

Stability in Gagliardo-Nirenberg-Sobolev inequalities 3/3: Entropy methods and stability

Jean Dolbeault

<http://www.ceremade.dauphine.fr/~dolbeault>

Ceremade, Université Paris-Dauphine

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*New Trends in Nonlinear Diffusion:
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Stability in Gagliardo-Nirenberg-Sobolev inequalities

- 🔵 Lecture 1 [Bruno] A **variational** point of view
Variational methods provide good stability results. The deficit functional is estimated by a relative entropy, or a relative Fisher information. However, as in the Bianchi-Egnell method, estimates are non-constructive.
- 🔵 Lecture 2 [Nikita] **Convergence in relative error** for the FDE
The fast diffusion equation (FDE) has great regularization properties. These quantities are constructive.
- 🔵 Lecture 3: Entropy methods and **stability**
*Gagliardo-Nirenberg-Sobolev inequalities can be reformulated as **entropy – entropy production inequalities**. Entropy methods are fully constructive. Using the FDE as a tool, we obtained **improved inequalities** that can be reinterpreted as constructive stability estimates.*

Joint work with Matteo Bonforte, Bruno Nazaret, Nikita Simonov

Joint work on ***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*** [arXiv:2007.03674](https://arxiv.org/abs/2007.03674) (Apr. 29, 2021) with

Matteo Bonforte

▷ *Universidad Autónoma de Madrid and ICMAT*



Bruno Nazaret

▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*



Nikita Simonov

▷ *Ceremade, Université Paris-Dauphine (PSL)*



Outline

- Chapter 2: Inequalities, **entropies**, flows
 - ▷ Rényi entropy powers, Gagliardo-Nirenberg-Sobolev inequalities and fast diffusion
 - ▷ The fast diffusion equation in self-similar variables:
 - relative entropy and the entropy – entropy production inequality
 - ▷ **Large time asymptotics** and increased spectral gaps
 - ▷ **Initial time layer** and improved entropy – entropy production estimates
- Chapter 5: **Stability in** (subcritical) **Gagliardo-Nirenberg inequalities**
 - ▷ The **threshold** time based on regularity results
 - ▷ Gluing the **initial and asymptotic time layer** estimates
 - ▷ Form an improved entropy – entropy production inequality to stability
- Chapter 6: **Stability in Sobolev's inequality** (critical case)
 - ▷ A constructive stability result
 - ▷ The main ingredient of the proof

From the carré du champ method to stability results

Carré du champ method (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad \frac{d\mathcal{F}}{dt} = -\mathcal{I}, \quad \frac{d\mathcal{I}}{dt} \leq -\Lambda \mathcal{I}$$

deduce that $\mathcal{I} - \Lambda \mathcal{F}$ is monotone non-increasing with limit 0

$$\mathcal{I}[u] \geq \Lambda \mathcal{F}[u]$$

▷ **Improved constant** means **stability**

Under some restrictions on the functions, there is some $\Lambda_\star \geq \Lambda$ such that

$$\mathcal{I} - \Lambda \mathcal{F} \geq (\Lambda_\star - \Lambda) \mathcal{F}$$

▷ **Improved entropy – entropy production inequality** (weaker form)

$$\mathcal{I} \geq \Lambda \psi(\mathcal{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathcal{I} - \Lambda \mathcal{F} \geq \Lambda (\psi(\mathcal{F}) - \mathcal{F}) \geq 0$$

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (\text{FDE})$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in \mathbb{R}^d , $d \geq 1$, $m \in (0, 1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0, x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 u_0 \, dx < +\infty$$

The large time behavior is governed by **the self-similar Barenblatt solutions**

$$B(t, x) := \frac{1}{(\kappa t^{1/\mu})^d} \mathcal{B}\left(\frac{x}{\kappa t^{1/\mu}}\right)$$

where $\mu := 2 + d(m-1)$ and \mathcal{B} is the Barenblatt profile

$$\mathcal{B}(x) := (C + |x|^2)^{-\frac{1}{1-m}}$$

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]

[JD, Toscani, 2016]

[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in L^1_+(\mathbb{R}^d)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in L^1_+(\mathbb{R}^d, (1+|x|^2) dx)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 u(t, x) dx = 2d \int_{\mathbb{R}^d} u^m(t, x) dx$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in L^1(\mathbb{R}^d)$ and $u_0 \in L^1_+(\mathbb{R}^d, (1+|x|^2) dx)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^m(t, x) dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u |\nabla u^{m-1}|^2 dx$$

Entropy functional and *Fisher information functional*

$$E[u] := \int_{\mathbb{R}^d} u^m dx \quad \text{and} \quad I[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u |\nabla u^{m-1}|^2 dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\text{GNS}}(p) \|f\|_{2p} \quad (\text{GNS})$$

$$p = \frac{1}{2m-1} \iff m = \frac{p+1}{2p} \in [m_1, 1)$$

$$u = f^{2p} \text{ so that } u^m = f^{p+1} \text{ and } u|\nabla u|^{m-1}|^2 = (p-1)^2 |\nabla f|^2$$

$$M = \|f\|_{2p}^{2p}, \quad E[u] = \|f\|_{p+1}^{p+1}, \quad I[u] = (p+1)^2 \|\nabla f\|_2^2$$

$$\text{If } u \text{ solves (FDE) } \frac{\partial u}{\partial t} = \Delta u^m$$

$$E' \geq \frac{p-1}{2p} (p+1)^2 \left(\mathcal{C}_{\text{GNS}}(p) \right)^{\frac{2}{\theta}} \|f\|_{2p}^{\frac{2}{\theta}} \|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}} = C_0 E^{1 - \frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d} u^m(t, x) dx \geq \left(\int_{\mathbb{R}^d} u_0^m dx + \frac{(1-m)C_0}{m-m_c} t \right)^{\frac{1-m}{m-m_c}} \quad \forall t \geq 0$$

$$\text{Equality case: } u(t, x) = \frac{c_1}{R(t)^d} \mathcal{B}\left(\frac{c_2 x}{R(t)}\right), \quad \mathcal{B}(x) := (1 + |x|^2)^{\frac{1}{m-1}}$$

Pressure variable and decay of the Fisher information

The t -derivative of the *Rényi entropy power* $E^{\frac{2}{d}} \frac{1}{1-m} - 1$ is

$$I^\theta E^{2 \frac{1-\theta}{\rho+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$P := \frac{m}{1-m} u^{m-1}$$

▷ *Fisher information*

$$I[u] = \int_{\mathbb{R}^d} u |\nabla P|^2 dx$$

If u solves (FDE), then

$$\begin{aligned} I' &= \int_{\mathbb{R}^d} \Delta(u^m) |\nabla P|^2 dx + 2 \int_{\mathbb{R}^d} u \nabla P \cdot \nabla \left((m-1) P \Delta P + |\nabla P|^2 \right) dx \\ &= -2 \int_{\mathbb{R}^d} u^m \left(\|D^2 P\|^2 - (1-m) (\Delta P)^2 \right) dx \end{aligned}$$

Rényi entropy powers and interpolation inequalities

► Integrations by parts and completion of squares

$$\begin{aligned}
 & -\frac{1}{2\theta} \frac{d}{dt} \log \left(I^\theta E^{2\frac{1-\theta}{\rho+1}} \right) \\
 & = \int_{\mathbb{R}^d} u^m \left\| D^2 P - \frac{1}{d} \Delta P \text{Id} \right\|^2 dx + (m - m_1) \int_{\mathbb{R}^d} u^m \left| \Delta P + \frac{1}{E} \right|^2 dx
 \end{aligned}$$

► Analysis of the asymptotic regime

$$\lim_{t \rightarrow +\infty} \frac{I[u(t, \cdot)]^\theta E[u(t, \cdot)]^{2\frac{1-\theta}{\rho+1}}}{M^{\frac{2\theta}{\rho}}} = \frac{I[\mathcal{B}]^\theta E[\mathcal{B}]^{2\frac{1-\theta}{\rho+1}}}{\|\mathcal{B}\|_1^{\frac{2\theta}{\rho}}} = (\rho + 1)^{2\theta} (\mathcal{C}_{\text{GNS}}(\rho))^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$I[u]^\theta E[u]^{2\frac{1-\theta}{\rho+1}} \geq (\rho + 1)^{2\theta} (\mathcal{C}_{\text{GNS}}(\rho))^{2\theta} M^{\frac{2\theta}{\rho}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t, x) = \frac{1}{\kappa^d R^d} v\left(\tau, \frac{x}{\kappa R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \frac{1}{2} \log R(t)$$

$\frac{\partial u}{\partial t} = \Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla u^{m-1} - 2x \right) \right] = 0 \quad (r\text{FDE})$$

Generalized entropy (free energy) and Fisher information

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m \mathcal{B}^{m-1} (v - \mathcal{B}) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2x \right|^2 dx$$

are such that $\mathcal{I}[v] \geq 4 \mathcal{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-4t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$(C_0 + |x|^2)^{-\frac{1}{1-m}} \leq v_0 \leq (C_1 + |x|^2)^{-\frac{1}{1-m}} \quad (\text{H})$$

Let $\Lambda_{\alpha,d} > 0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} f^2 d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_{\alpha} \quad \forall f \in H^1(d\mu_{\alpha}), \quad \int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0$$

with $d\mu_{\alpha} := (1 + |x|^2)^{\alpha} dx$, for $\alpha < 0$

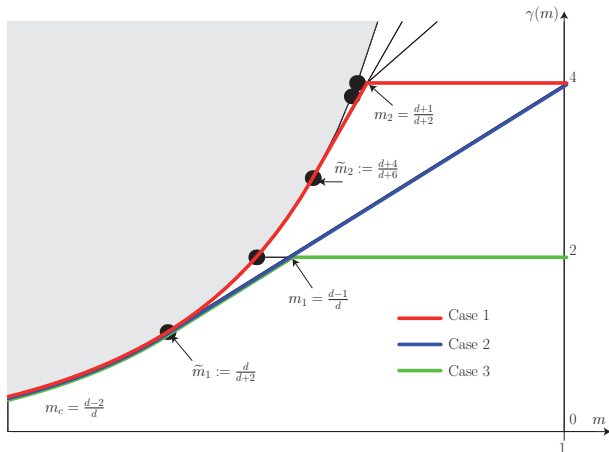
Lemma

Under assumption (H),

$$\mathcal{F}[v(t, \cdot)] \leq C e^{-2\gamma(m)t} \quad \forall t \geq 0, \quad \gamma(m) := (1-m) \Lambda_{1/(m-1),d}$$

Moreover $\gamma(m) := 2$ if $1 - 1/d \leq m < 1$

Spectral gap



[Denzler, McCann, 2005]

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Much more is know, e.g., [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

- ▶ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality
- ▶ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and linearized Fisher information

$$F[g] := \frac{m}{2} \int_{\mathbb{R}^d} g^2 \mathcal{B}^{2-m} dx \quad \text{and} \quad I[g] := m(1-m) \int_{\mathbb{R}^d} |\nabla g|^2 \mathcal{B} dx$$

Hardy-Poincaré inequality. Let $d \geq 1$, $m \in (m_1, 1)$ and $g \in L^2(\mathbb{R}^d, \mathcal{B}^{2-m} dx)$ such that $\nabla g \in L^2(\mathbb{R}^d, \mathcal{B} dx)$, $\int_{\mathbb{R}^d} g \mathcal{B}^{2-m} dx = 0$ and $\int_{\mathbb{R}^d} x g \mathcal{B}^{2-m} dx = 0$

$$I[g] \geq 4\alpha F[g] \quad \text{where} \quad \alpha = 2 - d(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2(dm - d + 1)$ and $\chi = m/(266 + 56m)$. If $\int_{\mathbb{R}^d} v dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x v dx = 0$ and

$$(1 - \varepsilon)\mathcal{B} \leq v \leq (1 + \varepsilon)\mathcal{B}$$

for some $\varepsilon \in (0, \chi\eta)$, then

$$\mathcal{F}[v] \geq (4 + \eta) \mathcal{F}[v]$$

The initial time layer improvement: backward estimate

A hint: for some strictly convex function ψ with $\psi(0) = \psi'(0) = 0$, we have

$$\mathcal{I} - 4\mathcal{F} \geq 4(\psi(\mathcal{F}) - \mathcal{F}) \geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}(\mathcal{Q} - 4)$$

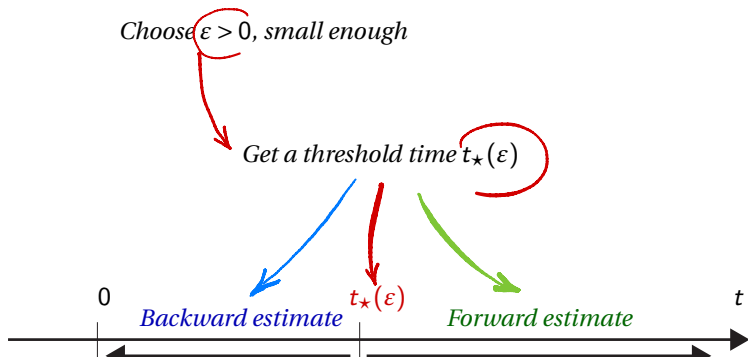
Lemma

Assume that $m > m_1$ and v is a solution to (r FDE) with nonnegative initial datum v_0 . If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T, \cdot)] \geq 4 + \eta$, then

$$\mathcal{Q}[v(t, \cdot)] \geq 4 + \frac{4\eta e^{-4T}}{4 + \eta - \eta e^{-4T}} \quad \forall t \in [0, T]$$

Stability in (subcritical) Gagliardo-Nirenberg inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results (Lecture 2) allow us to glue the initial time layer estimates with the asymptotic time layer estimates:

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

Uniform convergence in relative error

Theorem

Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit **threshold time** $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (2)$$

with nonnegative initial datum $u_0 \in L^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx \leq A < \infty \quad (H_A)$$

$\int_{\mathbb{R}^d} u_0 \, dx = \int_{\mathbb{R}^d} \mathcal{B} \, dx = \mathcal{M}$ and $\mathcal{F}[u_0] \leq G$, then

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t, x)}{\mathcal{B}(t, x)} - 1 \right| \leq \varepsilon \quad \forall t \geq t_\star$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$t_\star = c_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^a}$$

where $a = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d(m - m_c)$ and $\vartheta = \nu / (d + \nu)$

$$c_\star = c_\star(m, d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max \{ \varepsilon \kappa_1(\varepsilon, m), \varepsilon^a \kappa_2(\varepsilon, m), \varepsilon \kappa_3(\varepsilon, m) \}$$

$$\kappa_1(\varepsilon, m) := \max \left\{ \frac{8c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m} \kappa_\star}{1 - (1-\varepsilon)^{1-m}} \right\}$$

$$\kappa_2(\varepsilon, m) := \frac{(4\alpha)^{\alpha-1} K^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m} \frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon, m) := \frac{8\alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta) \mathcal{F}[v]$$

for any nonnegative function $v \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x v \, dx = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star), \quad \varepsilon_\star := \frac{1}{2} \min \{ \varepsilon_{m,d}, \chi \eta \} \quad \text{with} \quad T = \frac{1}{2} \log R(t_\star)$$

$$(1 - \varepsilon) \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon) \mathcal{B} \quad \forall t \geq T$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t, \cdot)] \geq (4 + \zeta) \mathcal{F}[v(t, \cdot)] \quad \forall t \in [0, T], \quad \text{where} \quad \zeta = \frac{4\eta e^{-4T}}{4 + \eta - \eta e^{-4T}}$$

Two consequences

$$\zeta = Z(A, \mathcal{F}[u_0]), \quad Z(A, G) := \frac{\zeta_\star}{1 + A(1-m)\frac{2}{\alpha} + G}, \quad \zeta_\star := \frac{4\eta c_\alpha}{4 + \eta} \left(\frac{\varepsilon_\star^a}{2\alpha c_\star} \right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (rFDE) with nonnegative initial datum $v_0 \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x v_0 \, dx = 0$ and v_0 satisfies (H_A) , then

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-(4+\zeta)t} \quad \forall t \geq 0$$

▷ The **stability in the entropy - entropy production estimate**

$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \zeta \mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \frac{\zeta}{4 + \zeta} \mathcal{I}[v]$$

Stability results

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell (as in Lecture 1)

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f] := \left(\frac{2d\kappa[f]^{p-1}}{p^2-1} \frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}, \quad \kappa[f] := \frac{\mathcal{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$A[f] := \frac{\mathcal{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}} \|f\|_{2p}^{2p}} \sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p} dx$$

$$E[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(\frac{\kappa[f]^{p+1}}{\lambda[f]^d \frac{p-1}{2p}} f^{p+1} - g^{p+1} - \frac{1+p}{2p} g^{1-p} \left(\frac{\kappa[f]^{2p}}{\lambda[f]^2} f^{2p} - g^{2p} \right) \right) dx$$

$$\mathfrak{G}[f] := \frac{\mathcal{M}^{\frac{p-1}{2p}}}{p^2-1} \frac{1}{C(p,d)} Z(A[f], E[f])$$

Theorem

Let $d \geq 1$, $p \in (1, p^*)$

If $f \in \mathcal{W}_p(\mathbb{R}^d) := \{f \in L^{2p}(\mathbb{R}^d) : \nabla f \in L^2(\mathbb{R}^d), |x|f^p \in L^2(\mathbb{R}^d)\}$,

$$\left(\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \right)^{2p\gamma} - (\mathcal{C}_{\text{GN}} \|f\|_{2p})^{2p\gamma} \geq \mathfrak{G}[f] \|f\|_{2p}^{2p\gamma} E[f]$$

With $\mathcal{K}_{\text{GNS}} = C(p, d) \mathcal{C}_{\text{GNS}}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2 \|\nabla f\|_2^2 + 4 \frac{d-p(d-2)}{p+1} \|f\|_{p+1}^{p+1} - \mathcal{K}_{\text{GNS}} \|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d \geq 1$ and $p \in (1, p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f]$ such that, for any $f \in L^{2p}(\mathbb{R}^d, (1+|x|^2) dx)$ such that $\nabla f \in L^2(\mathbb{R}^d)$ and $A[f^{2p}] < \infty$,

$$\delta[f] \geq \mathcal{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} \left| (p-1)\nabla f + f^p \nabla \varphi^{1-p} \right|^2 dx$$

- ▷ The dependence of $\mathcal{C}[f]$ on $A[f^{2p}]$ and $\mathcal{F}[f^{2p}]$ is explicit and does not degenerate if $f \in \mathfrak{M}$
- ▷ Can we remove the condition $A[f^{2p}] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▶ A constructive stability result
- ▶ The main ingredient of the proof

A constructive stability result

Let $2p^* = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^*}(\mathbb{R}^d) = \left\{ f \in L^{p^*+1}(\mathbb{R}^d) : \nabla f \in L^2(\mathbb{R}^d), |x|f^{p^*} \in L^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \geq 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^*}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2) f^{2^*} dx = \int_{\mathbb{R}^d} (1, x, |x|^2) g dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*} dx \leq A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - S_d^2 \|f\|_{2^*}^2 \geq \frac{\mathcal{C}_*(A)}{4 + \mathcal{C}_*(A)} \int_{\mathbb{R}^d} \left| \nabla f + \frac{d-2}{2} f^{\frac{d}{d-2}} \nabla g^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_*(A) = \mathfrak{C}_*(1 + A^{1/(2d)})^{-1}$ and $\mathfrak{C}_* > 0$ depends only on d

We can remove the normalization of f , use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*} (x + x_f) \quad \text{and} \quad Z[f] := \left(1 + \mu[f]^{-d} \lambda[f]^d A[f]\right)$$

the *Bianchi-Egnell type result*

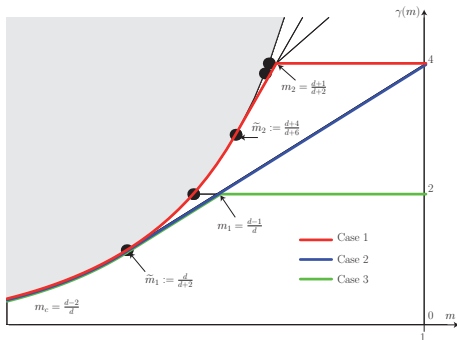
$$\delta[f] \geq \frac{c_* Z[f]}{4 + Z[f]} \inf_{g \in \mathfrak{M}} \mathcal{J}[f|g]$$

with x_f , $\lambda[f]$ and $\mu[f]$ as in the subcritical case

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2} \mathcal{B}(x/\sqrt{\lambda})$ in order to match $\int_{\mathbb{R}^d} |x|^2 v dx$ where the function v solves (rFDE) or to further rescale v according to

$$v(t, x) = \frac{1}{\mathfrak{R}(t)^d} w\left(t + \tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star} \int_{\mathbb{R}^d} |x|^2 v dx \right)^{-\frac{d}{2}(m-m_c)} - 1, \quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\tau(t)}$$

Lemma

$t \mapsto \tau(t)$ is bounded on \mathbb{R}^+

These slides can be found at

<http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/>
▷ Lectures

More related papers can be found at

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For final versions, use Dolbeault as login and Jean as password

E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention !

- 1 Fast diffusion equation and entropy methods
 - Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities
 - The fast diffusion equation in self-similar variables
 - Initial and asymptotic time layers
- 2 Stability in (subcritical) Gagliardo-Nirenberg inequalities
 - The threshold time
 - Improved entropy – entropy production inequality
 - First stability results
- 3 Stability in Sobolev's inequality (critical case)
 - A constructive stability result
 - The main ingredient of the proof