An abstract approach to the conjecture of Crouzeix

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Multivariable Operator Theory
and Function Spaces in Several Variables
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This talk is based on papers by Ransford–Schwenninger (2018) and Ostermann–Ransford (2020). These are short and beautifully written – go read them!

The starting point: spectral sets

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Let $\mathrm{A}(\Omega) \subset \mathrm{C}(\overline{\Omega})$ denote the algebra of functions that are holomorphic on Ω. Then, Ω is a (complete) $\mathcal{Q}$-spectral set if and only if the map

$$\Theta_T : p \mapsto p(T)$$

extends to a (completely) bounded homomorphism on $\mathrm{A}(\Omega)$ with norm at most $\mathcal{Q}$.

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What is another instance of a naturally occuring spectral set?

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Theorem (Delyon–Delyon 1999)

Let $T \in B(\mathcal{H})$ and let $\Omega \subset \mathbb{C}$ be a bounded open convex subset. Assume that $W(T) \subset \Omega$. Then, Ω is a $\mathcal{Q}$-spectral set for T for some constant $\mathcal{Q}$ that depends only Ω.

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Theorem (Crouzeix 2007)

Let $T \in B(\mathcal{H})$. Then, $W(T)$ is a complete (11.08)*-spectral set.*

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Example

Let $T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then, $W(T) = \overline{\mathbb{D}}$ and $\|T\| = 2$, so that $W(T)$ cannot be a $\mathcal{Q}$-spectral set for T for $\mathcal{Q} < 2$.

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Given a constant $Q > 0$, the following statements are equivalent.

1. *There is an invertible operator $X \in B(\mathcal{H})$ such that $\|X\|\|X^{-1}\| \leq Q$ and such that the operator XTX^{-1} admits a $\partial\Omega$-normal dilation.*

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1. *There is an invertible operator $X \in B(\mathcal{H})$ such that $\|X\|\|X^{-1}\| \leq \mathcal{Q}$ and such that the operator XTX^{-1} admits a $\partial\Omega$-normal dilation.*
2. *The set Ω is a complete $\mathcal{Q}$-spectral set for T.*

The basic insight of the lions

$\Omega \subset \mathbb{C}$ bounded open convex subset with smooth boundary
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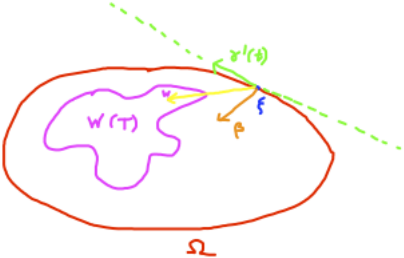

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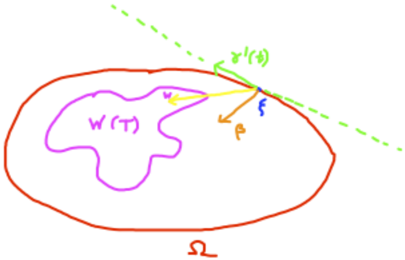

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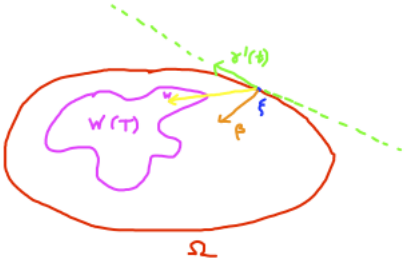

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We have

$$\mathrm{Re}\left(\frac{1}{2\pi i}\gamma'(t)(\gamma(t)I-T)^{-1}\right) = \frac{1}{4\pi}(\zeta I - T)^{-1}(\mathrm{Re}(T-\zeta I)\overline{\beta})(\zeta I - T)^{*-1} \geq 0$$

for every $0 \leq t \leq 1$.

The Cauchy transform and the key estimate

For $f \in \mathrm{A}(\Omega)$, Cauchy's formula gives

$$f(T) = \int_{\partial\Omega} f(\zeta) \left(\frac{1}{2\pi i} (\zeta I - T)^{-1} \right) d\zeta.$$

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Lemma (Classical?)

For each $f \in \mathrm{A}(\Omega)$, there is a function $\alpha(f) \in \mathrm{A}(\Omega)$ with $\|\alpha(f)\| \leq \|f\|$ such that

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$$f(T) + (\alpha(f))(T)^* = 2 \int_0^1 f(\gamma(t)) \left(\frac{1}{2\pi i} \operatorname{Re} \gamma'(t)(\gamma(t) I - T)^{-1} \right) dt$$

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Crouzeix's conjecture: $\|f(T)\| \leq 2\|f\|_{\mathrm{A}(\Omega)}$

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  This grouping of term is "best possible".

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Is all hope lost? No! We should also assume that α is unital.

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If α is unital, then $\|\theta\| \leq 2$.

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Theorem (Ostermann–Ransford 2020)

The following statements hold.

- *The map θ sends orthogonal projections to orthogonal projections.*
- *If $\mathcal{A}$ is a commutative von Neumann algebra, then $\|\theta\| = 1$.*

Exceptional matrices

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Proof.

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$$\|\theta\|^2 = \langle \theta(f)\xi, \theta(f)\xi\rangle = \langle (\theta(f) + \theta(\alpha(f))^*)\xi, \theta(f)\xi\rangle \leq 2\|\theta\|.$$

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Corollary (Okubo–Ando 1975)

Crouzeix's conjecture holds when $W(T) \subset \overline{\mathbb{D}}$.

Dilation theory?

The map

$$\frac{1}{2}(\theta + \theta^* \circ \overline{\alpha})$$

is a unital completely contractive map on $\mathcal{A}$. In particular, there is a unital $*$-homomorphism $\pi : \mathrm{C}(X) \to B(\mathcal{H})$ such that

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But then what?

Thank you!