On the Segre invariant for rank 2 vector bundles on $\mathbb{P}^2$

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Moduli, Motives and Bundles – New Trends in Algebraic Geometry

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Moduli space of vector bundles on surfaces

Segre Invariant

A stratification of the moduli space $M_{\mathbb{P}^2}(2; c_1, c_2)$

Applications to Brill-Noether Theory

Let X be a smooth, irreducible complex projective surface.

Definition (Mumford-Takemoto 1972)

Let H be an ample divisor on X. A vector bundle E on X is H-stable, if for all proper subbundle F,

$$\mu_H(F) < \mu_H(E)$$

where $\mu_H(\cdot) = \frac{\deg_H(\cdot)}{rank\,(\cdot)}$ and $\deg_H(\cdot) = c_1(\cdot) \cdot H$.

Theorem (Maruyama 1977)

Let H be an ample divisor on X. There exists a moduli space $M_{X,H}(n; c_1, c_2)$ for $H-$stable vector bundles of rank n and fixed Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$, for $i = 1, 2$.

Problems:

- Describe the geography of $M_{X,H}(n; c_1, c_2)$, i.e, given topological invariants n, c_1 and c_2 lying in the admissible range, does there exist a stable vector bundle having these invariants?

$$\Delta(n; c_1, c_2) := 2nc_2 - (n-1)c_1^2$$

- What does the moduli space look like, as an algebraic variety? Is it for example, connected, irreducible, rational or smooth?,
- What does it look as topological space? What is it geometry? What are the singularities of the moduli space?.

Segre invariant

Definition

Let H be an ample divisor on X. For a rank 2 vector bundle E on X the Segre invariant $S_H(E)$ is defined as

$$S_H(E) := \deg_H(E) - 2\max\{\deg_H(L)\},$$

where the maximum is taken over all subline bundles L of E

- $S_H(E) = S_H(E \otimes L)$ for all $L \in Pic(X)$.
- E is H-stable if and only if $S_H(E) > 0$.
- The Segre invariant is always a finite number. The set

  $$\{deg_H(L) : L \subset E, L \text{ a line bundle}\},$$

  is bounded from above.

- If $L \subset E$ is maximal, then
  - *(i)* $S_H(E) = deg_H(E) - 2\deg_H(L)$,
  - *(ii)* E can be written in the exact sequence

$$0 \to L \to E \to L' \otimes I_Z \to 0$$

where $L' \otimes I_Z$ is torsion free and I_Z denote the ideal sheaf of a subscheme Z of codimension 2.

Theorem (Maruyama 1976)

Let ξ be a family of rank 2 vector bundles parameterized by $M_{X,H}(2; c_1, c_2)$. The function

$$\begin{aligned} S_H : M_{X,H}(2; c_1, c_2) &\longrightarrow \mathbb{Z}^{>0} \\ t &\longmapsto S_H(\mathcal{E}_t) := deg_H(\mathcal{E}_t) - 2 \max_{L \subset \mathcal{E}_t} deg_H(L) \end{aligned}$$

is well defined and lower semicontinuous. , i.e, the set

$$\{E \in M_{X,H}(2; c_1, c_2) : S_H(E) > s\}$$

is open in $M_{X,H}(2; c_1, c_2)$.

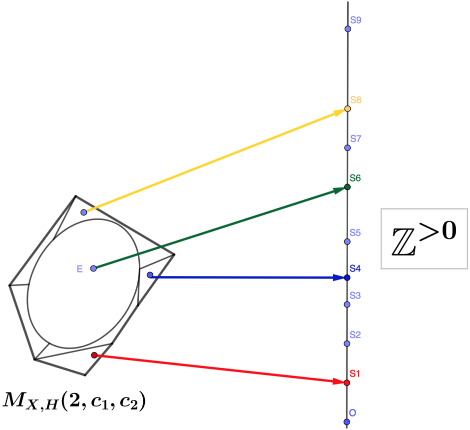
The function

$$S_H : M_{X,H}(2; c_1, c_2) \longrightarrow \mathbb{Z}^{>0}$$

induces a stratification of the moduli space $M_{X,H}(2; c_1, c_2)$ into locally closed subsets

$$M_{X,H}(2; c_1, c_2; s) := \{E \in M_{X,H}(2; c_1, c_2) : S_H(E) = s\}$$

according to the value of s.



A stratification of the moduli space $M_{\mathbb{P}^2}(2; c_1, c_2)$

- As $Pic\,(\mathbb{P}^2) \cong \mathbb{Z}$, there is a unique notion of stability for $\mathbb{P}^2$.
  - E is H-stable if and only if it is aH-stable, for $a \in \mathbb{N}$.
- We will use

$$c_1(\cdot) := deg_H(\cdot) = c_1(\cdot) \cdot \mathcal{O}_{\mathbb{P}^2}(1)$$

  to denote the degree with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$.
- We will write

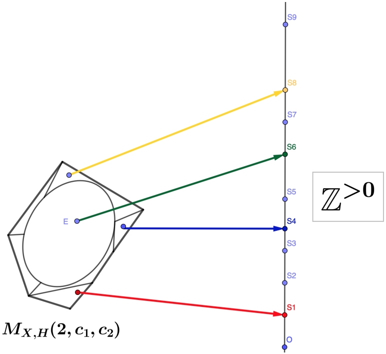
$$S(E) := c_1(E) - 2 \max_{\mathcal{O}_{\mathbb{P}^2}(k) \subset E} \{\mathcal{O}_{\mathbb{P}^2}(k)\}$$

  to denote the Segre invariant of the vector bundle E of rank 2 on $\mathbb{P}^2$.

- Since $S(E) = S(E \otimes L)$ for any line bundle L, we assume that E has degree $c_1 \in \{-1, 0\}$ and second Chern class c_2.

For convenience, we restrict our attention to the case $c_1 = 0$; and the Segre invariant can be written as

$$S(E) := c_1(E) - 2 \max_{\mathcal{O}_{\mathbb{P}^2}(k) \subset E} \{\mathcal{O}_{\mathbb{P}^2}(k)\}$$
$$= -2 \max_{\mathcal{O}_{\mathbb{P}^2}(k) \subset E} \{\mathcal{O}_{\mathbb{P}^2}(k)\}$$



$$S : M_{\mathbb{P}^2}(2;0,c_2) \longmapsto \mathbb{Z}^{>0}$$

$$t \longmapsto S(\xi_t) = -2 \max_{\mathcal{O}_{\mathbb{P}^2}(-k)\subset E} \mathcal{O}_{\mathbb{P}^2}(-k) = 2k$$

$$M_{\mathbb{P}^2}(2;0,c_2;2k) := \{E \in M_{\mathbb{P}^2}(2;0,c_2) : S_H(E) = 2k\}$$

Questions

- Q1: Which possible values can the function $S_H(E)$ take?
- Q2: For which values of k are the strata non-empty ?
- Q3: What is the dimension of the stratum $M_{X,H}(2;0,c_2;2k)$? Is it irreducible?
- Q4: What is the Segre invariant of a general bundle in the moduli space?
- Q5: Applications to Brill-Noether Theory.

Theorem (—, Torres-López, Zamora 2021)

Let $c_2 \geq 2$ and $k \in \mathbb{N}$. Then a vector bundle $E \in M_{\mathbb{P}^2}(2, 0, c_2)$ with $S(E) = 2k$ exists if and only if $k^2 + k \leq c_2$. Furthermore, E fits in an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-k) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z \longrightarrow 0,$$

with $Z \subset \mathbb{P}^2$ of codimension 2 and $\mathcal{O}_{\mathbb{P}^2}(-k) \subset E$ maximal. Moreover, $M_{\mathbb{P}^2}(2; 0, c_2; 2k)$ is an irreducible variety of dimension:

$$\begin{cases} 3c_2 + k^2 + 3k - 2, & \text{if } c_2 > k^2 + 3k + 1 \\ 4c_2 - 3, & \text{if } c_2 \leq k^2 + 3k + 1. \end{cases}$$

$$dim\, M_{\mathbb{P}^2}(2; 0, c_2) = 4c_2 - 3.$$

Sketch of the proof

Theorem (Serre correspondence)

Let $Z \subset X$ be a local complete intersection of codimension two in the projective non-singular surface X, and let L and M be line bundles on X. Then there exists an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \otimes I_Z \longrightarrow 0$$

such that E is locally free if and only if the pair $(L^{-1} \otimes M \otimes \omega_X, Z)$ satisfy the Cayley-Bacharach property:

(CB) if $Z' \subset Z$ is a sub-scheme with $l(\tilde{Z}) = l(Z) - 1$ and $s \in H^0(L^{-1} \otimes M \otimes \omega_X)$ with $s|_{\tilde{Z}} = 0$, then $s|_Z = 0$.

Claim: If $k^2 + k \leq c_2$, then there exists a vector bundle $E \in M_{\mathbb{P}^2}(2, 0, c_2)$ with $S(E) = 2k$.

- Let $Z \subset \mathbb{P}^2$ of codimension two such that Z is not contained in any curve of degree $2k - 1$.
- The pair $(\mathcal{O}_{\mathbb{P}^2}(2k - 3), Z)$ satisfies the Cayley-Bacharach property. We have an extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-k) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z \longrightarrow 0$$

where E is locally free. Moreover, since Z is not contained in any curve of degree $2k - 1$ it follows that

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z) = h^0(\mathbb{P}^2, E) = 0.$$

Therefore, the vector bundle E is stable.

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_{\mathbb{P}^2}(-l) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-k) & \longrightarrow & E & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z & \longrightarrow & 0 \end{array}$$

We assume that there exists $\mathcal{O}_{\mathbb{P}^2}(-l)$, $l < k$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-k+l) \to E(l) \to \mathcal{O}_{\mathbb{P}^2}(k+l) \otimes I_Z \to 0.$$

Since $l < k$ it follows that:

$$h^0(E(l)) = h^0(\mathcal{O}_{\mathbb{P}^2}(k+l) \otimes I_Z) \leq h^0(\mathcal{O}_{\mathbb{P}^2}(2k-1) \otimes I_Z) = 0.$$

This implies that $\mathcal{O}_{\mathbb{P}^2}(-l)$ is not a subbundle of E. Thus $\mathcal{O}_{\mathbb{P}^2}(-k)$ is maximal and

$$S(E) = -2c_1(\mathcal{O}_{\mathbb{P}^2}(-k)) = 2k.$$

If $E \in M_{\mathbb{P}^2}(2; c_1, c_2; 2k)$, then E can be written in the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-k) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z \longrightarrow 0$$

where $l(Z)$ denotes the length of Z. Therefore,

$$\begin{aligned} \dim\ M_{\mathbb{P}^2}(2; 0, c_2; 2k) &= \dim\ Hilb^{l(Z)}(\mathbb{P}^2) + \\ &\dim\ Ext^1(\mathcal{O}_{\mathbb{P}^2}(k) \otimes I_Z, \mathcal{O}_{\mathbb{P}^2}(-k)) - \dim \mathbb{P}H^0(E(k)) - 1. \end{aligned}$$

$$= \begin{cases} 3c_2 + k^2 + 3k - 2, & \text{if } c_2 > k^2 + 3k + 1 \\ 4c_2 - 3, & \text{if } c_2 \leq k^2 + 3k + 1. \end{cases}$$

Corollary

Let r, c_2 and $k \in \mathbb{N}$ such that $c_2 \geq r^2 + 2$. Then a vector bundle $E \in M_{\mathbb{P}^2}(2, 2r, c_2)$ with $S(E) = 2k$ exists if and only if $k^2 + k + r^2 \leq c_2$. Furthermore, E fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(r-k) \to E \to \mathcal{O}_{\mathbb{P}^2}(r+k) \otimes I_Z \to 0.$$

Moreover, $M_{\mathbb{P}^2}(2; 2r, c_2; 2k)$ is an irreducible variety of dimension

$$\begin{cases} 3c_2 - 3r^2 + k^2 + 3k - 2, & \text{if } c_2 > r^2 + k^2 + 3k + 1 \\ 4c_2 - 4r^2 - 3, & \text{if } c_2 \leq r^2 + k^2 + 3k + 1. \end{cases}$$

$$dim\, M_{\mathbb{P}^2}(2; 2r, c_2) = 4c_2 - 4r^2 - 3.$$

Applications to Brill-Noether Theory

Let $M_{\mathbb{P}^2}(2; c_1, c_2)$ be the moduli space of stable vector bundles of rank 2 on $\mathbb{P}^2$ with fixed Chern classes c_1, c_2.

For any $t \geq 0$, the subvariety of $M_{\mathbb{P}^2}(2; c_1, c_2)$ defined as

$$W^t(2; c_1, c_2) := \{E \in M_{\mathbb{P}^2}(2; c_1, c_2) : h^0(\mathbb{P}^2, E) \geq t\}$$

is called the $t-$Brill-Noether locus of the moduli space $M_{\mathbb{P}^2}(2; c_1, c_2)$.

(Costa, Miro-Roig 2008)

- For any $t \geq 0$, $W^t(2; c_1, c_2)$ has structure of determinantal variety.
- Each non-empty irreducible component of $W^t(2; c_1, c_2)$ has dimension greater or equal to the Brill-Noether number on $\mathbb{P}^2$

$$\rho^t(2; c_1, c_2) := 4c_2 - c_1^2 - 3 - t\left(t - \frac{c_1^2}{2} - \frac{3c_1}{2} + c_2 - 2\right).$$

Theorem (—, Torres-López, Zamora 2021)

Let r, k, c_2 be integers. Assume,

$$t = \frac{(r-k+2)(r-k+1)}{2}.$$

Let $E \in M_{\mathbb{P}^2}(2; 2r, c_2; 2k)$. Then

$$\begin{cases} E \notin W^1(2; 2r, c_2), & \text{if } r < k \\ E \in W^t(2; 2r, c_2), & \text{if } r \geq k. \end{cases}$$

Moreover, the Brill-Noether number

$$\rho^t(2, 2r, c_2) < \dim M_{\mathbb{P}^2}(2, 2r, c_2; 2k) \leq \dim W^t(2, 2r, c_2)$$

for $c_2 >> 0$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(r-k) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^2}(r+k) \otimes I_Z \longrightarrow 0,$$

Theorem (—, Torres-López, Zamora 2021)

Let $r, k, c_2 \in \mathbb{N}$

- Assume $r^2 + 2 \leq c_2$ and $k < r$. Then, a vector bundle $E \in M_{\mathbb{P}^2}(2; 2r, c_2, 2k)$ exists such that

$$h^0(\mathbb{P}^2, E) \geq (r-k)^2 + 4(r-k) + 3.$$

- Assume $r \geq 2$, $3k^2 - 4k + r^2 + 2 < c_2$ and $k < r$. Let $t = (r-k)^2 + 4(r-k) + 3$. Then,

$$dim\, W^t(2; 2r, c_2) \geq$$

$$\begin{cases} 2c_2 + 2k^2 - 2r^2 + 4k - 2, & \text{if } c_2 > k^2 + 3k + r^2 + 1 \\ k^2 + 3c_2 + k - r^2 - 3, & \text{if } c_2 \leq k^2 + 3k + r^2 + 1. \end{cases}$$

Weak Brill-Noether

Definition

The moduli space $M_{X,H}(n; c_1, c_2)$ satisfies weak Brill-Noether if the general sheaf in $M_{X,H}(n; c_1, c_2)$ has at most one nonzero cohomology group.

Theorem (Göttsche, Hirschowitz 1994)

Suppose that $c_1 > 0$ and $c_2 = 2 + \frac{c_1^2+3c_1}{2}$. Then the moduli space $M_{\mathbb{P}^2}(2, c_1, c_2)$ satisfies weak Brill-Noether.

Let $k = r + 1$, then the stratum $M_{\mathbb{P}^2}(2; 2r, c_2; 2k)$ is open and $h^0(\mathbb{P}^2, E) = 0$ for any $E \in M_{\mathbb{P}^2}(2; 2r, c_2; 2k)$.

Thank you for your attention!!