An algorithmic discrete gradient field

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Abstract

We introduce an algorithm that constructs a discrete gradient field on any simplicial complex. We show that, in all situations, the gradient field is maximal possible and, in a number of cases, optimal. We make a thorough analysis of the resulting gradient field in the case of Munkres' discrete model for $C(K_m, 2)$, the configuration space of two ordered non-colliding particles on the complete graph K_m on m vertices.

Introduction to the algorithm and some notation

Let K be a finite abstract ordered simplicial complex of dimension d with ordered vertex set $(V, \preceq)$. We describe and study an algorithm $\mathcal{A}$ that constructs a discrete gradient field W (which depends on $\preceq$) on K. As we will watch (both by explicit and generic examples), W is either optimal[a] (perhaps after a selection of $\preceq$) or close to being optimal (for generic $\preceq$), depending of course on the complex K. Furthermore, as observed in (1) below, W turns out to be maximal for any K. In fact, our algorithm can be thought of as a generalization of the inclusion-exclusion method (with respect to a fixed vertex) that yields an optimal gradient field on a full simplex. In the case of a general complex, the ordering $\preceq$ plays a heuristic role that guides the inclusion-exclusion process.

By the order-extension principle, we may as well assume $\preceq$ is linear from the outset. Let $\mathcal{F}^i$ denote the set of i-dimensional faces of K. Recall a face $\alpha^{(i)} \in \mathcal{F}^i$ is identified with the ordered tuple $[\alpha_0, \alpha_1, \cdots, \alpha_i]$, $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_i$, of its vertices. In such a setting, we say that α_r appears in position r of α. The ordered-tuple notation allows us to lexicographically extend $\preceq$ to a linear order (also denoted by $\preceq$) on the set $\mathcal{F}$ of faces of K. We write $\prec$ for the strict version of $\preceq$.

For a vertex $v \in V$, a face $\alpha \in \mathcal{F}^i$ and an integer $r \geq 0$, let

$$\iota_r(v, \alpha) = \begin{cases} \alpha \cup \{v\}, & \text{if } \alpha \cup \{v\} \in \mathcal{F}^{i+1}, v \text{ appears in position } r \text{ of } \alpha \cup \{v\}; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Algorithm

At the start of the algorithm we set $W := \varnothing$ and initialize auxiliary variables $F^i := \mathcal{F}^i$ for $0 \leq i \leq d$ which, at any moment of the algorithm, keep track of i-dimensional faces not taking part of a pairing in W. Throughout the algorithm $\mathcal{A}$, pairings $(\alpha, \beta) \in \mathcal{F}^i \times \mathcal{F}^{i+1}$ are added to W by means of a family of processes $\mathcal{P}^i$ running for $i = d-1, d-2, \ldots, 1, 0$ (in that order), where $\mathcal{P}^i$ is executed provided (at the relevant moment) both F^i and F^{i+1} are not empty (so there is a chance to add new pairings to W). Process $\mathcal{P}^i$ consists of three levels of nested subprocesses:

1. At the most external level, $\mathcal{P}^i$ consists of a family of processes $\mathcal{P}^{i,r}$ for $i+1 \geq r \geq 0$, executed in descending order with respect to r.
2. In turn, each $\mathcal{P}^{i,r}$ consists of a family of subprocesses $\mathcal{P}^{i,r,v}$ for $v \in V$, executed from the $\preceq$-largest vertex to the smallest one.
3. At the most inner level, each process $\mathcal{P}^{i,r,v}$ consists of a family of instructions $\mathcal{P}^{i,r,v,\alpha}$ for $\alpha \in \mathcal{F}^i$, executed following the $\preceq$-lexicographic order.

Instruction $\mathcal{P}^{i,r,v,\alpha}$ checks whether, at the moment of its execution, $(\alpha, \iota_r(v, \alpha)) \in F^i \times F^{i+1}$, i.e., whether $(\alpha, \iota_r(v, \alpha))$ is "available" as a new pairing. If so, the pairing $\alpha \nearrow \iota_r(v, \alpha)$ is added to W, while α and $\iota_r(v, \alpha)$ are removed from F^i and F^{i+1}, respectively. By construction, at the end of the algorithm, the resulting family of pairs W is a partial matching in $\mathcal{F}$. Furthermore, from its construction,

all faces and cofaces of an unpaired cell are involved in a W-paring, (1)

so that W is maximal. Most importantly:

Proposition 0.1. *W is a gradient field.*

This algorithm can be modified to be more computational-efficient, even though we have shown this version due to its theoretical advantages.

Example 0.1. Figure 1 gives a triangulation of the projective plane $\mathbb{R}P^2$. The gradient field shown by the heavy arrows is determined by $\mathcal{A}$ using the indicated ordering of vertices. The only critical faces are $[6]$ (in dimension 0), $[2, 5]$ (in dimension 1) and $[1, 3, 4]$ (in dimension 2), so optimality of the field follows from the known mod-2 homology of $\mathbb{R}P^2$. Although the gradient field depends on the ordering of vertices, we have verified with the help of a computer that, in this case, all possible 720 gradient fields (coming from the corresponding $6!$ possible orderings of vertices) are optimal. A corresponding optimal gradient field on the 2-torus (and the vertex-order rendering it) is shown in Figure 2. This time the critical faces are $[9]$ (in dimension 0), $[2, 8]$ and $[5, 8]$ (in dimension 1) and $[1, 3, 7]$ (in dimension 2). The torus case is interesting in that there are vertex orderings that yield non-optimal gradient fields. In general, a plausible strategy for choosing a convenient ordering of vertices consists on assuring the largest possible number of vertices with high $\preceq$-tag so that no two such vertices lie on a common face. For instance, in our torus example, no pair of vertices taken from 7, 8 and 9 lie on a single face.

[a] Optimality refers to the property that the number of critical cells in a given dimension agrees with the corresponding Betti number.

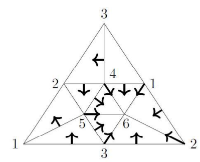

Figure 1: Algorithmic gradient field in the projective plane

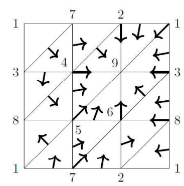

Figure 2: Algorithmic gradient field in the 2-torus

Collapsibility conditions

In this section we identify a set of conditions implying collapsibility of a given face.

Definition 0.1. *A vertex α_i of a face $\alpha = [\alpha_0, \ldots, \alpha_k] \in \mathcal{F}^k$ is said to be maximal in α if $\partial_{\alpha_i}(\alpha) \cup \{v\} \notin \mathcal{F}^k$ for all vertices v with $\alpha_i \prec v$. When α_i is non-maximal in α, we write*

$$\alpha^i := \max\{v \in V : \alpha_i \prec v \text{ and } \partial_{\alpha_i}(\alpha) \cup \{v\} \in \mathcal{F}^k\} \quad \text{and} \quad \alpha(i) := \partial_{\alpha_i}(\alpha) \cup \{\alpha^i\}.$$

Note that α^i is maximal in $\alpha(i)$, and that α^i is not a vertex of α.

Iterating the construction, for $\alpha = [\alpha_0, \ldots, \alpha_k] \in \mathcal{F}^k$ and a sequence of integers $0 \leq i_1 < i_2 < \cdots < i_p \leq k$, we say that the ordered vertices $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p}$ are non-maximal in α provided:

- *α_{i_1} is non-maximal in α, so we can form the face $\alpha(i_1)$;*
- *α_{i_2} is non-maximal in $\alpha(i_1)$, so we can form the face $\alpha(i_1, i_2) := \alpha(i_1)(i_2)$;*

etcetera,

- *α_{i_p} is non-maximal in $\alpha(i_1, \ldots, i_{p-1})$, so we can form the face $\alpha(i_1, \ldots, i_p) := \alpha(i_1, \ldots, i_{p-1})(i_p)$.*

When $p = 0$ (so there is no constructing process), $\alpha(i_1, i_2, \ldots, i_p)$ is interpreted as α.

Lemma 0.1. *No vertex of a redundant k-face $\alpha \in \mathcal{F}^k$ is maximal in α.*

Corollary 0.1. *The following conditions are equivalent for a k-face $\alpha = [\alpha_0, \ldots, \alpha_k] \in \mathcal{F}^k$:*

(1) α_k is maximal in α.

(2) $\partial_{\alpha_k}(\alpha) \nearrow \alpha$.

Proposition 0.2. *For a face $\alpha = [\alpha_0, \ldots, \alpha_k] \in \mathcal{F}^k$ and an integer $r \in \{0, 1, \ldots, k\}$ with α_r maximal in α, the pairing $\partial_{\alpha_r}(\alpha) \nearrow \alpha$ holds provided*

$\mathtt{for\ any\ sequence}\ r+1 \leq t_1 < \cdots < t_p \leq k,$
$\mathtt{the\ ordered\ vertices}\ \alpha_{t_1}, \ldots, \alpha_{t_p}\ \mathtt{are\ non-maximal\ in}\ \alpha.$

Definition 0.2. *A vertex α_r of a face $\alpha = [\alpha_0, \ldots, \alpha_k] \in \mathcal{F}^k$ is said to be collapsing in α provided:*

(i) The face α is not redundant.

(ii) Condition (0.2) holds.

(iii) For every v with $\alpha_r \prec v$ and $\partial_{\alpha_r}(\alpha) \cup \{v\} \in \mathcal{F}^k$, there is a vertex α_j of α with $v \prec \alpha_j$ such that α_j is collapsing in $\partial_{\alpha_r}(\alpha) \cup \{v\}$.

The first and third conditions in Definition 0.2 hold when α_r is maximal in α. Note the recursive nature of Definition 0.2.

Theorem 0.1. *If α_r is collapsing in α, then $\partial_{\alpha_r}(\alpha) \nearrow \alpha$.*

Application to configuration spaces

We use the gradient field in the previous section in order to describe the cohomology ring of the configuration space of 2 ordered points on a complete graph.

Definition 0.3. *Munkres' homotopy simplicial model*

Let K_m be the 1-dimensional skeleton of the full $(m-1)$-dimensional simplex on vertices $V_m = \{1, 2, \ldots, m\}$. Thus $|K_m|$ is the complete graph on the m vertices. The homotopy type of $\mathrm{Conf}(|K_m|, 2)$ is well understood for $m \leq 3$, so we assume $m \geq 4$ from now on. We think of K_m as an ordered simplicial complex with the natural order on V_m, and study $\mathrm{Conf}(|K_m|, 2)$ through its simplicial homotopy model C_m [3, Lemma 70.1]. The condition $m \geq 4$ implies that C_m is a pure 2-dimensional complex, i.e., all of its maximal faces have dimension 2. Furthermore, 2-dimensional faces of C_m have one of the forms

$$\begin{bmatrix} a & a & d \\ b & c & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a' & c' & c' \\ b' & b' & d' \end{bmatrix} \tag{2}$$

where

$$d > a \notin \{b, c\}, \quad b < c \neq d, \quad d' > b' \notin \{a', c'\} \quad \text{and} \quad a' < c' \neq d'. \tag{3}$$

Proposition 0.3. *Let W_m be the gradient field on C_m constructed by the algorithm in Section **??** with respect to the lexicographic order on the vertices $\substack{a \\ b} = (a, b) \in V_m \times V_m \setminus \Delta_{V_m}$ of C_m. The full list of W_m-pairings is:*

(a) $\left[\begin{smallmatrix} a & a \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & m \\ b & d & d \end{smallmatrix}\right]$, *for $a < m > d$.*

(b) $\left[\begin{smallmatrix} a & a \\ b & m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & m-1 \\ b & m & m \end{smallmatrix}\right]$, *for $a < m - 1$.*

(c) $\left[\begin{smallmatrix} a & c \\ b & b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & c & c \\ b & b & m \end{smallmatrix}\right]$, *for $b < m > c$.*

(d) $\left[\begin{smallmatrix} a & m \\ b & b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m & m \\ b & b & m-1 \end{smallmatrix}\right]$, *for $b < m - 1$.*

(e) $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & c & c \\ b & b & d \end{smallmatrix}\right]$, *for $a < c$, $b < d$, $b \neq c$ and ($c < m > d$ or $c = m > d + 1$).*

(f) $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$, *for $a < c$, $b < d$, $a \neq d$ and ($b = c < m > d$ or $c + 1 < m = d$).*

(g) $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m \\ b & m-1 \end{smallmatrix}\right]$, *for either $b < m - 1$ or $a < m - 1 = b$.*

(h) $\left[\begin{smallmatrix} a \\ m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m-1 \\ m & m \end{smallmatrix}\right]$, *for $a < m - 1$.*

(i) $\left[\begin{smallmatrix} m-1 \\ m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} m-1 & m-1 \\ m-2 & m \end{smallmatrix}\right]$.

In particular, the critical faces are:

(j) In dimension 0, the vertex $\left[\begin{smallmatrix} m \\ m-1 \end{smallmatrix}\right]$.

(k) In dimension 1, the simplices:

(k.1) $\left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right]$, *with either $a = m-1 > b+1$ or $a < m-1 \geq b$.*

(k.2) $\left[\begin{smallmatrix} m & m \\ b & d \end{smallmatrix}\right]$, *with $d < m - 1$.*

(k.3) $\left[\begin{smallmatrix} a & c \\ m & m \end{smallmatrix}\right]$, *with $c < m - 1$.*

(l) In dimension 2, the simplices $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ *with $b \neq c < m > d$.*

The Morse coboundary map $\delta\colon \mu^0(C_m) \to \mu^1(C_m)$ is forced to vanish since $c_0 = 1$. More interestingly:

Proposition 0.4. *The coboundary $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$ vanishes on the duals of the critical faces of types (k.2) and (k.3) in Proposition 0.3. For the duals of the critical faces of type (k.1) we have*

$$\delta\left(\begin{bmatrix} a & m-1 \\ b & m \end{bmatrix}\right) = \sum \begin{bmatrix} a & a & x \\ y & b & b \end{bmatrix} - \sum \begin{bmatrix} a & a & x \\ b & y & y \end{bmatrix} + \sum \begin{bmatrix} x & x & a \\ b & y & y \end{bmatrix} - \sum \begin{bmatrix} x & x & a \\ y & b & b \end{bmatrix}, \tag{4}$$

where all four summands run over all integers x and y that render critical 2-faces. Explicitly, $a < x < m$ in the first and second summations, $x < a$ in the third and fourth summations, $b < y < m$ in the second and third summations, $y < b$ in the first and fourth summations, and $b \neq x \neq y \neq a$ in all four summations.

The full cohomology R-algebra $H^*(\mathrm{Conf}(K_m, 2); R)$ for any commutative unital ring R will be described in a next paper which is close to be send for publishing.

References

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