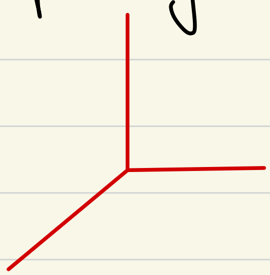
Polyhedral & tropical geometry of flag positroids

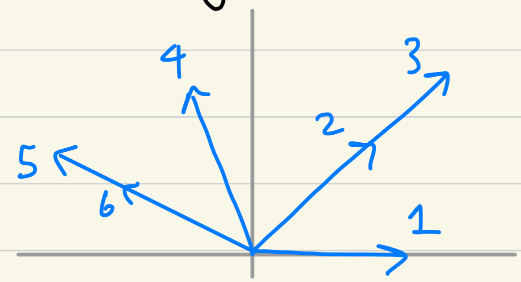
Joint w/ Jonathan Boretsky & Chris Eur

Overview:

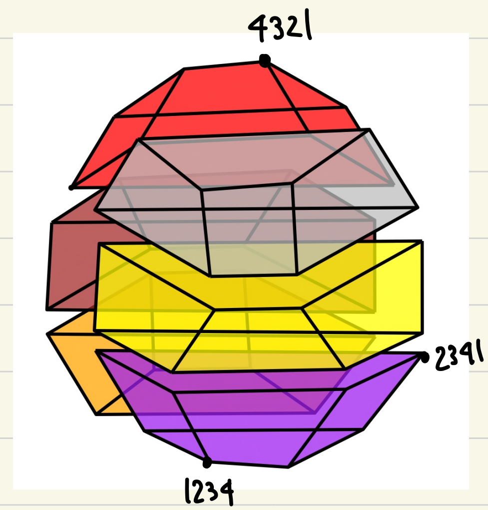
Tropical geometry



(Positive) (Flag) Matroids



Subdivisions of (flag) positroid polytopes



In honor of the birthday of Bernard Leclerc

Figure from Nadeau–Tewari
2208.04128
"Remixed Eulerian numbers"

Outline

- Intro to pos. flag variety
- Flag positroids (special class of flag matroids) & their moment polytopes
- The pos. tropical flag variety (pos Dressian)
- Theorem connecting the above objects
- Applications to realizability questions & Bruhat interval polytopes
- Example of $\mathrm{Tr}\,Fl_4^{>0}$ (cluster connection, relation to Lara Bossinger's talk)

Positive Flag Variety (Type A)

Def: Let $R = \{r_1 < \dots < r_k\} \subset [n] = \{1, \dots, n\}$
The flag variety $Fl_{R;n}$ is variety of partial flags of subspaces
$\{(V_1, \dots, V_k) : 0 \subset V_1 \subset \dots \subset V_k \subset \mathbb{R}^n$ and $\dim V_i = r_i \ \forall i\}$

Can represent an element of $Fl_{R;n}$ by an $r_k \times n$ matrix s.t.
span of top r_i rows $\rightsquigarrow V_i$.

Special cases: ① If $R = [n]$: complete flag var Fl_n
② If $R = \{r\}$: Grassmannian $Gr_{r;n}$

Have projection $\pi : Fl_n \to Fl_{R;n}$ obtained by forgetting some V_i's

Ex: $A = \begin{pmatrix} 1 & a+c & bc \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ represent element of Fl_3 where V_i = span of top i rows

For $I \subset [n]$, Plucker coord $p_I(A) :=$ det of submatrix in columns I and rows $1, 2, \dots, |I|$.

Two notions of positivity. Fix $R = \{r_1 < \dots < r_k\} \subset [n]$

Plucker-positivity: We say (a matrix representing) an element $(V_1 \subset \dots \subset V_k) \in Fl_{R;n}$ is Plucker-positive / nonnegative iff all Plucker coords p_I for $|I| = r_i$ are pos/nonnegative.

Lusztig-positivity: $GL_n^{>0} = \{n \times n$ matrices s.t. all square submatrices have pos. det$\}$

$Fl_n^{>0}$ = image of $GL_n^{>0}$ inside GL_n/B ← Borel subgp

$Fl_n^{\geq 0}$ = closure $\overline{Fl_n^{>0}}$

For R as above, $Fl_{R;n}^{>0}$ = projection $\pi(Fl_n^{>0})$

$Fl_{R;n}^{\geq 0}$ = " " $Fl_n^{\geq 0}$

Thm (Bloch–Karp): The two notions of positivity for $Fl_{R;n}$ coincide iff R is a set of consecutive integers.
(complete flag case indep proved by Boretsky)

(*) We now restrict attention to case where $R = \{a, a+1, \dots, b\}$

Matroids, Flag matroids, Flag positroids

Def: Given subset $S \subseteq [n]$, let $e_S = \sum_{i \in S} e_i \in \mathbb{R}^n$

Given collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) =$ conv hull $\{e_B : B \in \mathcal{B}\}$ in $\mathbb{R}^n$
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$, then say $\mathcal{B}$ is set of bases of matroid $M_{\mathcal{B}}$ and that $P(\mathcal{B})$ is a matroid polytope. (Gelfand-Goresky-MacPherson-Serganova)

If $\exists$ $d \times n$ matrix A s.t. $p_I(A) \neq 0$ iff $I \in \mathcal{B}$, say A realizes M.

Def: Let $R = (r_1 < \dots < r_k) \subset [n]$.
A flag matroid of ranks R on $[n]$ is sequence $\underline{M} = (M_1 \dots M_k)$ of matroids of ranks R on $[n]$ s.t. all vertices of the Minkowski sum $P(\underline{M}) = P(M_1) + \dots + P(M_k)$ are equidistant from the origin.
$P(\underline{M})$ called flag matroid polytope.

If $(M_1, \dots, M_k)$ has a realiz by a real $r_k \times n$ matrix A s.t. for $1 \leq i \leq k$ the top $r_i \times n$ submatrix of A has its max'l minors positive, say that $(M_1, \dots, M_k)$ is a flag positroid & $P(M)$ a flag pos. polytope.
[Note: defining flag positroid s.t. it's automatically realiz]

Torus $(\mathbb{C}^*)^n$ acts on $Gr_{r;n}$ by scaling columns of matrix.

Note: If matroid M is realiz by matrix A then matroid polytope $P(M)$ = moment map image of closure of torus orbit of A.

(Kodama–W)

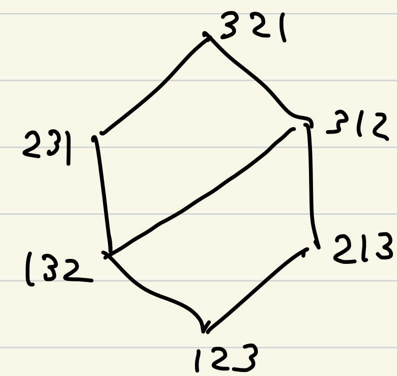
Ex: If $R=[n]$, the flag pos. polytopes are Bruhat interval polytopes

These have form

$$P_{u,v} := \mathrm{conv}\{(x(1),\dots,x(n)) \mid u \leq x \leq v\} \subset \mathbb{R}^n$$

strong Bruhat order

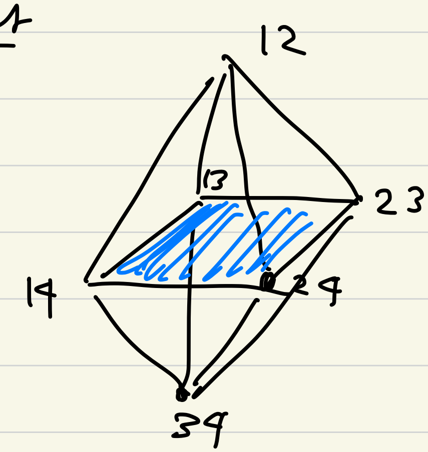
$u \leq v$ in S_n



Ex: If $R=\{r\}$, the flag pos. polytopes are positroid polytopes

connection to noncrossing partitions (Ardila – Rincon – W)

Eg hypersimplex $\longrightarrow$



Q: when & how can we subdivide matroid polytope into smaller matroid polytopes?

Same Q for flag matroid polytopes & pos. analogues...

(Kapranov, Lafforgue, Speyer... matroid subdivisions & connection to trop Grassmannian)

Thm (Lukowski–Parisi–W, Speyer–W, Arkani-Hamed–Lam–Spradlin)

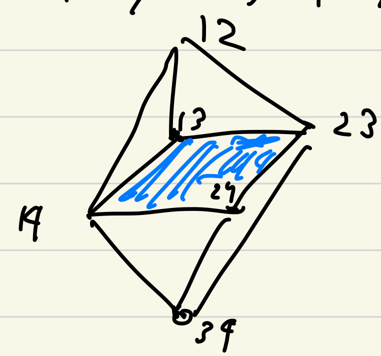
Let $\mu = (\mu_I : I \in \binom{[n]}{r}) \in \mathbb{R}^{\binom{n}{r}}$. TFAE:

- μ lies in pos. trop. Grassmannian $\mathrm{Tr}\,\mathrm{Gr}_{r,n}^{>0}$ (closure of coord-wise valuation of $\mathrm{Gr}_{r,n}^{>0}$ over Puiseux series)
- μ obeys the pos. trop. 3-term Plucker relation: for $i<j<k<l$ and S disjoint from them, $|S| = r-2$, $\mu_{Sik} + \mu_{Sjl} = \min(\mu_{Sij} + \mu_{Skl}, \mu_{Sil} + \mu_{Sjk})$
- Every face in coherent (regular) subdiv of hypersimplex $\Delta_{r,n} = \mathrm{conv}\{e_I \mid I \in \binom{[n]}{r}\}$ induced by μ is a positroid polytope.

Coherent subdiv obtained by "lift" each vertex e_I of $\Delta_{r,n}$ to "ht" μ_I: then proj lower facets of $\mathrm{conv}\{(e_I, \mu_I)\}$ back to $\mathbb{R}^n$

Ex: $n=4$, $r=2$. Consider $(\mu_I : I \in \binom{[4]}{2})$ s.t. $\mu_{13} + \mu_{24} = \mu_{23} + \mu_{14} < \mu_{12} + \mu_{34}$.

Then get this subdiv $\rightsquigarrow$



Thm (Boretsky, Joswig-Loho-Luber-Olarte): Let $\mu = \{\mu_I : I \subsetneq [n]\} \in \mathbb{R}^{2^n}$. TFAE

- μ lies in pos. trop complete flag variety $\mathrm{Tr}\,Fl_n^{>0}$
- μ obeys pos. trop 3-term Plucker & incidence Plucker relations.

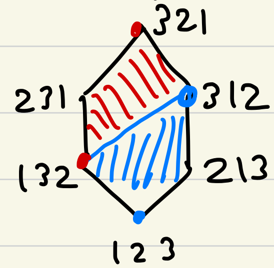
  Ex: $\mu_2 + \mu_{13} = \min(\mu_1 + \mu_{23}, \mu_3 + \mu_{12})$

Boretsky $\mathrm{Tr}\,Fl_n^{\geq 0}$

- Every face in coherent subdiv of permutohedron Perm_n induced by μ is a Bruhat interval polytope.

(Now eg. we lift vertex $(3,1,2,5,4)$ to ht $\mu_4 + \mu_{45} + \mu_{145} + \mu_{1345}$)

Ex: $n=3$. Consider $(\mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{13}, \mu_{23})$ s.t. $\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$



Give thm which generalizes these in 2 ways:

- extend to flag varieties $Fl_{R;n}$ where R is consec set of integers
- replace "positive" by "nonnegative" which allows us to look at subdiv of more general polytopes.

tropical hyperfield

<u>Thm (Boretsky–Eur–W):</u> Let R be seq of consec integers $(a, a+1, \dots, b)$.
Let $\mu = (\mu^a, \dots, \mu^b) \in \prod_{i=a}^{b} \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{i}}\right)$ "=" $\prod_{i=a}^{b} \left(\mathbb{R} \cup \{\infty\}\right)^{\binom{[n]}{i}}$. TFAE

- μ lies in nonneg trop. of $Fl_{R;n}$ $\;$ (= closure of coord-wise val of $Fl_{R;n}^{\geq 0}(\mathcal{R})$)
- μ satisfies all pos. trop 3-term Plucker & incidence Plucker rel
- Every face in coherent subdiv of flag matroid polytope $\underbrace{P(\underline{\mu}^a) + \dots + P(\underline{\mu}^b)}$ is a flag positroid polytope.

Applications

Cor (BEW): For flag matroid $\underline{M} = (M_a, M_{a+1} \ldots M_b)$ of consec ranks $a, a+1 \ldots b$, its flag matroid polytope $P(\underline{M})$ is a flag pos. polytope iff all its (≤ 2)-dim'l faces are.

Thm (Tsukerman-W): Every faces of a Bruhat interval polytope (BIP) is a BIP.

Cor (BEW): A complete flag matroid poly is BIP iff all 2-dim'l faces are.

Q: When does sequence of positroids of diff ranks have a realization by one matrix?

Cor (BEW): Suppose $(M_a, M_{a+1} \ldots M_b)$ is sequence of positroids of consec ranks. Then, when considered as sequence of pos. oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Example: $\mathrm{Tr}\, Fl_4^{>0} = \{(\mu_I : I \subsetneq [4])\} \subset \mathbb{R}^{15}$ "tropical Plucker vector"

Bossinger computed Grobner fan structure

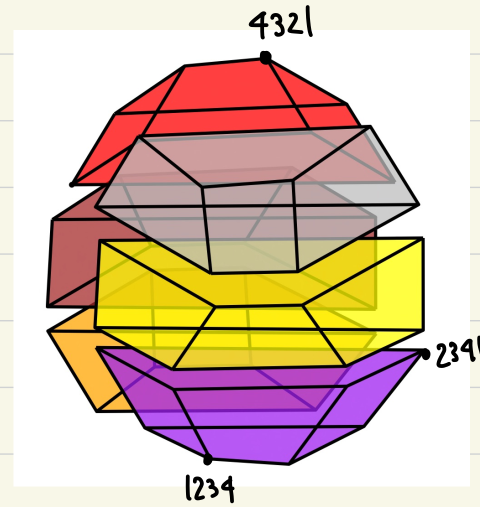
- 14 max'l cones, 9 rays, dual to 3D associahedron

Note: Fl_4 is cluster variety of finite type A_3.

Secondary fan structure. Got same fan.

Height function $(P_1, P_2, P_3, P_4; P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}; P_{123}, P_{124}, P_{134}, P_{234})$	Bruhat interval polytopes in subdivision	f-vector
$(15, -1, -7, -7; 4, -2, -2, -2, -2, 4; -7, -7, -1, 15)$	$P_{3214,4321}, P_{3124,4231}, P_{2314,3421}, P_{2134,3241}, P_{1324,2431}, P_{1234,2341}$	$(24, 46, 29, 6)$
$(15, 3, -9, -9; 4, -8, -8, -4, -4, 20; -1, -1, -1, 3)$	$P_{2413,4321}, P_{3124,4231}, P_{2314,4231}, P_{2134,3241}, P_{1324,2431}, P_{1234,2341}$	
$(15, -7, -1, -7; -2, 4, -2, -2, 4, -2; -7, -1, -7, 15)$	$P_{3142,4321}, P_{3124,4312}, P_{2143,3421}, P_{2134,3412}, P_{1243,2431}, P_{1234,2413}$	
$(-1, -1, -1, 3; 4, -8, -4, -8, -4, 20; 15, 3, -9, -9)$	$P_{2413,4321}, P_{1423,4231}, P_{1342,4231}, P_{1324,4213}, P_{1243,4132}, P_{1234,4123}$	
$(-7, -7, -1, 15; 4, -2, -2, -2, -2, 4; 15, -1, , -7, -7)$	$P_{1432,4321}, P_{1423,4312}, P_{1342,4231}, P_{1324,4213}, P_{1243,4132}, P_{1234,4123}$	
$(-1, -7, -7, 15; -2, -2, 4, 4, -2, -2; 15, -7, -7, -1)$	$P_{3142,4321}, P_{2143,4312}, P_{2134,4213}, P_{1342,3421}, P_{1243,3412}, P_{1234,2413}$	
$(-9, -9, 3, 15; 20, -4, -8, -4, -8, 4; 3, -1, -1, -1)$	$P_{1432,4321}, P_{1423,4312}, P_{1342,4231}, P_{1324,4213}, P_{1324,4132}, P_{1234,3142}$	
$(11, -7, -7, 3; -6, -6, 4, 4, 2, 2; 11, -7, -7, 3)$	$P_{3142,4321}, P_{2143,4312}, P_{2134,4213}, P_{2143,3421}, P_{1243,2431}, P_{1234,2413}$	
$(3, 3, -3, -3; 20, -10, -10, -10, -10, 20; -3, -3, 3, 3)$	$P_{2413,4321}, P_{3124,4231}, P_{2314,4231}, P_{1324,2431}, P_{1324,3241}, P_{1234,3142}$	
$(3, -1, -1, -1; 20, -4, -4, -8, -8, 4; -9, -9, 3, 15)$	$P_{3214,4321}, P_{3124,4231}, P_{2314,3421}, P_{1324,3241}, P_{1324,2431}, P_{1234,3142}$	
$(-3, -3, 3, 3; 20, -10, -10, -10, -10, 20; 3, 3, -3, -3)$	$P_{2413,4321}, P_{1423,4231}, P_{1342,4231}, P_{1324,4132}, P_{1324,4213}, P_{1234,3142}$	
$(3, -7, -7, 11; 2, 2, 4, 4, -6, -6; 3, -7, -7, 11)$	$P_{3142,4321}, P_{3124,4312}, P_{1342,3421}, P_{2134,3412}, P_{1243,3412}, P_{1234,2413}$	
$(11, -1, -7, -3; -2, -8, -4, -4, 0, 18; 11, -1, -7, -3)$	$P_{2413,4321}, P_{2143,4231}, P_{2134,4213}, P_{1243,2431}, P_{1234,2413}$	$(24, 45, 27, 5)$
$(-3, -7, -1, 11; 18, 0, -4, -4, -8, -2; -3, -7, -1, 11)$	$P_{3142,4321}, P_{3124,4312}, P_{1342,3421}$ $P_{1324,3412}, P_{1234,3142}$	

TABLE 1. Table documenting the 14 finest coherent subdivisions of Perm_4 into Bruhat interval polytopes. There are two possible f-vectors, each of which can be realized in multiple ways.



Thank you!

(w/ Jon Boretsky & Chris Eur) Polyhedral & trop geom of positroids arXiv: 2208.09131