Monday 28th November:

- Local Fields
- p-adic groups { Algebraic structure / Topological structure }
- Smooth representations

§1 Local Fields

F a non-archimedean local field

$\mathcal{O} = \mathcal{O}_F$ the ring of integers

$\mathfrak{p} = \mathfrak{p}_F$ its unique maximal ideal

$k = k_F = \mathcal{O}/\mathfrak{p}$ the residue field

p characteristic of k

q cardinality of k

$|\cdot|$ normalized absolute value

$\varpi = \varpi_F$ uniformizer

$\underline{(F,+)}$ has a fundamental system of neighbourhoods of 0 consisting of pro-p subgroups $\{\mathfrak{p}^n : n \in \mathbb{Z}\}$

$$\bigcap_{n\in\mathbb{Z}} \mathfrak{p}^n = \{0\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} \mathfrak{p}^n = F$$

$\underline{(F^\times,\cdot)}$ has

$$\underset{\text{locally compact}}{F^\times} \supseteq \underset{\text{compact}}{\mathcal{O}^\times} \supseteq U_F^{n+1} = 1+\mathfrak{p}_F^{n+1}, \quad n \geq 0$$

filtration by open pro-p subgroups

Characters

C an algebraically closed field of characteristic $\ell \neq p$ (allow $\ell = 0$)

[• $p \in C$ is invertible
• C contains all p-power roots of unity]

Fix $\psi : F \longrightarrow C^\times$ a non-trivial continuous additive character

such that $\psi|_{\mathfrak{p}} = \mathbb{1}$ but $\psi|_{\mathcal{O}} \neq \mathbb{1}$

Then
$$F \xrightarrow{\sim} \hat{F} := \{\text{continuous additive characters}\}$$
$$a \longmapsto (x \longmapsto \psi(ax))$$

and, for $n \geq m$, we get

$$\mathfrak{p}^{-n}/\mathfrak{p}^{-m} \xrightarrow{\sim} \left(\mathfrak{p}^{m+1}/\mathfrak{p}^{n+1}\right)^{\wedge}$$

Multiplicative Characters

$\widehat{F^\times} = \{$ continuous $\chi : F^\times \longrightarrow \mathbb{C}^\times$ group homomorphism $\}$

- smooth since $\ker \chi = \chi^{-1}(1)$ is open
- depth $\ell(\chi) = \min\{n \geq 0 : \ker(\chi) \supseteq U_F^{n+1}\}$
- unramified if $\ker \chi \supseteq \mathcal{O}^\times$.

If $0 \leq m < n \leq 2m+1$ then

$$\mathfrak{p}^{m+1}/\mathfrak{p}^{n+1} \xrightarrow{\sim} U_F^{m+1}/U_F^{n+1}$$

$$x + \mathfrak{p}^{n+1} \longmapsto (1+x)U_F^{n+1}$$

so we get an isomorphism

$$\mathfrak{p}^{-n}/\mathfrak{p}^{-m} \xrightarrow{\sim} \left(U_F^{m+1}/U_F^{n+1}\right)^\wedge$$

$$a + \mathfrak{p}^{-m} \longmapsto \psi_a : (1+x)U_F^{n+1} \longmapsto \psi(ax)$$

§2 p-adic groups

$G = \mathbb{G}(F)$ the group of F-rational points of a (connected) reductive group defined over F.

e.g. $GL_n(F)$, $SL_n(F)$, $Sp_{2n}(F) = \{ g \in GL_{2n}(F) : g^T J g = J \}$

where $J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & -1 & & & \\ -1 & & & & \end{pmatrix}$

G has two structures:

- Algebraic structure from $\mathbb{G}$
- Topological structure from F

Algebraic Structure

$G = GL_n(F)$ has

$$T = \mathbb{T}(F) = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} : t_i \in F^\times \right\}$$

the standard maximal F-split torus in G.

- All maximal F-split tori are conjugate to T
- There are other maximal tori — e.g.

E/F a separable extension of degree n

$$G \supseteq \mathrm{Aut}_F(E) \longleftarrow E^\times$$

There is a maximal torus $\mathbb{T}'$ such that $\mathbb{T}'(F) = E^\times$

(conjugate to $\mathbb{T}'$ in $GL_n(E)$)

$\underline{G = GL_n(F)}$

$T = \mathbb{T}(F) = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$ standard maximal F-split torus.

$B = \mathbb{B}(F) = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$ the standard Borel subgroup

- All maximal connected solvable subgroups are conjugate to B

$P = \mathbb{P}(F) = \begin{pmatrix} \overbrace{\square}^{n_1} & \overbrace{*}^{n_2} & \cdots & \overbrace{*}^{n_r} \\ & \square & & \vdots \\ 0 & & \ddots & \\ & & & \square \end{pmatrix}$ a standard parabolic subgroup

- A parabolic subgroup is a subgroup containing a Borel
- A parabolic subgroup is a closed algebraic subgroup s.t. G/P is a projective variety

$G = GL_n(F)$

$T = \mathbb{T}(F) = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$ standard maximal F-split torus

$B = \mathbb{B}(F) = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$ standard Borel subgroup

$$P = \mathbb{P}(F) = \begin{array}{cl} \begin{pmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \vdots \\ 0 & & & * \end{pmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \\ \}n_r \end{matrix} \end{array}$$

standard parabolic subgroup

$\bigtriangledown$

$$N = \mathbb{N}(F) = \begin{array}{cl} \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \\ \}n_r \end{matrix} \end{array}$$

the unipotent radical of P

(maximal connected normal unipotent subgroup)

$$M = \mathbb{M}(F) = \begin{array}{cl} \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \\ \}n_r \end{matrix} \end{array}$$

$$\simeq \prod_{i=1}^{r} GL_{n_i}(F)$$

the standard Levi component of P

(complement to N containing T)

$$P = M \ltimes N$$

$G = GL_n(F)$

$T = \mathbb{T}(F) = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$B = \mathbb{B}(F) = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$

$P = \mathbb{P}(F) = \begin{pmatrix} * & * & \cdots & * \\ & * & & \vdots \\ & & \ddots & * \\ 0 & & & * \end{pmatrix}$

$\triangledown$

$N = \mathbb{N}(F) = \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$

$M = \mathbb{M}(F) = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$

$\simeq \prod_{i=\ldots} GL_{n_i}(F)$

$SL_n(F)$

Just intersect

everything

with

SL_n

$\{(g_i) : \prod_{i=\ldots} \det(g_i) = 1\}$

$Sp_{2n}(F)$

$\left\{ \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} : t_i \in F^\times \right\}$

Intersect

$\begin{pmatrix} {}^t g_r^{-1} & & & & 0 \\ & \ddots & & & \\ & & g_0 & & \\ & & & \ddots & \\ 0 & & & & g_r \end{pmatrix}$

is

$Sp_{2n_0}(F) \times \prod_{i=1}^{r} GL_{n_i}(F)$

$G = GL_n(F)$

General group G

$T = \mathbb{T}(F) = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

Maximal F-split torus

[May not be a maximal torus.
If it is then G is split.]

$B = \mathbb{B}(F) = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$

Minimal parabolic subgroup defined over F

[May not be the F-points of a Borel.
If it is then G is quasi-split.
Split $\Rightarrow$ quasi-split.]

$P = \mathbb{P}(F) = \begin{pmatrix} \boxed{\ } & & * \\ & \ddots & \\ 0 & & \boxed{\ } \end{pmatrix}$

$G/\mathbb{P}$ is projective

$\triangledown$

$N = \mathbb{N}(F) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Unipotent radical

$M = \mathbb{M}(F) = \begin{pmatrix} \boxed{*} & & 0 \\ & \ddots & \\ 0 & & \boxed{*} \end{pmatrix}$

Complement

Topological Structure / Integral Structure

$G = \mathbf{G}(F)$ is a locally compact totally disconnected group.

$G = GL_n(F)$

$K = GL_n(\mathfrak{o})$ maximal compact (open) subgroup
[all are conjugate in G]

$\triangledown$

$K^m = 1 + \mathrm{Mat}_n(\mathfrak{p}^m)$, for $m \geq 1$ pro-p subgroup

- $\bigcap_{m\geq 1} K^m = \{1\}$ fundamental system of neighbourhoods of 1
- $[K^m, K^r] \subseteq K^{m+r}$

$\tilde{K} = N_G(K) = ZK$ maximal compact-mod-centre subgroup.

$G = GL_n(F)$

$K = GL_n(\mathcal{O})$ maximal compact open

$K^m = 1 + \mathrm{Mat}_n(\mathfrak{p}^m)$ filtration by pro-p normal subgroups

$$\begin{tikzcd}
K^{0+} = K^1 \arrow[r, hook] & K \arrow[r, two heads] & GL_n(k_F)
\end{tikzcd}$$

pro-p radical (maximal normal pro-p) — reductive quotient

$$\begin{tikzcd}
 & K & GL_n(k_F) \\
\begin{pmatrix} \, & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \, \end{pmatrix}^{\times} & \mathcal{P} \arrow[u, phantom, "\cup" description] \arrow[r, two heads] & \mathcal{P}(k_F) \arrow[u, phantom, "\cup" description] \\
1 + \begin{pmatrix} \, & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \, \end{pmatrix} & \mathcal{P}^{0+} \arrow[u, phantom, "\cup" description] \arrow[r, two heads] & N(k_F) \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

$\mathcal{P}(k_F)$: parabolic

$\mathcal{P}$: parahoric

$N(k_F)$: unipotent radical

$\mathcal{P}^{0+}$: pro-p-radical

$$\begin{tikzcd}
\mathcal{P}^{0+} \arrow[r, hook] & \mathcal{P} \arrow[r, two heads] & M(k_F)
\end{tikzcd}$$

Levi subgroup

$G = GL_n(F)$

$K = GL_n(\mathcal{O})$

$$\begin{array}{ccccc} K^{0+} & \hookrightarrow & K & \twoheadrightarrow & GL_n(k_F) \\ & & \cup\!\!\mid & & \cup\!\!\mid \\ & & I & \twoheadrightarrow & \mathbb{B}(k_F) \end{array}$$ Borel subgroup

$\begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}$ Iwahori subgroup

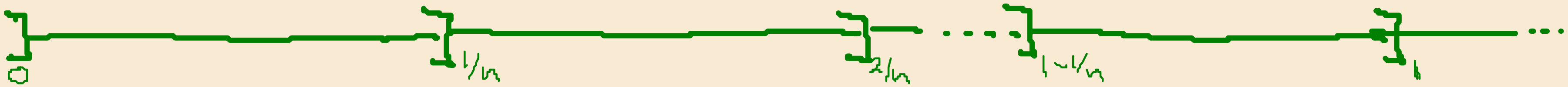
$\tilde{I} = N_G(I)$ is another maximal compact-mod-centre subgroup

[All such are conjugate to $\tilde{\mathfrak{P}} = N_G(\mathfrak{P})$ for $\mathfrak{P}$ with equal block sizes.]

Filtration

$$\begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix} \supseteq 1 + \begin{pmatrix} \mathfrak{p} & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathfrak{p} \end{pmatrix} \supseteq 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & & \mathcal{O} \\ & \ddots & \ddots & \\ \mathfrak{p} & & \ddots & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & & \mathfrak{p} \end{pmatrix} \supseteq \cdots \supseteq 1 + \begin{pmatrix} \mathfrak{p} & & \mathfrak{p} \\ & \ddots & \\ \mathfrak{p}^2 & & \mathfrak{p} \end{pmatrix} \supseteq \cdots$$

$$I = I^0 \qquad I^{0+} = I^{1/n} \qquad I^{1/n+} = I^{2/n} \qquad \cdots \qquad I^{1-1/n+} = I^1 \quad \cdots$$



$G = GL_n(F)$

$K = GL_n(\mathcal{O})$

$$K^{0+} \hookrightarrow K \twoheadrightarrow GL_n(k_F)$$
$$\text{parahoric} \quad \mathcal{P} \subseteq K \twoheadrightarrow \mathbb{P}(k_F) \subseteq GL_n(k_F) \quad \text{parabolic}$$

Filtration

$\mathcal{P}$ has filtrations $(\mathcal{P}^r)_{r\geq 0}$ by pro-p subgroups, with

- $\bigcap_{r\geq 0} \mathcal{P}^r = \{1\}$
- $r \longmapsto \mathcal{P}^r$ left continuous (and $\mathcal{P}^{r+} := \bigcup_{s>r} \mathcal{P}^s$)
- $[\mathcal{P}^r, \mathcal{P}^s] \subseteq \mathcal{P}^{r+s}$
- $\mathcal{P}/\mathcal{P}^{0+} \simeq \mathbb{M}(k_F)$ connected reductive $/k_F$

There are many filtrations, indexed by a point x in the Bruhat-Tits building (or by a lattice function Λ).

$G = GL_n(F)$

$K = GL_n(\mathcal{O})$ maximal compact

$\cup|$

$\mathcal{P} = \begin{pmatrix} L_L & \mathcal{O} \\ \mathfrak{p} & L \end{pmatrix}^\times$ parahoric

$\cup|$

$I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}$ Iwahori

Filtration $(\mathcal{P}^r)_{r>0}$

$\mathcal{P}/\mathcal{P}^{0+} \simeq \prod_i GL_{n_i}(k_F)$

$G = SL_n(F)$

$SL_n(\mathcal{O})$ and conjugates by $\begin{pmatrix} \varpi & & & & & \\ & \ddots & & & & \\ & & \varpi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$

Intersect and conjugate

Intersect (all conjugate)

Intersect the filtration

$\{(g_i) : \prod_i \det(g_i) = 1\}$

$G = Sp_{2n}(F)$

$$K_i = \begin{array}{c} \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} \\ \begin{array}{ccc} \underbrace{\quad}_{i} & \underbrace{\qquad}_{2(n-i)} & \underbrace{\quad}_{i} \end{array} \end{array} \cap G$$

$K_i/K_i^{0+} \simeq Sp_{2i}(k_F) \times Sp_{2(n-i)}(k_F)$

Intersect

Intersect the filtration

$Sp_{2n_0}(k_F) \times \prod_i GL_{n_i}(k_F) \times Sp_{2n_0'}(k_F)$

s.t. $n_0 + \sum n_i + n_0' = n$

$G = GL_n(F)$

$K = GL_n(\mathcal{O})$ maximal compact

$\cup\!\shortmid$

$$\mathcal{P} = \begin{pmatrix} \boxed{} & \mathcal{O} \\ \mathfrak{p} & \ddots \end{pmatrix}^\times \quad \text{parahoric}$$

$\cup\!\shortmid$

$$I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix} \quad \text{Iwahori}$$

Filtration $(\mathcal{P}^r)_{r \geq 0}$

$$\mathcal{P}/\mathcal{P}^{0+} = \prod_i GL_{n_i}(k_F)$$

General group G

the "connected part" of the maximal compact subgroup of the stabilizer of a point in the Bruhat-Tits building of G.

Moy-Prasad filtration

Connected reductive group over k_F

Haar measure

Because $p \in \mathbb{C}^\times$, the group G has a Haar measure with values in $\mathbb{C}$:

- choose a pro-p open subgroup (e.g. $\mathfrak{P}^{0+}$) and declare it to have measure 1.

Note that compact open subgroups can have measure 0 !

e.g. On $F^\times$ if $\mu(U_F^1) = 1$

then $\mu(\mathcal{O}_F^\times) = q-1$ which may be 0 in $\mathbb{C}$.

§3 Smooth representations

G a group as above, or any closed subgroup

C algebraically closed field of characteristic $\ell \neq p$ (allow $\ell = 0$)

Definition A smooth representation of G is a pair (π, V) with

- V a C-vector space
- $\pi : G \longrightarrow \mathrm{Aut}_G(V)$ homomorphism s.t. $\mathrm{Stab}_G(v)$ is open, for $v \in V$.

Invariants $V^H = \{ v \in V : \pi(h)v = v \ \forall h \in H \}$, for $H \leq G$

Then $$V = \bigcup_{K \text{ compact open}} V^K.$$

(π, V) is admissible if $\dim_C V^K$ is finite, for all compact open K.

G a group as above, or any closed subgroup

C algebraically closed field of characteristic $\ell \neq p$

$\mathrm{Rep}_C(G)$ the category of smooth representations of G

- morphisms are G-equivariant linear maps
- abelian category
- not just finite-length representations
- not semisimple in general — it is if G is compact and $\ell = 0$

$\mathrm{Irr}_C(G)$ the set of equivalence classes of irreducible representations

Examples

(1) $\chi : F^\times \longrightarrow C^\times$ a continuous character

(2) If $\pi \in \mathrm{Irr}_C(GL_n(F))$ is finite-dimensional then

$$\pi = \chi \circ \det \quad \text{is 1-dimensional.}$$

Theorem Every $\pi \in \mathrm{Irr}_C(G)$ is admissible.

Adjunctions and Harish-Chandra Theory

For H a closed subgroup of G we have

$$\mathrm{Res}_H^G : \mathrm{Rep}_c(G) \longrightarrow \mathrm{Rep}_c(H) \qquad \text{restriction}$$

which has a right adjoint (smooth) induction

$$\mathrm{Ind}_H^G : \mathrm{Rep}_c(H) \longrightarrow \mathrm{Rep}_c(G)$$

i.e.

$$\mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho)$$

$\mathrm{Ind}_H^G \rho$ given by the right regular action of G on

$$\{ f : G \to V_\rho \ : \ f(hg) = \rho(h) f(g) \ \forall h \in H, g \in G \text{ and } f \text{ is smooth} \}$$

If H is also open then there is also a left adjoint ind_H^G, compact induction, i.e.

$$\mathrm{Hom}_H(\rho, \mathrm{Res}_H^G \pi) = \mathrm{Hom}_G(\mathrm{ind}_H^G \rho, \pi)$$

$\mathrm{ind}_H^G \rho$ given by the right regular action on

$$\{ f \in \mathrm{Ind}_H^G \rho \; : \; \mathrm{supp}(f) \subseteq HY \text{ with } Y \text{ compact} \}$$

<u>Example</u>

$$\mathrm{ind}_{\mathcal{O}^\times}^{F^\times} \mathbb{1} = \{ f : F^\times/\mathcal{O}^\times \to \mathbb{C} \text{ with finite support} \} \simeq \mathbb{C}[X, X^{-1}]$$

- has infinite length
- every unramified character is a quotient
- no irreducible subrepresentation.

<u>Remark</u> If P is a parabolic of G then $\mathrm{Ind}_P^G = \mathrm{ind}_P^G$ as $P \backslash G$ is compact.

Parabolic Induction

If $P = MN$ is a parabolic subgroup of $G = \mathbb{G}(F)$ then we have a parabolic induction functor

$$\begin{tikzcd}
\mathrm{Rep}_c(M) \arrow[rr, "\mathrm{Ind}^G_{M,P}"] \arrow[dr, "\mathrm{Infl}^P_M"'] & & \mathrm{Rep}_c(G) \\
& \mathrm{Rep}_c(P) \arrow[ur, "\mathrm{Ind}^G_P"'] &
\end{tikzcd}$$

$P/N \cong M$

which is right adjoint to the Jacquet (parabolic restriction) functor

$$\begin{tikzcd}
\mathrm{Rep}_c(G) \arrow[rr, "\mathrm{Res}^G_{P,M}"] \arrow[dr, "\mathrm{Res}^G_P"'] & & \mathrm{Rep}_c(M) \\
& \mathrm{Rep}_c(P) \arrow[ur, "\mathrm{Coinv}^P_M"'] &
\end{tikzcd}$$

$V_N = V/V(N)$

(π, V) $\quad V(N) = \langle \pi(n)v - v : n \in N, v \in V \rangle$

Parabolic Induction

$P = MN$ a parabolic subgroup of $G = \mathbb{G}(F)$

$$\mathrm{Rep}_C(M) \xrightarrow{\mathrm{Ind}_{M,P}^{G}} \mathrm{Rep}_C(G)$$

$\bar{P} = M\bar{N}$ the opposite parabolic with $P \cap \bar{P} = M$.

Theorem (2nd adjunction, Bernstein ($C = \mathbb{C}$), Dat–Helm–Moss–Kurinczuk (C any $\mathbb{Z}[\frac{1}{p}]$-algebra))

$\mathrm{Ind}_{M,P}^{G}$ is left adjoint to $\delta_P \cdot \mathrm{Res}_{\bar{P},M}^{G}$

where $\delta_P : M \to C^\times$ is the modulus character of P.

[$\delta_P(m) = \underbrace{(mKm^{-1} : K)}_{\text{generalized group index}}$ for any compact open subgroup of P.]

Definition

- (π, V) is cuspidal if $\mathrm{Res}^G_{P,M}\,\pi = 0$ for all proper parabolics P.

If $\pi \in \mathrm{Irr}_c(G)$ this is equivalent (by adjunction) to

π is not a subrepresentation of any $\mathrm{Ind}^G_{M,P}\,\rho$, for P a proper parabolic.

Theorem (Harish-Chandra)

For any $\pi \in \mathrm{Irr}_c(G)$ there are a

- parabolic subgroup $P = MN$ and
- $\rho \in \mathrm{Irr}_c(M)$ cuspidal such that

$$\pi \longrightarrow \mathrm{Ind}^G_{M,P}\,\rho$$

If (M', ρ') is another such then it is conjugate to (M, ρ) in G.

The conjugacy class of (M, ρ) is called the cuspidal support of π.

Cuspidal and Supercuspidal

$\pi \in \mathrm{Irr}_c(G)$ is

- cuspidal if it is not a subrepresentation of any proper $\mathrm{Ind}_{M,P}^{G}\,\rho$
- supercuspidal if it is not a subquotient of any proper $\mathrm{Ind}_{M,P}^{G}\,\rho$

Remark If $\ell = 0$ then cuspidal $\Longleftrightarrow$ supercuspidal

Theorem For any $\pi \in \mathrm{Irr}_c(G)$ there are a

- parabolic subgroup $P = MN$ and
- $\rho \in \mathrm{Irr}_c(M)$ supercuspidal such that

$$\pi \leq \mathrm{Ind}_{M,P}^{G}\,\rho$$

is a subquotient of

But there is no uniqueness in general. (There is for GL_n, SL_n.)

Invariants

K a compact open subgroup of G.

$\mathcal{H}_c(G,K) = \{K\text{-bi-invariant functions } f: G \to \mathbb{C} \text{ with compact support}\}$

$\cong \mathrm{End}_G(\mathrm{ind}_K^G \mathbb{1})$

Hecke algebra with convolution product

Theorem Suppose $\mu(K) \neq 0$. Then there is a bijection

$$\{(\pi,V) \in \mathrm{Irr}_c(G) : V^K \neq 0\} \xleftrightarrow{\sim} \{\text{simple right } \mathcal{H}_c(G,K)\text{-modules}\}/\sim$$

$$V \longmapsto V^K \cong \mathrm{Hom}_G(\mathrm{ind}_K^G \mathbb{1}, V)$$

Iwahori Invariants

$G = GL_n(F) \supseteq I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}$ Iwahori subgroup

$C = \mathbb{C}$

Theorem The map $(\pi, V) \longmapsto V^I$ induces an equivalence of categories

$$\underbrace{\mathrm{Rep}_{\mathbb{C}}^{I}(G)}_{\text{representations } (\pi,V) \text{ generated (as } G\text{-rep.) by } V^I} \longrightarrow \mathrm{Mod}\text{-}\mathcal{H}_{\mathbb{C}}(G, I)$$

Moreover:

- $(\pi, V) \in \mathrm{Rep}_{\mathbb{C}}^{I}(G) \iff$ all irreducible subquotients of π embed in some $\mathrm{Ind}_{T,B}^{G} \chi$ with χ unramified
- $\mathrm{Rep}_{\mathbb{C}}^{I}(G)$ is an indecomposable summand of $\mathrm{Rep}_{\mathbb{C}}(G)$.