# STABLE CONES IN THE THIN ONE-PHASE FREE BOUNDARY PROBLEM

Xavier Ros-Oton

ICREA and Universitat de Barcelona

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- Also known as the Bernoulli problem, it appears in flame propagation, jet flows, ...
- Regularity theory ⇝ initiated by Alt-Caffarelli in the 1980s,

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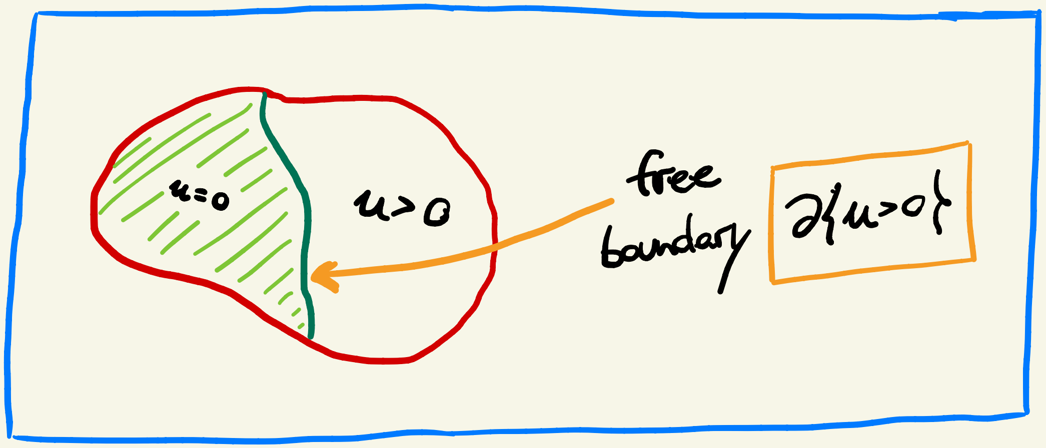

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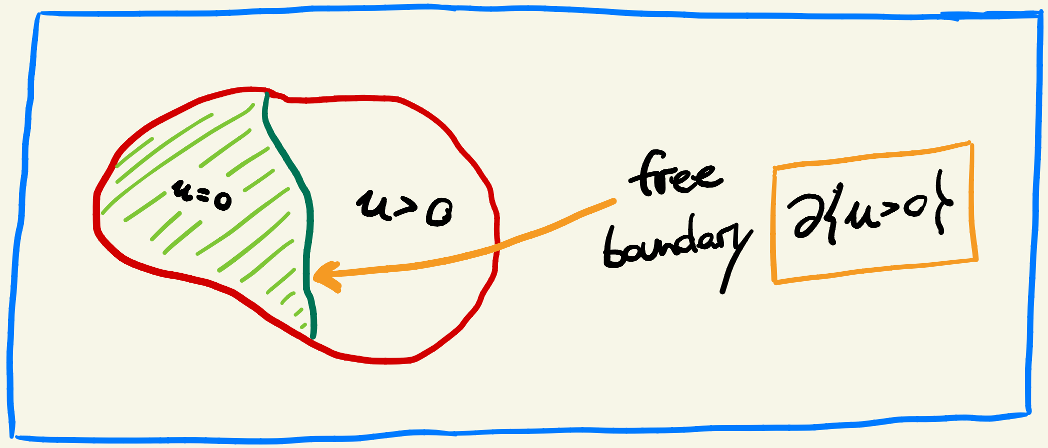


- Regularity of free boundaries: strong analogy with MINIMAL SURFACES.

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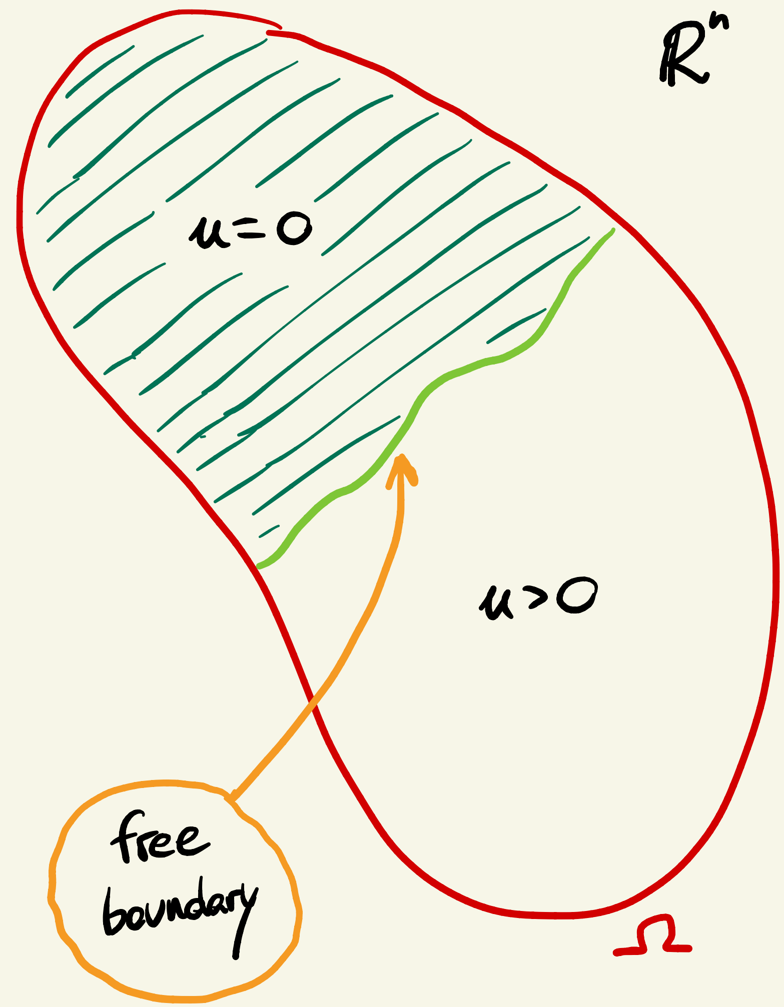

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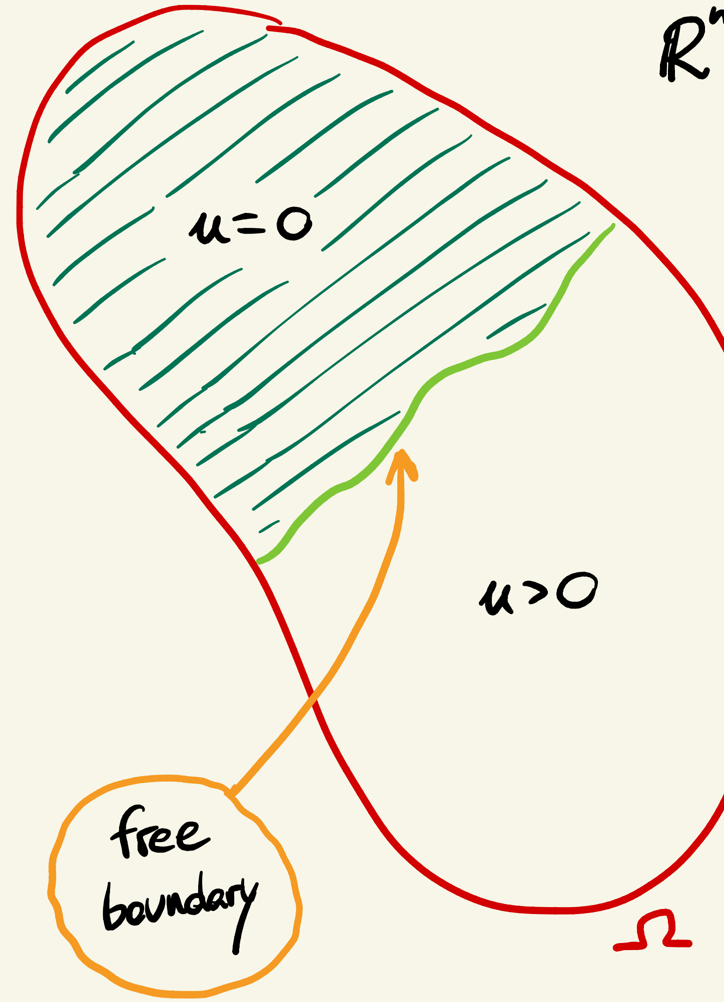


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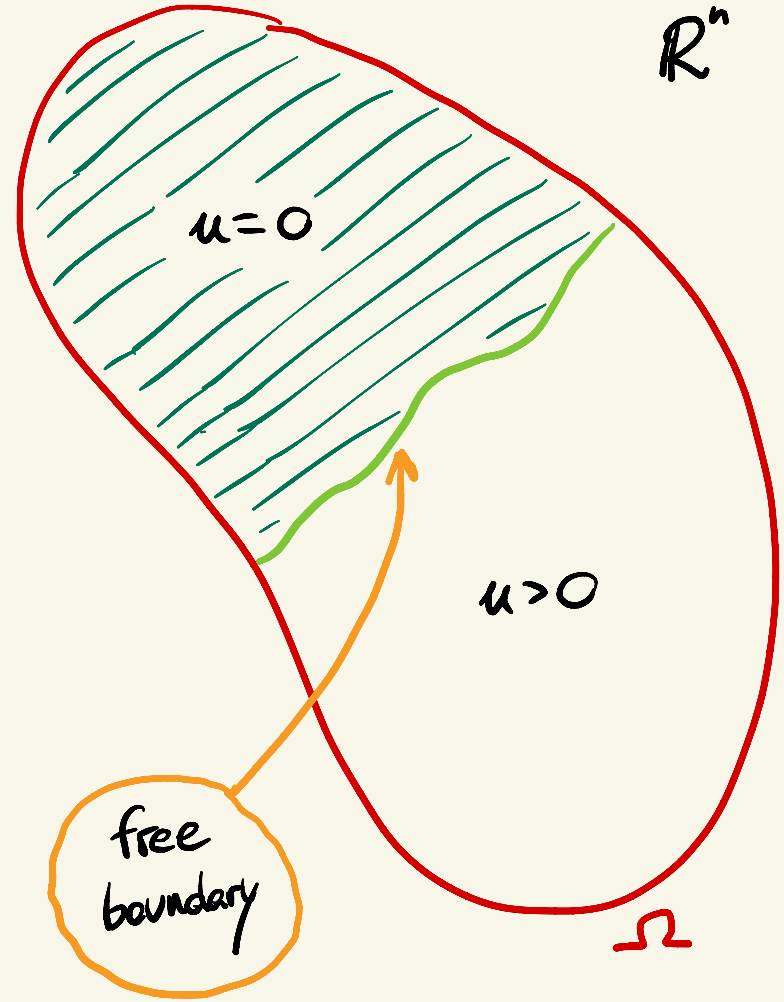


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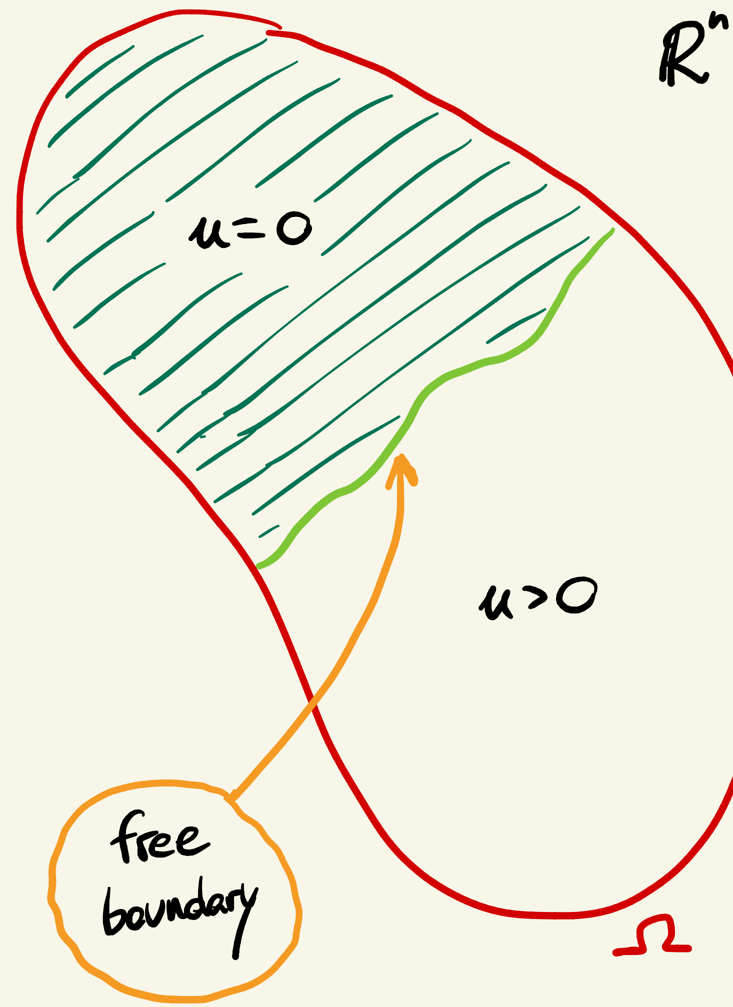


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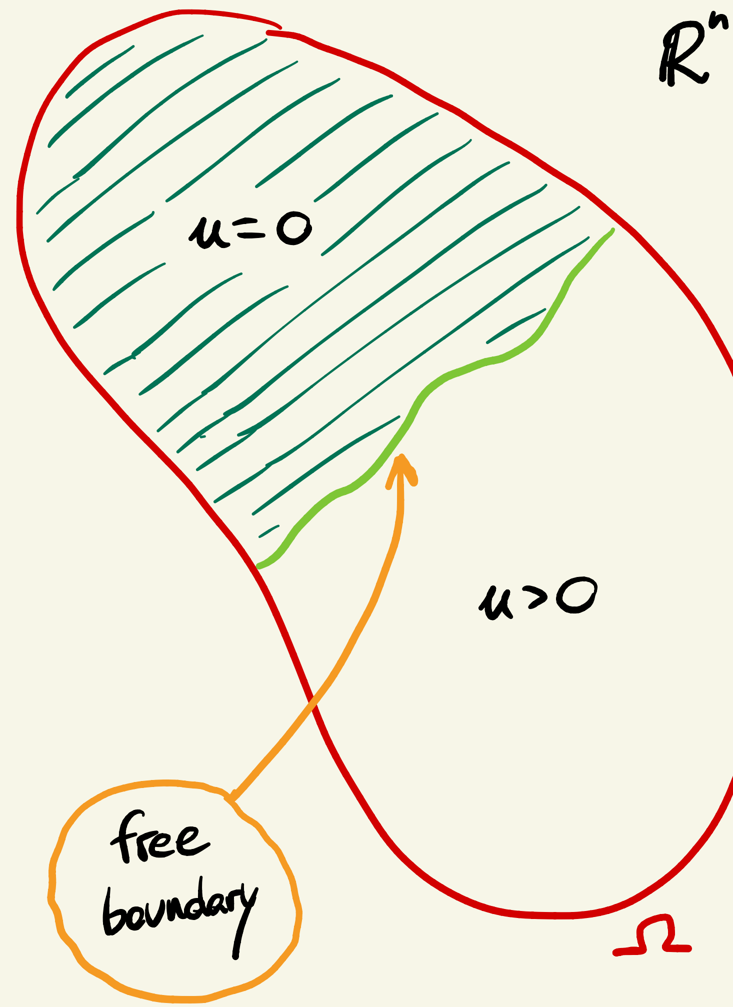


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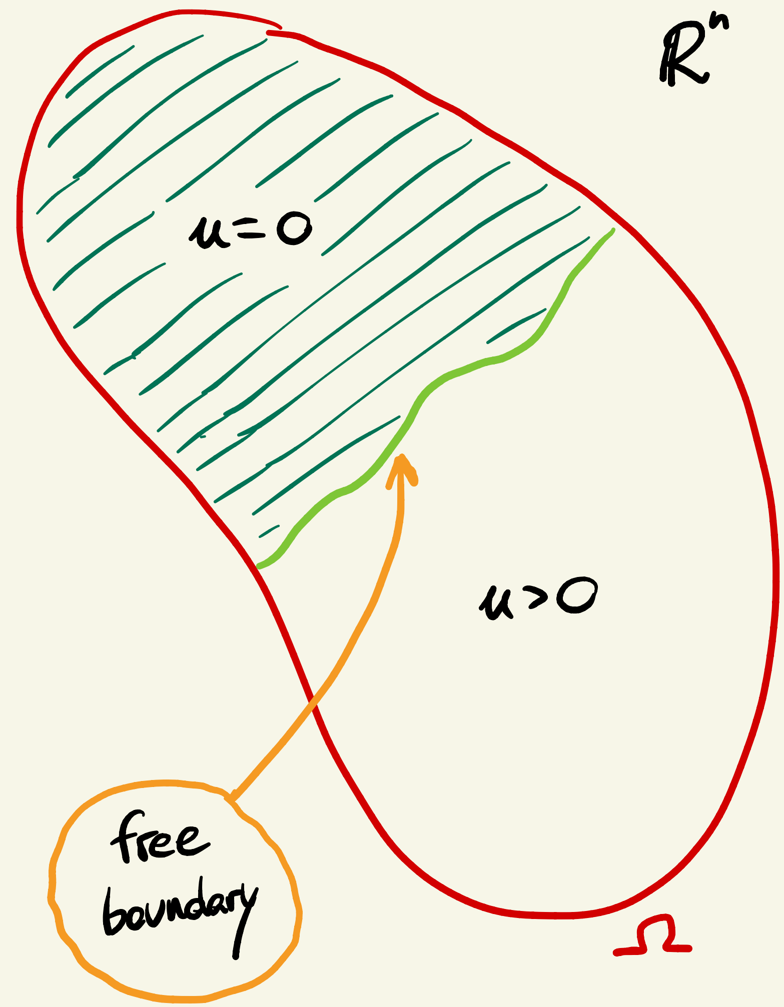


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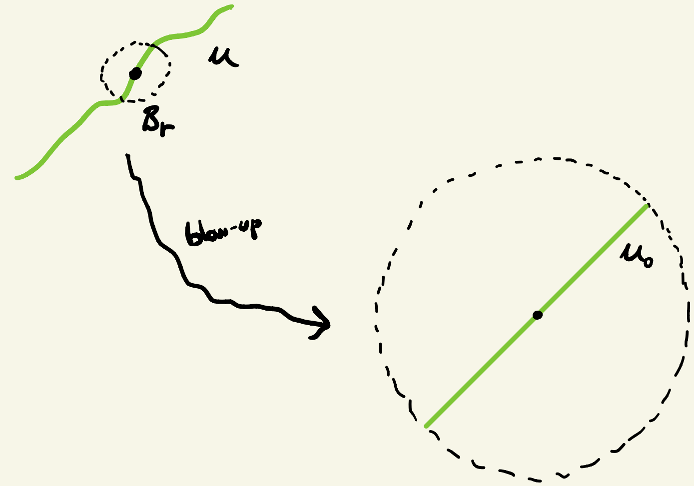
  - In dimensions $n \geq 7$, there may be singular points.
  - OPEN QUESTION: What about dimensions $n=5$ and $n=6$?

## Overall strategy:

- The problem is scale-invariant, so we can study blow-ups.



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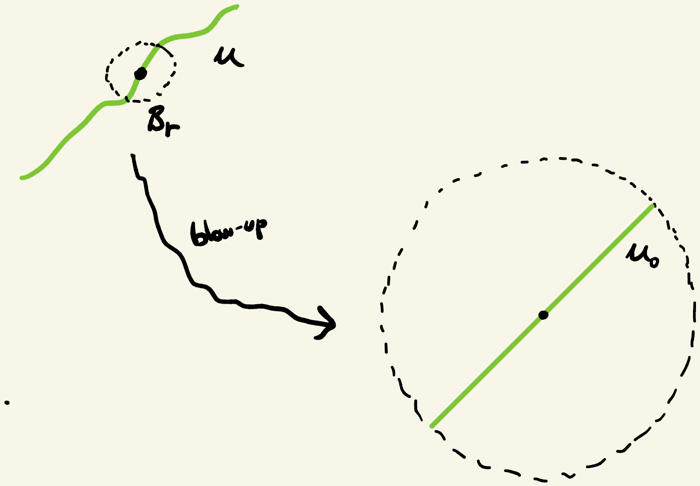


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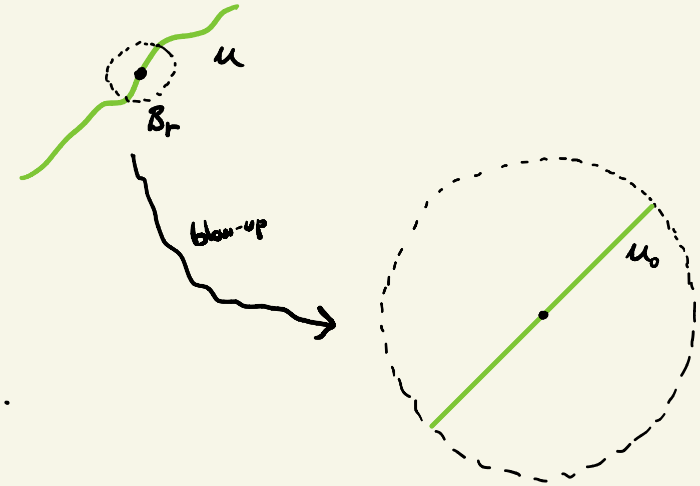


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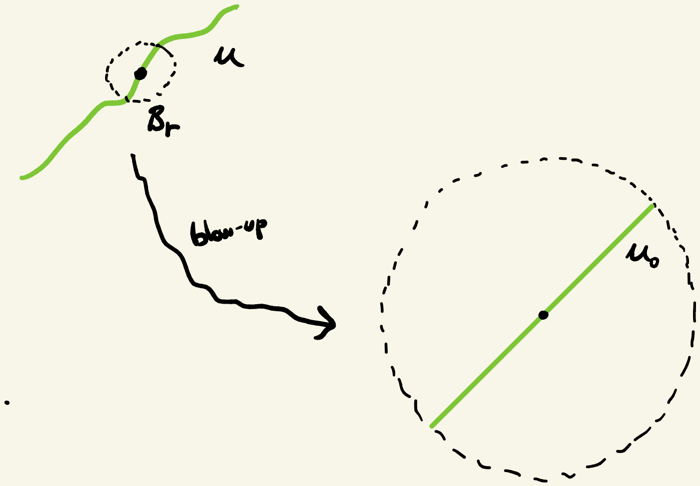


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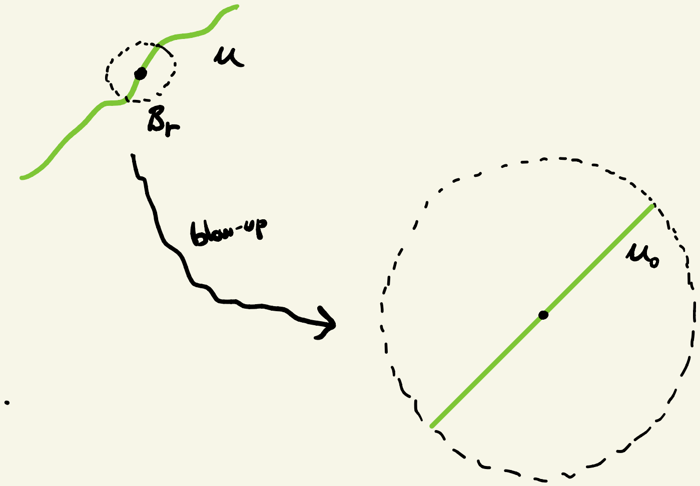


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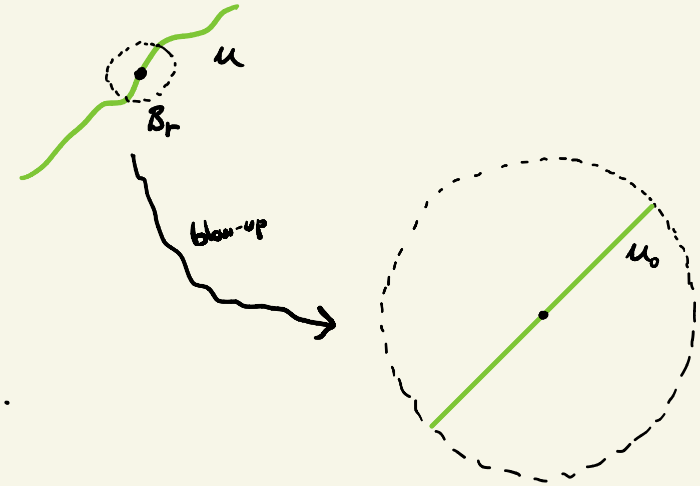


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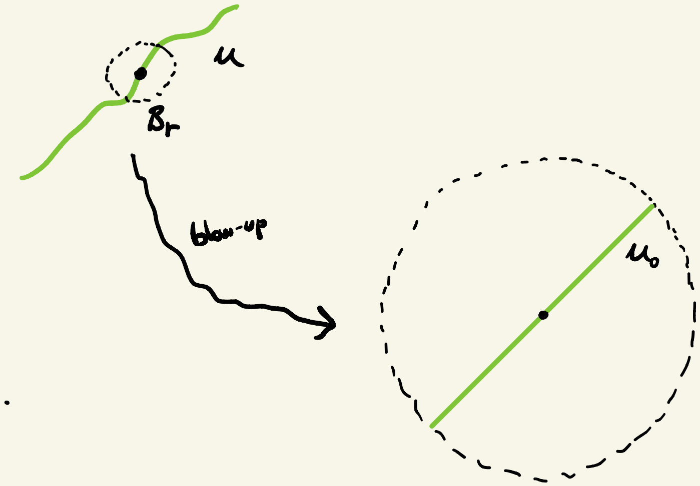


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This strategy is inspired by MINIMAL SURFACES.

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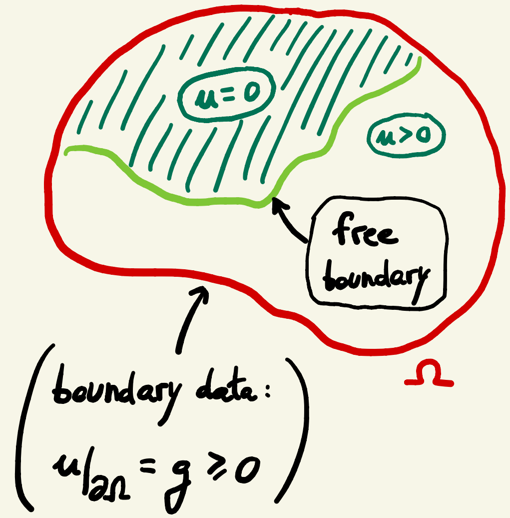

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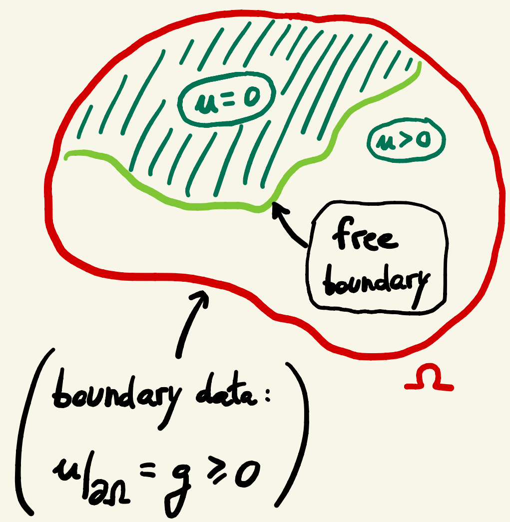


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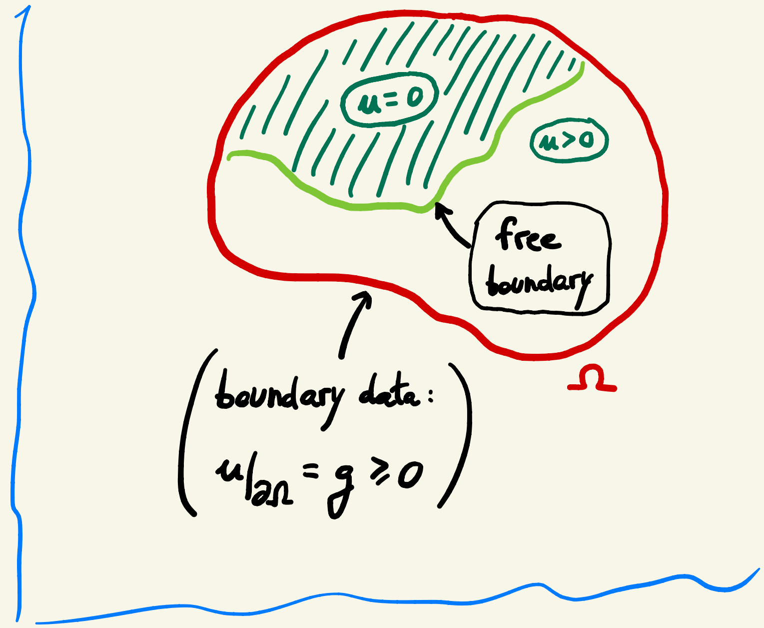


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- It is not difficult to show that $u$ then solves

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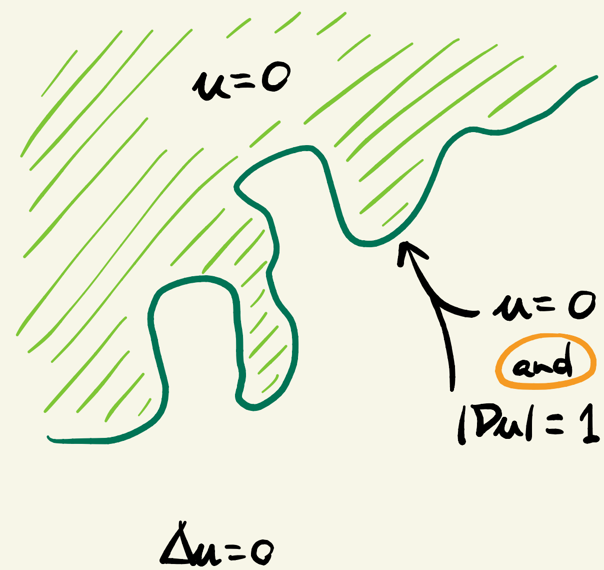

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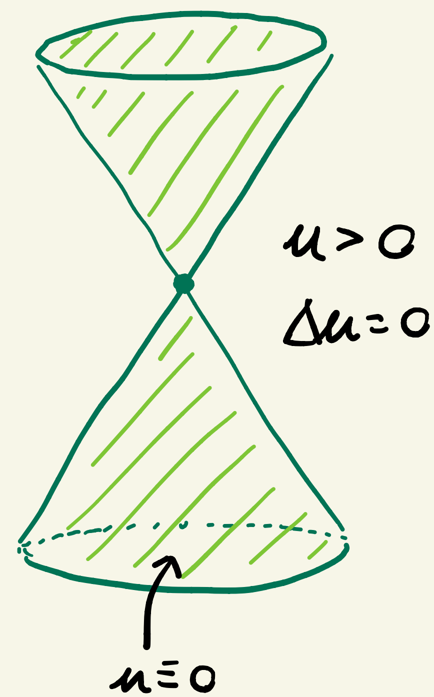

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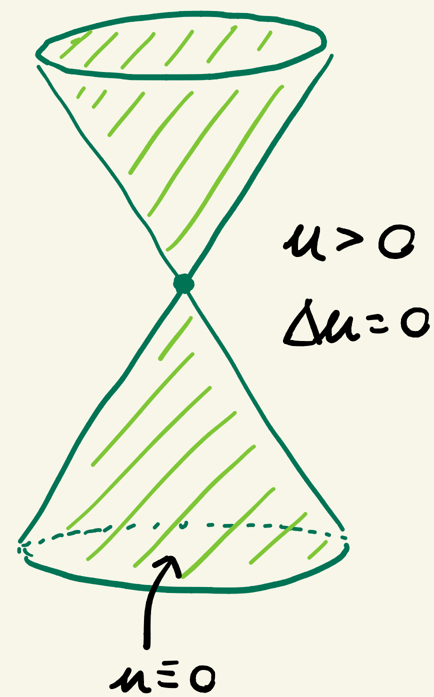

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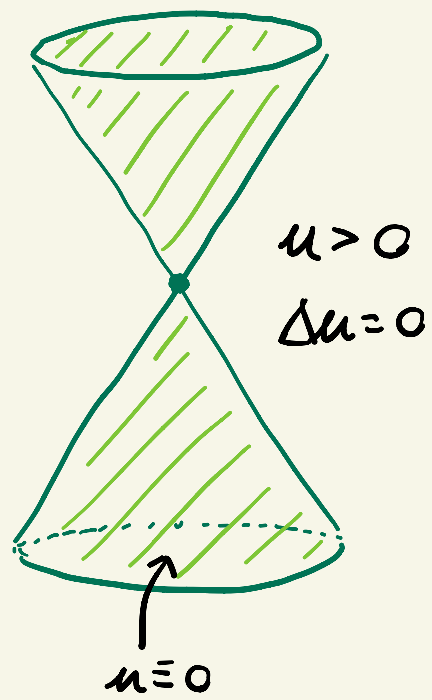
- In $\mathbb{R}^7$ it is a minimizer.



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$$\boxed{\int_{\{u>0\}} |\nabla \eta|^2 \geq \int_{\partial\{u>0\}} H \cdot \eta^2}$$

for all $\eta \in C_c^\infty(\Omega)$.

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where $H$ is the mean curvature of the free boundary.

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- This is crucial in order to
  - classify blow-ups in dimensions $n\leq 4$
  - understand axially symmetric solutions for $n\leq 6$

The thin (or fractional)

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**Thm.** Free boundaries are smooth outside a set of singular points $\Sigma$, with

$$\boxed{\dim(\Sigma) \leq n - k_s^*}$$

The value of $k_s^*$ is the lowest dimension in which nontrivial blow-ups appear.

[De Silva, Savin, JEMS '15]

[Engelstein, Kauranen, Prats, Sakellaris, Sire, CPAM '20]

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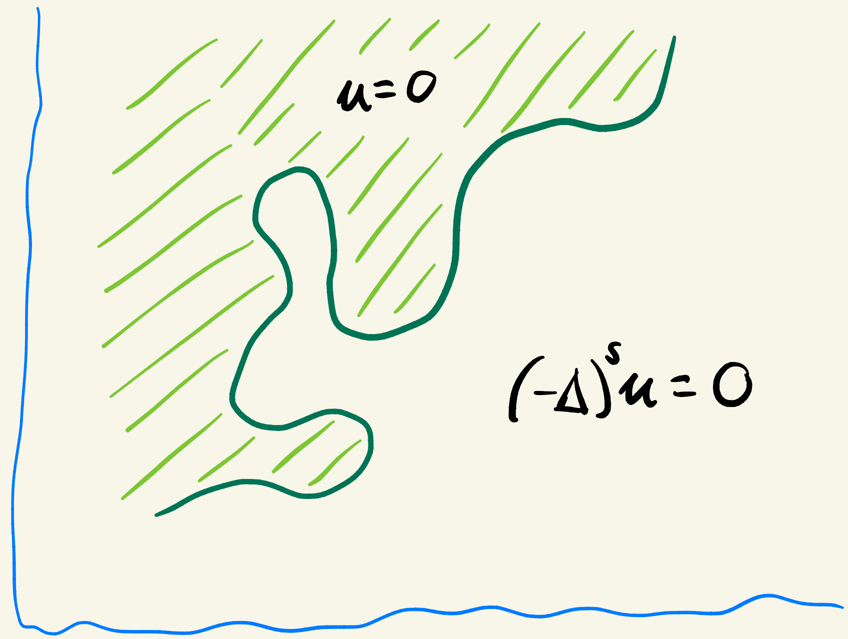


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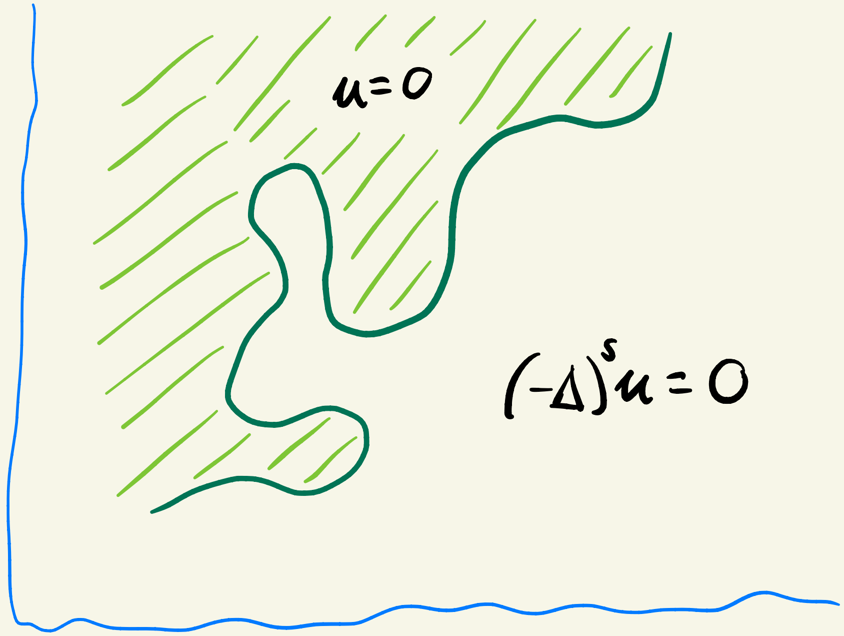


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- This is the question that motivates our work.

- Only known result:

[ **Thm.** In $\mathbb{R}^2$, all blow-ups are 1D. ]

In particular, there are no singular points in 2D.

$$\boxed{\mathcal{E}(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \left|\{u>0\}\cap\Omega\right|}$$

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- What is the PDE satisfied by minimizers?

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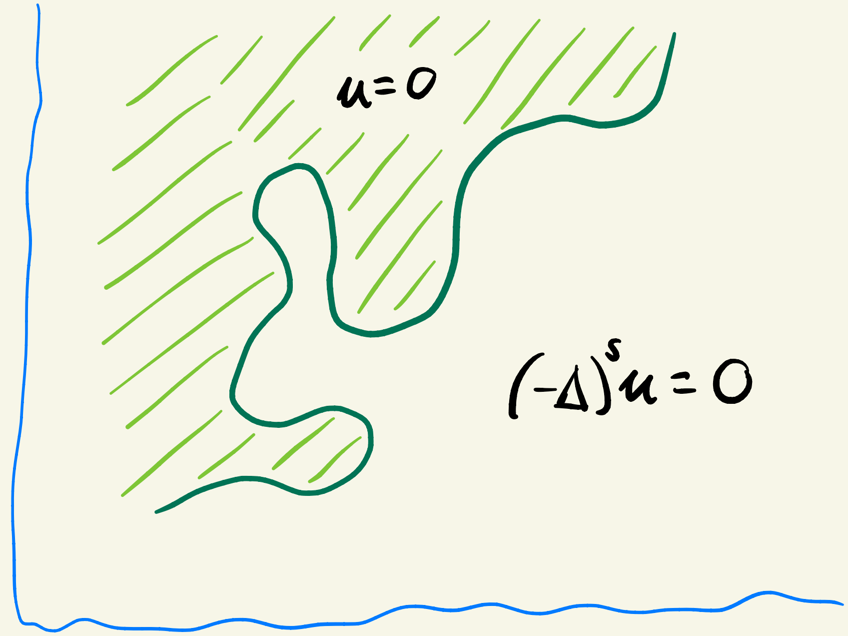

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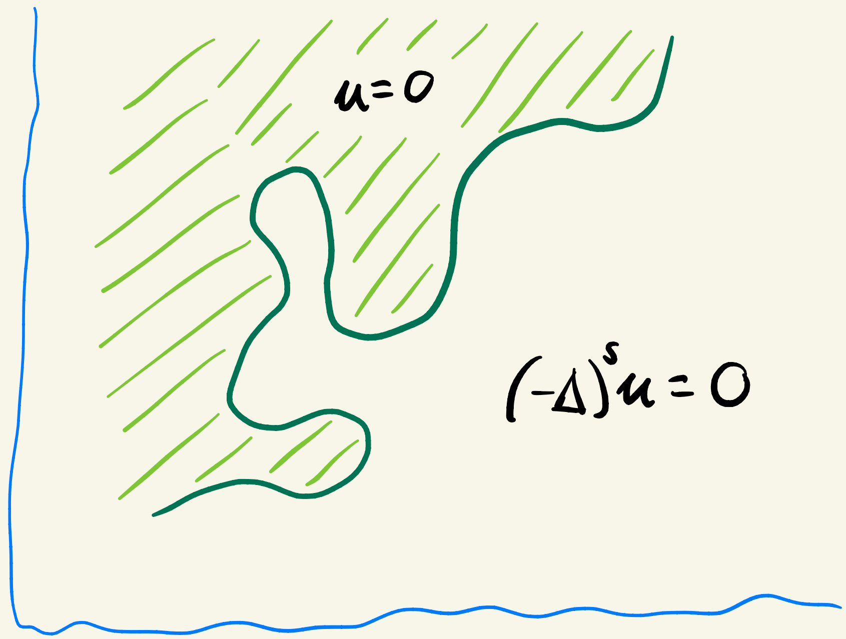


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- Moreover, solutions are $C^s$ and blow-ups $u_0$ are homogeneous of degree $s$.

- Thus, blow-ups $u_0$ satisfy

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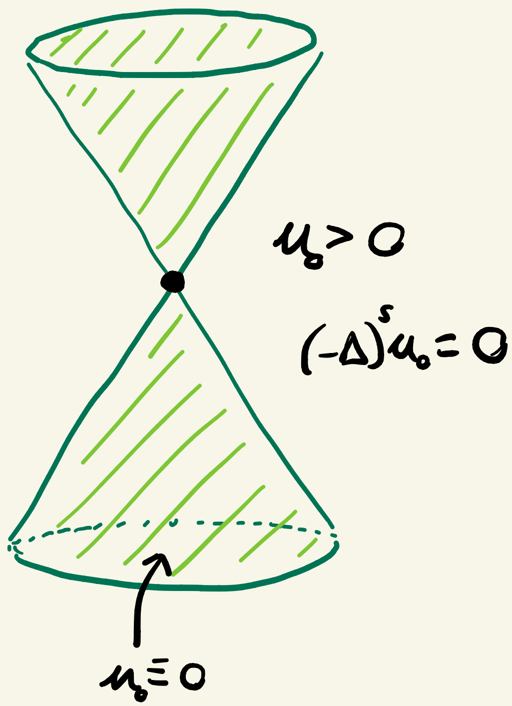

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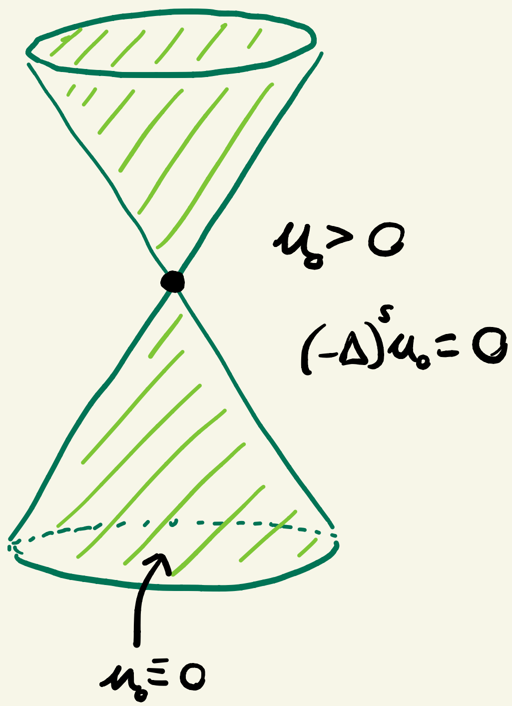


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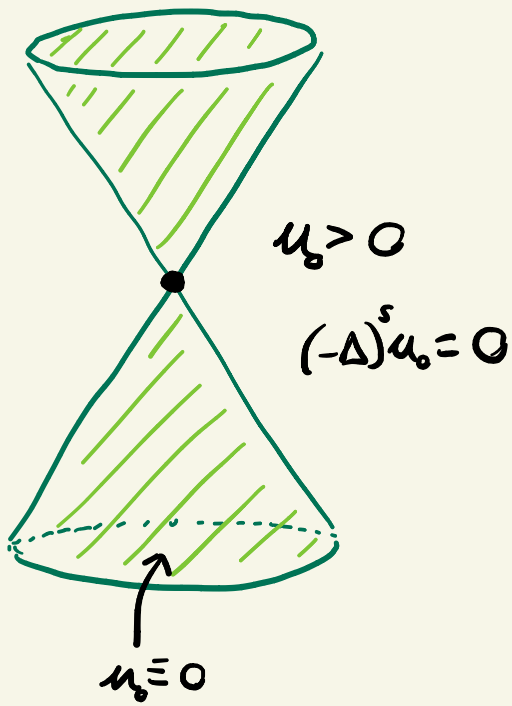


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- For this, we need the following:

> What is the STABILITY CONDITION for the thin one-phase problem?

[X.R., X. Fernández-Real, '20]

- Our first main result is the following:

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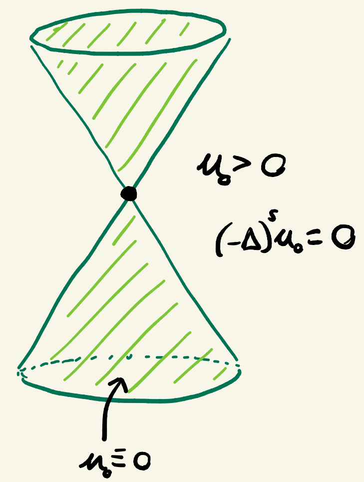

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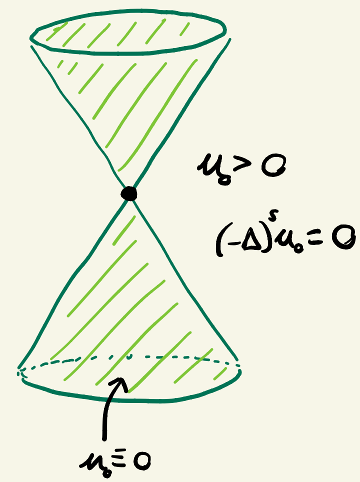

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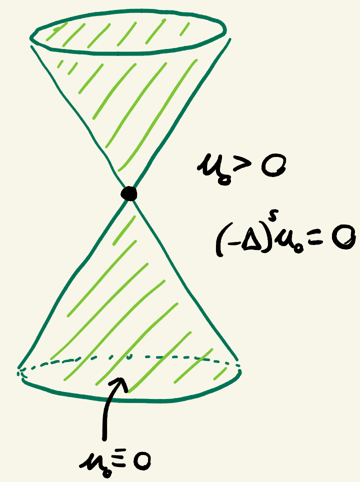


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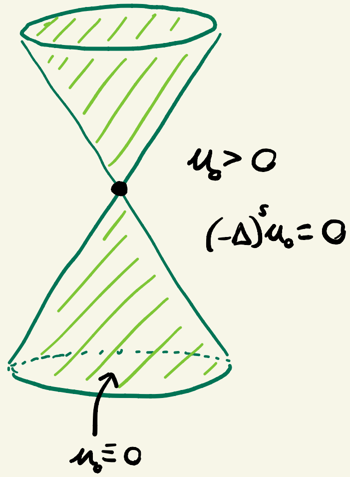


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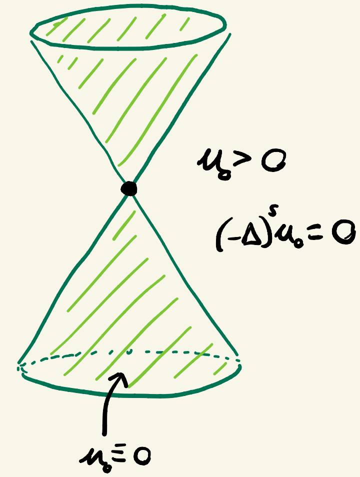


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(If one tries their approach in the nonlocal case, it requires delicate numerical computations!)

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Moreover,

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- They are given by

$$\boxed{\mathcal{K}_{\Gamma,s}(x,y) := \lim_{\substack{\bar{x}\to x\\ \bar{y}\to y}} \frac{G_{\{u>0\},s}(\bar{x},\bar{y})}{d^s(\bar{x})\cdot d^s(\bar{y})}} \quad \text{and} \quad \boxed{\mathcal{H}_{\Gamma,s}(x) := \int_{\Gamma} |\nu(x)-\nu(y)|^2 \cdot \mathcal{K}_{\Gamma,s}(x,y)}$$

($G_{U,s}$ is the Green function for $(-\Delta)^s$ in $U$)

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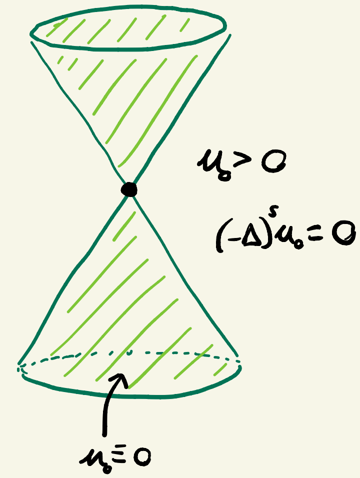

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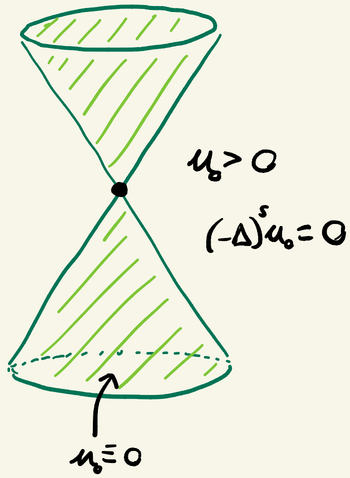


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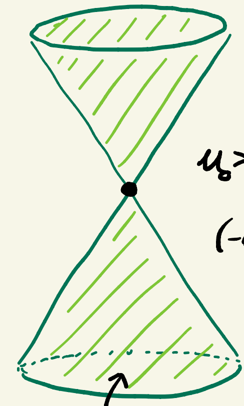


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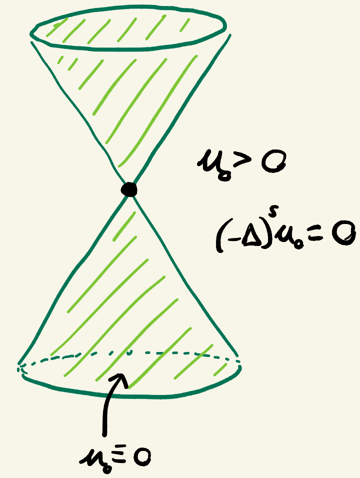


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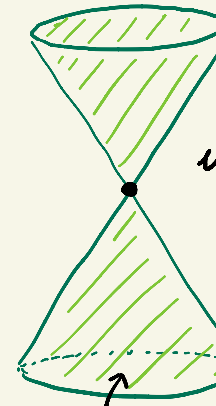


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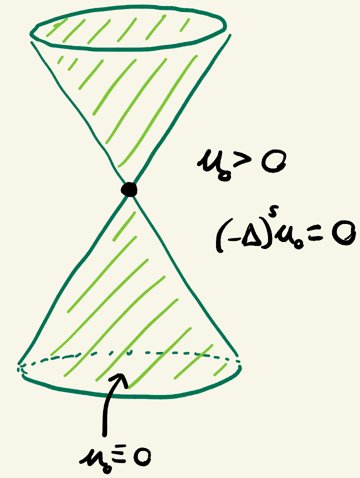


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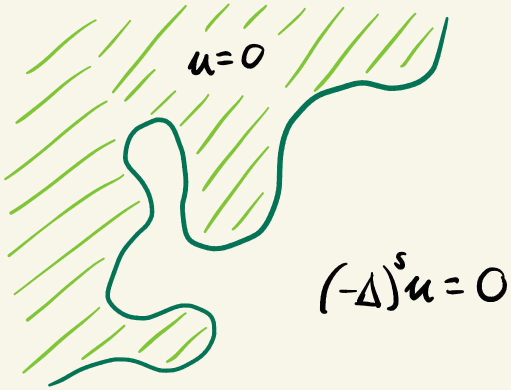
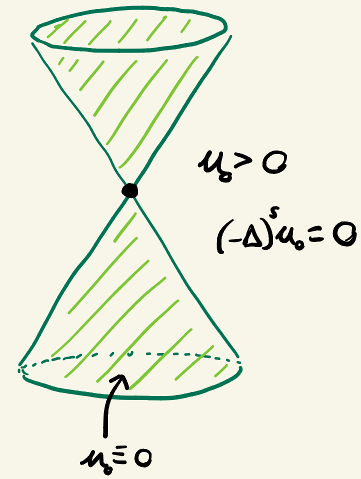
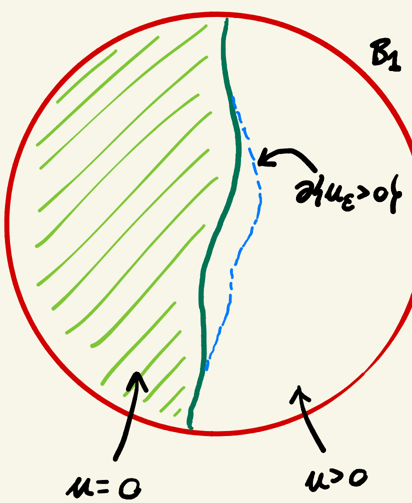
- This is somehow related to the problem of classifying nonlocal minimal cones.


$u=0$
$(-\Delta)^s u = 0$


$u_0 > 0$
$(-\Delta)^s u_0 = 0$
$u_0 \equiv 0$

Thank you!


$B_1$
$\partial\{u_\varepsilon > 0\}$
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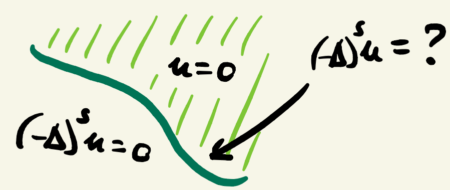
$u>0$


$u=0$
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