Geometric analysis on Finsler manifolds

Shin-ichi Ohta

Osaka University

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§Outline of the talk

Aim: Geometric analysis on various curved spaces (*comparison theorems*).

Today

- Develop "nonlinear Γ-calculus" on Finsler mfds using the Bochner inequality (by O.–Sturm 2014).
- Show some functional inequalities.

§1 Finsler manifolds

§2 Weighted Ricci curvature

§3 Nonlinear Γ-calculus

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Finsler manifolds

A Finsler manifold will be an n-dimensional connected C^∞-manifold M equipped with $F : TM \longrightarrow [0, \infty)$ s.t.

(1) $F \in C^\infty(TM \setminus \{0\})$;

(2) $F(cv) = cF(v)$ $\forall v \in TM$, $c > 0$ (*positive homog.*);

(3) $\forall v \in T_x M \setminus \{0\}$, the $n \times n$-symmetric matrix

$$g_{ij}(v) := \frac{1}{2} \frac{\partial^2 [F^2]}{\partial v^i \partial v^j}(v), \quad \text{where } v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_x,$$

is positive-definite (*strong convexity*).

Note: $F(-v) \neq F(v)$ is allowed.

Riemannian approximation g_v

For each $v \in T_xM \setminus \{0\}$, $g_{ij}(v)$ defines an inner product g_v of T_xM (*fundamental tensor*) by

$$g_v\biggl(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j \frac{\partial}{\partial x^j}\biggr) := \sum_{i,j=1}^{n} a_i b_j g_{ij}(v).$$

This is an approximation of $F|_{T_xM}$ in the direction v (up to the second order).

Ricci curvature

Instead of the precise definition, we explain a useful interpretation of the Ricci curvature (or *Ricci scalar*) $\mathrm{Ric}(v)$ of $v \in T_xM \setminus \{0\}$:

"Riemannian characterization"

(1) Extend v to a C^∞-vector field V on a neighborhood U of x such that every integral curve is geodesic.

(2) Consider the Riem. str. g_V on U induced from V.

(3) Then $\mathrm{Ric}(v)$ coincides with the Ricci curvature of v w.r.t. g_V (indep. of the choice of V).

Measure?

To develop analysis on Finsler manifolds, we would like to equip a Finsler manifold with a measure.
However, there is no unique canonical measure like the Riemannian volume measure.

Thus we start with an arbitrary measure $\mathfrak{m}$ on M and modify the Ricci curvature according to $\mathfrak{m}$, inspired by the weighted Ricci curvature for Riemannian manifolds equipped with measures.

§2 Weighted Ricci curvature

Weighted Ricci curvature

We fix an arbitrary positive C^∞-measure $\mathfrak{m}$ on M and take $v \in T_x M \setminus \{0\}$ and V as above.

Decompose $\mathfrak{m}$ as $\mathfrak{m} = \mathrm{e}^{-\psi} \operatorname{vol}_{g_V}$ and let η be the geodesic with $\dot{\eta}(0) = v$.

For $N \in (-\infty, 0] \cup (n, \infty)$ $(n = \dim M)$, define

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + (\psi \circ \eta)''(0) - \frac{(\psi \circ \eta)'(0)^2}{N - n},$$

$$\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + (\psi \circ \eta)''(0), \quad \mathrm{Ric}_n(v) := \lim_{N \downarrow n} \mathrm{Ric}_N(v).$$

Remarks

- Monotonicity: For $N \in (n, \infty)$ and $N' \leq 0$,

$$\mathrm{Ric}_n \leq \mathrm{Ric}_N \leq \mathrm{Ric}_\infty \leq \mathrm{Ric}_{N'} .$$

- $\mathrm{Ric}_N \geq K$ (i.e., $\mathrm{Ric}_N(v) \geq KF^2(v)\ \forall v \in TM$) is equivalent to the curvature-dimension condition $\mathrm{CD}(K, N)$ à la Lott–Sturm–Villani (O. 2009, 2016).

- A typical example satisfying $\mathrm{Ric}_\infty \geq 0$ is a normed space endowed with a log-concave measure.

We are interested in $(M, F, \mathfrak{m})$ with $\mathrm{Ric}_N \geq K$.

Three useful techniques

- The curvature-dimension condition $\mathrm{CD}(K, N)$ via the L^2*-optimal transport theory*.
- The Γ-calculus based on the Bochner inequality (O.–Sturm 2014). $\longrightarrow$ *this talk*
- The localization (a.k.a. needle decomposition) via the L^1- & L^2*-optimal transport theory* (O. 2018).

§3 Nonlinear Γ-calculus

Nonlinear Laplacian

- For $u : M \longrightarrow \mathbb{R}$ differentiable at $x \in M$, define

  $$\nabla u(x) := \text{the } \textit{Legendre transform} \text{ of } du(x),$$

  i.e., $F^*(du) = F(\nabla u)$ and $du[\nabla u] = F^*(du)^2$.

  Note: The Legendre transf. $T_x^*M \longrightarrow T_xM$ is linear only when $F|_{T_xM}$ comes from an inner product.

- For $u \in H^1_{\mathrm{loc}}(M)$, define the nonlinear Laplacian $\Delta u := \mathrm{div}_{\mathfrak{m}}(\nabla u)$ in the weak sense that

  $$\int_M \phi \Delta u \, d\mathfrak{m} = -\int_M d\phi(\nabla u) \, d\mathfrak{m} \quad \forall \phi \in C_c^\infty(M).$$

Nonlinear heat semigroup

$\boldsymbol{\Delta}$ is nonlinear but locally uniformly elliptic (by the strong convexity of F), this helps us to analyze the nonlinear heat equation $\partial_t u_t = \boldsymbol{\Delta} u_t$ as follows.

Existence & regularity (Ge–Shen 2001, O.–Sturm 2009)

$\forall f \in H_0^1(M)$, $\exists$a unique solution $(u_t)_{t \ge 0}$ to $\partial_t u_t = \boldsymbol{\Delta} u_t$ with $u_0 = f$, which is H^2_{loc} in x and $C^{1,\alpha}$ in t & x. And $\boldsymbol{\Delta} u_t \in H_0^1(M)$ if M is compact (or *unif. smooth*).

Note: The $C^{1,\alpha}$-regularity cannot be improved.

The heart of the Γ-calculus:

Bochner inequality (O.–Sturm 2014)

For $u \in C^\infty(M)$ and $N \in (-\infty, 0) \cup [n, \infty]$, we have

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - d(\Delta u)(\nabla u) \geq \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}$$

point-wise on $\{\nabla u \neq 0\}$ and in the weak sense on M.

Here $\Delta^{\nabla u}$ is the linearized Laplacian (w.r.t. $g_{\nabla u}$):

$$\Delta^{\nabla u} f := \mathrm{div}_{\mathfrak{m}}(\nabla^{\nabla u} f), \quad \nabla^{\nabla u} f := \sum_{i,j=1}^{n} g^{ij}(\nabla u) \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Note: This is *not* the Bochner inequality for $g_{\nabla u}$.

Applications under $\mathrm{Ric}_N \geq K$

Let M be compact for simplicity. $(u_t)_{t\geq 0}$: sol. to heat eq.

Gradient estimates (O.–Sturm 2014)

- *L^2-gradient estimate* $(N = \infty)$:

$$F^2(\boldsymbol{\nabla} u_t) \leq \mathrm{e}^{-2Kt} P_t^{\boldsymbol{\nabla} u}(F^2(\boldsymbol{\nabla} u_0)) \quad \forall t > 0,$$

where $f_t = P_t^{\boldsymbol{\nabla} u}(f)$ is the solution to $\partial_t f_t = \Delta^{\boldsymbol{\nabla} u_t} f_t$, $f_0 = f$.

- *Li–Yau gradient estimate* $(K \leq 0, N \in [n, \infty))$:

$$F^2(\boldsymbol{\nabla}(\log u_t)) - \theta \cdot \partial_t(\log u_t) \leq N\theta^2 \left(\frac{1}{2t} - \frac{K}{4(\theta - 1)} \right) \ \forall t > 0, \theta > 1.$$

Poincaré–Lichnerowicz inequality (O. 2009, 2017)

$\mathfrak{m}(M) = 1$, $K > 0$, $N \in (-\infty, 0) \cup [n, \infty]$: For $f \in H^1(M)$,

$$\int_M f^2 \, d\mathfrak{m} - \left(\int_M f \, d\mathfrak{m} \right)^2 \leq \frac{2(N-1)}{KN} \mathcal{E}(f).$$

Logarithmic Sobolev inequality (O. 2009, 2017)

$\mathfrak{m}(M) = 1$, $K > 0$, $N \in [n, \infty]$:

For nonnegative $f \in H^1(M)$ with $\int_M f \, d\mathfrak{m} = 1$,

$$\int_M f \log f \, d\mathfrak{m} \leq \frac{N-1}{2KN} \int_M \frac{F^2(\nabla f)}{f} \, d\mathfrak{m}.$$

Sobolev inequality (O. 2017)

$\mathfrak{m}(M) = 1$, $K > 0$, $N \in [n, \infty)$, $p \in [2, 2(N+1)/N]$ (or $p \in [2, 2N/(N-2)]$ if *reversible* $F(-v) = F(v)$): For $f \in H^1(M)$,

$$\frac{\|f\|_{L^p}^2 - \|f\|_{L^2}^2}{p-2} \le \frac{N-1}{KN} \int_M F^2(\nabla f)\, d\mathfrak{m}.$$

Beckner inequality (O. 2021; cf. Gentil–Zugmeyer 2021)

$\mathfrak{m}(M) = 1$, $K > 0$, $N \in (-\infty, -2) \cup [n, \infty)$, $p \in [1, 2]$ for $N \in [n, \infty)$ and $p \in \left[1, \frac{2N^2+1}{(N-1)^2}\right]$ for $N < -2$:

For $f \in H^1(M)$, the same inequality as above holds.

Final remarks

The proofs essentially follow the lines of the Riemanninan case up to some technical differences.

- One can also generalize *Bakry–Ledoux's Gaussian isoperimetric inequality*.
- In some results, the *noncompact case* is yet to be fully understood, due to the lack of *higher order regularity* and the *Wasserstein contraction*.
- The localization is useful in the noncompact case, however, it does not give sharp estimates in the *non-reversible* case (at present).

If you are interested $\downdownarrows$

Reference: “Comparison Finsler Geometry”
(to appear in *Springer Monographs in Mathematics*).

Thank you for your attention!