcal{R}_1 := \mathcal{R}^1_{p_{23}}(p_{12}^*\mathcal{U}_1 \otimes p_{13}^*\mathcal{U}_2^*) \\ & & \downarrow & & \downarrow \\ \mathcal{U}_1 & & X \times M_1 \times M_2 & \xrightarrow{p_{23}} & M_1 \times M_2 \\ \downarrow & \swarrow^{p_{12}} & & \searrow^{p_{13}} & \\ X \times M_1 & & & & X \times M_2. \end{array} \tag{4}$$

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- If $E_2 = E_2$, then the restriction of

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to the complement Δ^c of the diagonal $\Delta \subset M_1 \times M_1$ will be the fine moduli space for isomorphic (indecomposable) simple semistable bundles S-equivalent to $E_1 \oplus E\_1$.

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- We want to describe $M(F,G) := H^1(C, F \otimes G^*)/Aut(F) \times Aut(G)$.

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- We can control it using flattering stratifications.

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- and in her Ph. D. thesis she consider the case of rank3.
- E is unstable and is an extension $\rho : 0 \to E_1 \to E \to E_2 \to 0$ with E_1 and E_2 semistables
- $0 \subset E_1 \subset E$ the HN-filtration and

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- Hence $\dim Hom(E_2, E_1) = h^0(E_2^* \otimes E_1)$ is a problem on the "twisted Brill-Noether theory".

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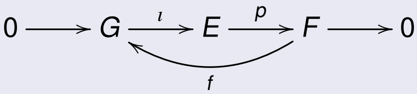

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- Hence, $\dim END(E) \geq 1 + h^0(F^* \otimes G)$.

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- If $E \in U_{\mu_1}(n, d, k)$ and $0 \to E_1 \xrightarrow{\imath} E \xrightarrow{p} F_1 \to 0$ then $k = h^0(F_1^* \otimes E_1)$ i.e. $\dim End(E) = 1 + h^0(F_1^* \otimes E_1)$.

Unstable bundles of HN-lenght 2

Theorem ——, R. Rios Sierra (Theorem A)

- $U_{\mu_1}(n, d, k)$ the subset of indecomposable vector bundles of HN-length 2 of coprime type $\sigma = (\mu_1, \mu_2)$ and $\dim END(E) = 1 + k$
- If $E \in U_{\mu_1}(n, d, k)$ and $0 \to E_1 \xrightarrow{\imath} E \xrightarrow{p} F_1 \to 0$ then $k = h^0(F_1^* \otimes E_1)$ i.e. $\dim End(E) = 1 + h^0(F_1^* \otimes E_1)$.
- Moreover,

$$End(E) \cong \mathbb{C}[x_1, \ldots, x_k]/(x_1, \ldots, x_k)^2.$$

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$$B^k(\mathcal{U}_1, \mathcal{U}_2) = \{(E_1, E_2^*) \in M_1 \times M_2 : h^0(E_1 \otimes E_2^*) \geq k\}$$

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- $Sing(B^k(\mathcal{U}_1, \mathcal{U}_2)) \subset B^{k+1}(\mathcal{U}_1, \mathcal{U}_2)$.
- In some cases, $Sing(B^k(\mathcal{U}_1, \mathcal{U}_2)) = B^{k+1}(\mathcal{U}_1, \mathcal{U}_2)$.

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- We prove that the universal twisted Brill-Noether loci gives a flattering stratification of

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$$\mathcal{R}_\mu := \mathcal{R}^1_{p_{23}}(p_{12}^*\mathcal{U}_1 \otimes p_{13}^*\mathcal{U}_2^*)$$

- such that in each strata Y_k, $\mathcal{R}_1$ is a vector bundle $\mathcal{V}_k$.

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- Recall that fixing $h^0(E \otimes F)$, $h^1(E \otimes F)$ is also fixed.

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- Moreover, if $\mathcal{Y}_k$ is irreducible and smooth of dimension ρ, then $U_{\mu_1}(n,d,k)$ is irreducible and smooth of dimension $\rho + h^1 - 1$ with $h^1 = k - d_0 + n_0(g-1)$.
- If $U_{\mu_1}(n,d,k)$ is non-empty, $B^k(\mathcal{U}_1, \mathcal{U}_2^*)$ is non-empty.
- If $\mathcal{Y}_k$ is irreducible and smooth then $H^i(\mathcal{U}_{\mu_1}(n,d,k), \mathbb{C}) \cong H^i(\mathcal{Y}_k, \mathbb{C})$ for $i \geq 1$.

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- Parameterize a diagram, not just one extension.

- If $\dim End(E) \leq 3$, $End(E) =$
- $$\begin{array}{l} \mathbb{C} \\ \mathbb{C}[t]/t^2 \\ \mathbb{C}[t]/t^3 \\ \mathbb{C}[r,s]/(r,s)^2 \end{array}$$
- —— Algebras of endomorphisms of semistable vector bundles of rank 3 over a Riemann surface. J. Algebra 123 (1989), no. 2, 414 – 425.

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- —- Moduli of endomorphisms of semistable vector bundles over a compact Riemann surface. Glasgow J. 32 (1990).

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- —- If E is semistable and indecomposable $\dim End(E) \leq 1 + \dfrac{n(n-1)}{2}$.
- —— and Rocío Ríos we give also a bound for $\dim End(E)$ using the HN-filtration.

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- —— Vector bundles of type T_3 over a curve. J. Algebra 169 (1994).

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- If they are stable a no coprime we an use the étale covering of the moduli where the universal family exists and proceed as before.

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- E is an extension of two semistable bundles

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- If e.g.E_1 is no simple and $1 < \dim End(E) \leq 3$, we can use the previous results and when there is a universal family we proceed as before.
- Note that in some cases $H^1(C, F \otimes G^*)/Aut(F)$ is a grassmannian and we consider the universal bundle over it.

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$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E \quad (5)$$

1. $F_i = E_i/E_{i-1}$ is semistable and
2. $\mu(E_1) > \mu(E_2/E_1) > \mu(E_3/E_2)$.

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- By Theorem B, there exists a universal family $\mathcal{G}$ of simple unstable bundles parameterised by $U_{\mu_1}(n, d, 0)$.
- Since F_2 is stable, we have the universal family $\mathcal{U}$ parameterized by $M(d(F_2), rk(F_2))$.

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- As before we use the families $\mathcal{G}$ and $\mathcal{U}$ to give a stratification of $U_{\mu_1}(n, d, 0) \times M(d(F_2), rk(F_2))$.
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- Under certain conditions, the simple bundles will have a fine moduli.
- We can use induction to construct the moduli space of simple bundles for any HN-length ≥ 2 (to appear in arxiv).

Thanks !!!