Stochastic wave equation with Lévy white noise

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Outline

1. Introduction

Wave equation

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(t,x) = c^2 \Delta w(t,x) & t > 0, x \in \mathbb{R}^d \quad (d \leq 3) \\ w(0,x) = u_0(x), \quad \dfrac{\partial w}{\partial t}(0,x) = v_0(x) & x \in \mathbb{R}^d \end{cases}$$

Description of waves (sound, water, seismic, light,...)

$w(t,x)$ is the displacement of point x at time t; c^2 propagation speed

$d = 1$: string; $d = 2$: membrane; $d = 3$: elastic solid

Solution ($c = 1$)

$$w(t,x) = \int_{\mathbb{R}^d} G_t(x-y) v_0(y) dy + \frac{\partial}{\partial t} \int_{\mathbb{R}^d} G_t(x-y) u_0(y) dy$$

Fundamental solution G

$$G_t(x) = \begin{cases} \frac{1}{2} 1_{\{|x|<t\}} & \text{if } d = 1, \\ \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}} & \text{if } d = 2, \\ \frac{1}{4\pi\sigma_t} & \text{if } d = 3 \end{cases}$$

$|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$
σ_t is the surface measure on $\{x \in \mathbb{R}^3; |x| = t\}$

History

$d = 1$: D'Alembert formula (1746)
$d = 3$: Kirkhhoff formula (Euler, 1756)
$d = 2$: Poisson formula; Hadamard (1923, method of descent)

Inhomogeneous wave equation (f is smooth)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + f(t,x) & t>0, x \in \mathbb{R}^d \quad (d \leq 3) \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x) & x \in \mathbb{R}^d \end{cases}$$

f is a "source function" (describes the effect of the source of the waves on the medium which carries them)

Solution

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y) dyds$$

Justification: Let $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$. Then $\mathcal{L}w = 0$

$$\mathcal{L}G = \delta \quad \text{and} \quad \mathcal{L}(f * G) = f * (\mathcal{L}G) = f * \delta = f$$

$*$ denotes space-time convolution

Stochastic wave equation with additive noise

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(t,x) & t>0, x \in \mathbb{R}^d \quad (d \leq 2) \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x) & x \in \mathbb{R}^d \end{cases} \tag{1}$$

Definition

A (random-field) **solution** of (1) satisfies

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) W(ds,dy), \tag{2}$$

RHS contains a stochastic integral with respect to the noise W

Stochastic heat equation with additive noise

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \dot{W}(t,x) & t>0, x \in \mathbb{R}^d \quad (d \geq 1) \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases} \quad (3)$$

Definition

A (random-field) **solution** of (3) satisfies

$$u(t,x) = (g_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} g_{t-s}(x-y) W(ds,dy), \quad (4)$$

where $(g_t)_{t\geq 0}$ is the heat semigroup:

$$g_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right)$$

Space-time Gaussian White Noise

$\{W(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is a Gaussian process with

$$\mathbb{E}[W(A)] = 0 \quad \text{and} \quad \mathbb{E}[W(A)W(B)] = |A \cap B|$$

$|A|$ is Lebesgue measure of A.
Define $W(1_A) := W(A)$.
By linearity, we extend W to the set $\mathcal{E}$ simple functions.
The map $\mathcal{E} \ni \varphi \mapsto W(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$:

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x)\psi(t,x)dxdt$$

Existence of solution: wave equation

Solution to the wave equation (1) exists iff the stochastic integral on the RHS of (2) is well-defined, i.e.

$$\int_0^{\infty} \int_{\mathbb{R}^d} G_{t-s}^2(x-y)dyds < \infty.$$

This forces $d = 1$.

Existence of solution: heat equation

Solution to the wave equation (3) exists iff the stochastic integral on the RHS of (4) is well-defined, i.e.

$$\int_0^{\infty} \int_{\mathbb{R}^d} g_{t-s}^2(x-y)dyds = c_d \int_0^t (t-s)^{-d/2} ds < \infty.$$

This forces $d = 1$.

2. SPDEs with Gaussian white noise ($d = 1$)

Wave equation with multiplicative nose

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0, x \in \mathbb{R} \tag{5}$$

Initial conditions $u_0 = v_0 = 0$; σ is a Lipschitz function

Definition

A (random-field) **solution** of (5) satisfies

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)$$

Heat equation

In the case of heat equation, we replace G_t by g_t.

Existence of solution (using Picard's iterations)

$$u_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u_n(s,y))W(ds,dy) \qquad u_0(t,x)=0$$

Main calculation: let $H_n(t) = \sup_{x\in\mathbb{R}} \mathbb{E}|u_n(t,x) - u_{n-1}(t,x)|^2$

$$\begin{aligned}
&\mathbb{E}|u_{n+1}(t,x) - u_n(t,x)|^2 \\
&= \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y)|\sigma(u_n(s,y)) - \sigma(u_{n-1}(s,y))|^2 dyds\right] \\
&\leq C_\sigma^2 \int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}|u_n(s,y) - u_{n-1}(s,y)|^2 \left(\int_{\mathbb{R}} G_{t-s}^2(x-y)dy\right) ds \\
&= C_\sigma^2 \int_0^t H_n(s) J(t-s) ds \quad \text{with} \quad J(t-s) = \frac{1}{2}(t-s)
\end{aligned}$$

Heat equation: $J(t-s) = \int_{\mathbb{R}} g_{t-s}^2(x-y)dy = c(t-s)^{-1/2}$.

Extension of Gronwall Lemma (Dalang, 1999)

Let $f_n : [0,T] \to \mathbb{R}_+$. If $f_0(t) \leq M$ for any $t \in [0,T]$ and

$$f_{n+1}(t) \leq \int_0^t f_n(s) J(t-s) ds \quad \text{for any } t \in [0,T]$$

where $J \geq 0$ is integrable on $[0,T]$, then

$$\sum_{n \geq 1} \sup_{t \in [0,T]} f_n(t)^{1/p} < \infty \quad \text{for any } p > 0.$$

Existence of solution

$\{u_n(t,x)\}_{n \geq 0}$ is a Cauchy sequence in $L^2(\Omega)$, uniformly in $[0,T] \times \mathbb{R}$:

$$\sum_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \|u_n(t,x) - u_{n-1}(t,x)\|_{L^2(\Omega)} = \sum_{n \geq 1} \sup_{t \in [0,T]} H_n(t)^{1/2} < \infty.$$

Its limit $u(t,x)$ is the unique solution to equation (5).

3. Lévy white noise

Space-time Lévy white noise $L = \{L(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$

$$L(A) = b|A| + \int_{A\times\{|z|\leq 1\}} z\widetilde{J}(dt,dx,dz) + \int_{A\times\{|z|>1\}} zJ(dt,dx,dz)$$

J is Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ of intensity $dtdx\nu(dz)$
$\widetilde{J}$ is compensated version of J and ν is a Lévy measure on $\mathbb{R}$, i.e.

$$\int_{\mathbb{R}} (z^2 \wedge 1)\nu(dz) < \infty \quad \text{and} \quad \nu(\{0\}) = 0$$

Infinitely-divisible independently-scattered random measure

$$\mathbb{E}[e^{-iuL(A)}] = \exp\left\{|A|\left[iub + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz1_{|z|\leq 1})\nu(dz)\right]\right\}$$

(Rajput and Rosinski, 1989)

Example 1: finite variance case, $v := \int_{\mathbb{R}} z^2 \nu(dz) < \infty$

If $b = -\int_{\{|z|>1\}} z\nu(dz)$, then $L(A) = \int_{A\times\mathbb{R}} z\widetilde{J}(dt,dx,dz)$
Stochastic integral:

$$\int_0^\infty \int_{\mathbb{R}^d} X(t,x)L(dt,dx) = \int_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}} X(t,x)z\widetilde{J}(dt,dx,dz)$$

Isometry property:

$$\mathbb{E}\left|\int_0^\infty \int_{\mathbb{R}^d} X(t,x)L(dt,dx)\right|^2 = \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} |X(t,x)z|^2 \nu(dz)dxdt\right]$$
$$= v \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}|X(t,x)|^2 dxdt.$$

Example 2: α-stable random measure (Samorodnitsky-Taqqu, 1994)

$$\nu(dz) = c_1 z^{-\alpha-1} 1_{(0,\infty)}(z) + c_2(-z)^{-\alpha-1} 1_{(-\infty,0)}(z)$$

Integral with respect to L

Let $L(1_A) = L(A)$ and extend by linearity to simple functions. Set

$$L(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t,x) L(dt,dx) \stackrel{P}{:=} \lim_{n\to\infty} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi_n(t,x) L(dt,dx)$$

if the limit exists for some simple functions $(\varphi_n)_{n\geq}$ with $\varphi_n \to \varphi$ a.e.
$S(L)$ is set of functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ s.t. $1_{[0,t]}\varphi$ is integrable w.r.t. L

SPDEs with Lévy noise ($\mathcal{L}$ is wave operator or heat operator)

$$\mathcal{L}u(t,x) = \sigma(u(t,x))\dot{L}(t,x), \quad t > 0, x \in \mathbb{R} \tag{6}$$

Picard's iterations

$$u_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u_n(s,y))L(ds,dy) \qquad u_0(t,x) = 0$$

(for wave equation). For heat equation, replace G_t by g_t.

Existence of solution: finite variance case ($d = 1$)

$H_n(t) = \sup_{x\in\mathbb{R}} \mathbb{E}|u_n(t,x) - u_{n-1}(t,x)|^2$

$$H_{n+1}(t) \leq \color{red}{v}\color{black}{} C_\sigma^2 \int_0^t H_n(s)J(t-s)ds$$

As for W, $u_n(t,x) \to u(t,x)$ in $L^2(\Omega)$; u is the unique solution of (6).

4. Heat equation with general Lévy noise

Heat equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{L}(t,x) \qquad t>0, x \in \mathbb{R}^d \qquad (7)$$

Initial condition $u_0 = 0$

Theorem 1. (Saint Loubert Bié, 1998)

If the measure ν satisfies

$$(S) \quad \int_{\mathbb{R}} |z|^p \nu(dz) < \infty \quad \text{for some} \quad p \in [1,2]$$

and $p < 1 + \frac{2}{d}$, then equation (7) has a unique solution u and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|u(t,x)|^p < \infty.$$

Remarks

- Condition $p < 1 + \frac{2}{d}$ is equivalent to

$$\frac{d}{2}(1-p) + 1 > 0,$$

  which comes from the requirement:

$$\int_0^t \int_{\mathbb{R}^d} g_{t-s}^p(x-y)dyds = c_d \int_0^t (t-s)^{\frac{d}{2}(1-p)}ds < \infty$$

  Recall that $(g_t)_{t \geq 0}$ is the heat semigroup:

$$g_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right)$$

  $|\cdot|$ is the Euclidean norm

- Condition (S) excludes the α-stable Lévy noise.

New idea (Chong, 2017)

Assumption A

There exist $0 < q \leq p$ such that

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < \infty \quad \text{and} \quad \int_{\{|z|>1\}} |z|^q \nu(dz) < \infty$$

If $p < 1$, assume that $b = \int_{|z|\leq 1} z\nu(dz)$.

Truncated noise

$$L_N(A) = b|A| + \int_{A\times\{|z|\leq 1\}} z\widetilde{J}(dt, dx, dz) + \int_{A\times\{1<|z|\leq Nh(x)\}} zJ(dt, dx, dz)$$

Truncation function: $h(x) = 1 + |x|^\eta$ with $\eta > 0$

Remark If $p < 1$, $L_N(A) = \int_{A\times\{|z|\leq Nh(x)\}} zJ(dt, dx, dz)$

Lemma 1. (Chong, 2017)

Suppose Assumption A holds. If $p < 1$, then for any $t \in [0,T], x \in \mathbb{R}^d$,

$$\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} g_{t-s}(x-y)X(s,y)L_N(ds,dy)\right|^p \leq$$

$$C_T \int_0^t \int_{\mathbb{R}^d} g_{t-s}^p(x-y)\mathbb{E}|X(s,y)|^p h(y)^{p-q} dyds$$

If $p \geq 1$, for any $t \in [0,T], x \in \mathbb{R}^d$,

$$\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} g_{t-s}(x-y)X(s,y)L_N(ds,dy)\right|^p \leq$$

$$C_T \int_0^t \int_{\mathbb{R}^d} (g_{t-s}^p(x-y) + g_{t-s}(x-y))\mathbb{E}|X(s,y)|^p h(y)^{p-q} dyds$$

Remark This lemma holds also for G_t (wave equation).

Theorem 2. (Chong, 2017)

Suppose Assumption A holds,

$$\frac{p}{2+2/d-p} < q \leq p < 1 + \frac{2}{d} \quad \text{and} \quad \frac{d}{q} < \eta < \frac{2-d(p-1)}{p-q}.$$

a) Equation (7) **with L replaced by L_N** has a unique solution u_N and

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}|u_N(t,x)|^p < \infty \quad \forall T>0, \forall R>0.$$

Moreover, $u_N(t,x) = u_{N+1}(t,x)$ if $t \leq \tau_N$, where

$$\tau_N = \inf\{t>0; J([0,t]\times\{(x,z);|z|>Nh(x)\})>0\} \uparrow \infty.$$

b) Define $u(t,x)=u_N(t,x)$ if $t\leq \tau_N$. Then u is a solution of (7) and

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\Big[|u(t,x)|^p 1_{\{t\leq \tau_N\}}\Big] < \infty.$$

Sketch of the proof (part (a))

- Picard interations (for fixed N):

$$u_N^{(n+1)}(t,x)=\int_0^t\int_{\mathbb{R}^d} g_{t-s}(x-y)\sigma(u_N^{(n)}(s,y))L_N(ds,dy), \quad n\geq 0$$

- Lemma 1 ($p<1$) and Hölder's inequality: $T_n(t)=\{t_1<\ldots<t_n<t\}$

$$\mathbb{E}|(u_N^{(n+1)}-u_N^{(n)})(t,x)|^p \leq C^n \int_{T_n(t)}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n g_{t_{i+1}-t_i}^p(x_{i+1}-x_i)h(x_i)^{p-q}d\mathbf{x}d\mathbf{t}$$

$$\leq C^n \int_{T_n(t)}\prod_{i=1}^n \left(\int_{\mathbb{R}^{nd}}\prod_{j=1}^n g_{t_{j+1}-t_j}^p(x_j)h(x-\sum_{j=i}^n x_j)^{n(p-q)}d\mathbf{x}\right)^{\frac{1}{n}} d\mathbf{t}$$

- Key properties of the heat kernel:

$$g_t^p(x)=Ct^{\frac{d(1-p)}{2}}g_{t/p}(x) \quad \text{and} \quad g_t * g_s = g_{t+s}$$

Path properties of the solution

Fix interval $[0, T]$

Define

$$\tau_N = \inf\{t \in [0, T]; J([0, t] \times \{(x, z); |z| > Nh(x)\}) > 0\}$$

With probability 1, for N large enough, $\tau_N = \infty$ and $u(t, x) = u_N(t, x)$
It suffices to study the path properties of $\{u_N(t, \cdot)\}_{t\in[0,T]}$.

Justification

With probability 1, for any $N \in \mathbb{N}$, J has finitely many points in the set

$$S_T = [0, T] \times \{(x, z); |z| > Nh(x)\}$$

Hence, with probability 1, for N large enough, J has no points in S_T

Fractional Sobolev Space ($r \in \mathbb{R}$)

$H^r(\mathbb{R}^d)$ is the set of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ s.t. $\mathcal{F}f$ is a function and

$$\|f\|^2_{H^r(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|^2 (1+|\xi|^2)^r d\xi < \infty$$

Remark: $\delta_x \in H^r(\mathbb{R}^d)$ if and only if $r < -d/2$ (note: $\mathcal{F}\delta_x(\xi) = e^{-i\xi\cdot x}$)

Local Fractional Sobolev Space

$H^r_{\mathrm{loc}}(\mathbb{R}^d)$ is the set of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\varphi f \in H^r(\mathbb{R}^d) \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

We say that $f_n \to f$ in $H^r_{\mathrm{loc}}(\mathbb{R}^d)$ if

$$\|\varphi f_n - \varphi f\|_{H^r(\mathbb{R}^d)} \to 0 \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

Basic properties of the heat kernel g

- For $t > 0$, $g_t \in \mathcal{S}(\mathbb{R}^d)$.
- For $t = 0$,

$$g_0 = \delta_0$$

since

$$g_0(x) = \lim_{t \to 0+} g_t(x) = \lim_{t \to 0+} \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)} = 0 \quad \text{if} \quad x \neq 0$$

$$g_0(0) = \lim_{t \to 0+} g_t(0) = \lim_{t \to 0+} \frac{1}{(2\pi t)^{d/2}} = \infty \quad \text{if} \quad x = 0$$

Theorem 3. (Chong, Dalang and Humeau, 2019)

Suppose Assumption A holds,

$$\frac{p}{2+2/d-p} < q \leq p < 1 + \frac{2}{d} \quad \text{and} \quad \frac{d}{q} < \eta < \frac{2-d(p-1)}{p-q}$$

Then the processes $\{u_N(t,\cdot)\}_{t\in[0,T]}$ and $\{u(t,\cdot)\}_{t\in[0,T]}$ have càdlàg modifications with values in $H^r_{\mathrm{loc}}(\mathbb{R}^d)$, for any $r < -d/2$.

Sketch of proof (case $p \geq 1$)

Recall the decomposition of the truncated noise L_N:

$$L_N(A) = b|A| + \int_{A\times\{|z|\leq 1\}} z\widetilde{J}(dt,dx,dz) + \int_{A\times\{1<|z|\leq Nh(x)\}} zJ(dt,dx,dz)$$
$$=: b|A| + L^M(A) + L^P_N(A)$$

$$
\begin{aligned}
u_N(t,x) = & \int_0^t \int_{|y|\leq 2A} g_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy) + \\
& \int_0^t \int_{|y|>2A} g_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy) + \\
& \int_0^t \int_{|y|\leq 2A} g_{t-s}(x-y)\sigma(u_N(s,y))L_N^P(ds,dy) + \\
& \int_0^t \int_{|y|>2A} g_{t-s}(x-y)\sigma(u_N(s,y))L_N^P(ds,dy) + \\
& b\int_0^t \int_{\mathbb{R}^d} g_{t-s}(x-y)\sigma(u_N(s,y))dyds \\
& =: u_N^{1,1}(t,x) + u_N^{1,2}(t,x) + u_N^{2,1}(t,x) + u_N^{2,2}(t,x) + u_N^{3}(t,x)
\end{aligned}
$$

The term $u_N^{2,1}(t,x)$

If J has points (T_i, X_i, Z_i) in $[0,T] \times \mathbb{R}^d \times \mathbb{R}$, then

$$u_N^{2,1}(t,x) = \sum_{i \geq 1} g_{t-T_i}(x - X_i)\sigma(u_N(T_i, X_i))Z_i 1_{\{T_i \leq t, |X_i| \leq 2A, 1<|Z_i| \leq Nh(X_i)\}}$$

Points (T_i, X_i, Z_i) are in a bounded set.
Therefore, the series above has **finitely many** terms.
So, it suffices to analyze one of these terms.

Fix $i \geq 1$. If $r < -d/2$, the map

$$[0,T] \ni t \mapsto F_i(t) := g_{t-T_i}(x - X_i) \in H^r(\mathbb{R}^d) \quad \text{is càdlàg}$$

Remark: F_i is smooth for $t > T_i$ and zero for $t < T_i$. But $F_i(T_i) = \delta_{X_i}$.

5. Wave equation with general Lévy noise

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{L}(t,x) \qquad t>0, x \in \mathbb{R}^d \quad (d \leq 2) \quad (8)$$

Initial conditions $u_0 = v_0 = 0$

Assumption A

There exist $0 < q \leq p$ such that

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < \infty \quad \text{and} \quad \int_{\{|z|>1\}} |z|^q \nu(dz) < \infty$$

If $p < 1$, assume that $b = \int_{|z|\leq 1} z\nu(dz)$.

Theorem 3. (B. 2021)

Suppose Assumption A holds, and $p < 2$ if $d = 2$. (If $d = 1$, we can take any $p \geq 2$.) Take $\eta > d/q$.
a) Equation (8) with L **replaced by** L_N has a unique solution u_N and

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}|u_N(t,x)|^p < \infty \quad \forall T > 0, \forall R > 0.$$

Moreover, $u_N(t,x) = u_{N+1}(t,x)$ if $t \leq \tau_N$, where τ_N is the same as in **Theorem 2** (for heat equation):

$$\tau_N = \inf\{t > 0; J([0,t] \times \{(x,z); |z| > Nh(x)\}) > 0\} \uparrow \infty.$$

b) Define $u(t,x) = u_N(t,x)$ if $t \leq \tau_N$. Then u is a solution of (8) and

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\big[|u(t,x)|^p 1_{\{t\leq\tau_N\}}\big] < \infty.$$

Sketch of proof (part (a), case $d = 2$)

- Picard iterations for fixed level N (as for heat equation)
- Basic properties of G:

$$\int_{\mathbb{R}^d} G_t^p(x)dx = c_p t^{2-p} \quad \text{for any} \quad p \in (0,2)$$

$$G_t^p(x) \leq (2\pi t)^{q-p} G_t^q(x) \quad \text{for any } p < q$$

- Convolutions of G (Bolaños-Guerrero, Nualart and Zheng, 2021):

for any $q \in (\frac{1}{2}, 1)$, $\delta \in [1, 1/q]$ and $p \in (0,1)$ with $p + 2q \leq 3$

$$\int_r^t (G_{t-s}^{2q} * G_{s-r}^{2q})^{\delta}(x)ds \leq C_q(t-r)^{1-\delta(2q-1)} G_{t-r}^{\delta(2q-1)}(x)$$

$$\int_r^t (G_{t-s}^{2q} * G_{s-r}^{p})(x)ds \leq C_{p,q}(t-r)^{3-p-2q} 1_{\{|x|<t-r\}}$$

Path properties of the solution

Basic properties of the wave kernel G

$$\mathcal{F}G_t(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} G_t(x) = \frac{\sin(t|\xi|)}{|\xi|}$$

- If $t > 0$, $G_t \in H^r(\mathbb{R}^d)$ for any $r < 1 - d/2$:

$$\int_{\mathbb{R}^d} |\mathcal{F}G_t(\xi)|^2 (1 + |\xi|^2)^r d\xi \leq C_t \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2}\right)^{1-r} d\xi$$

- If $t = 0$: (a) $G_0 = \frac{1}{2} 1_{\{0\}}$ if $d = 1$; (b) $G_0 = \delta_0$ if $d = 2$, since:

$$G_0(x) := \lim_{t \to 0+} G_t(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Theorem 4. (B. 2021)

Suppose tha Assumption A holds and $p < 2$ if $d = 2$. Take $\eta > d/q$. Let u_N be the solution to equation (8) with L replaced by L_N, and u be the solution of (8) constructed in Theorem 3.(b), but for the stopping times:

$$\tau_N = \inf\{t \in [0, T]; J([0, t] \times \{(x, z); |z| > Nh(x)\}) > 0\}$$

a) If $d = 1$, the processes $\{u_N(t, \cdot)\}_{t\in[0,T]}$ and $\{u(t, \cdot)\}_{t\in[0,T]}$ have càdlàg modifications with values in $H^r_{\mathrm{loc}}(\mathbb{R})$, for any $r < 1/4$.

b) If $d = 2$, the processes $\{u_N(t, \cdot)\}_{t\in[0,T]}$ and $\{u(t, \cdot)\}_{t\in[0,T]}$ have càdlàg modifications with values in $H^r_{\mathrm{loc}}(\mathbb{R})$, for any $r < -1$.

Sketch of proof

$$u_N(t,x) = \int_0^t \int_{|y|\leq 2A} G_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy)+$$
$$\int_0^t \int_{|y|>2A} G_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy)+$$
$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_N(s,y))L_N^P(ds,dy)+$$
$$b\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_N(s,y))dyds$$
$$=: u_N^{1,1}(t,x) + u_N^{1,2}(t,x) + u_N^2(t,x) + u_N^3(t,x)$$

Note: $u^{1,2}(t,\cdot)1_{\{|x|\leq A\}} = 0$ for any $A \geq T$ (G_t contains $1_{\{|x|<t\}}$)

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Thank you!