BIRS - CMO Workshop

Multi-Stage Stochastic Optimization for Clean Energy Transition

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Towards a Decomposition Method for Linear Multi-Stage Stochastic Integer Programs with Discrete Distributions

Rüdiger Schultz (University of Duisburg-Essen)

The potential offered by Discrete Mathematics for solving stochastic integer programs is widely under-exploited

.

Gordan, Dickson, Maclagan and

Aschenbrenner, Hemmecke, Nash-Williams

Graver

or

- Solution of linear two-stage stochastic (pure) integer programs
- by successive augmentation of feasible vectors.

Augmentation with Tailored Generating Sets (Bases)

Solve $\min\{f(x) : x \in X\}$.

There is a finite set B containing improving vectors, if any:

Either $\exists\, b \in B : \; x_{n+1} := x_n + b \in X, \; f(x_{n+1}) < f(x_n)$ or x_n is optimal.

Issues:

1. Tailored Ground Set $\mathcal{S}$
2. Tailored Partial Order $\sqsubseteq$
3. Existence of B – Finite Antichain
4. Computation of B – Critical Pair/Completion (Buchberger)
5. SP Algorithm – Augmentation $\rightarrow$ scenario-wise !!
6. Bonus: $card\, B$ “stabilizes” with growing number of scenarios

Issues – IP:

- **Ground Set:** $\mathcal{S} := \mathbb{Z}^n$
- **Partial Order on $\mathbb{Z}^n$:** $u \sqsubseteq v$, if

$$u^{(j)} \cdot v^{(j)} \geq 0 \quad \text{and} \quad |u^{(j)}| \leq |v^{(j)}| \quad \text{for all components } j.$$

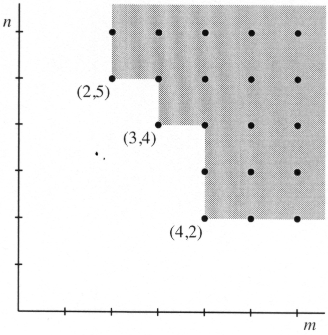
Commonly said "u reduces v'

- **The Set B**

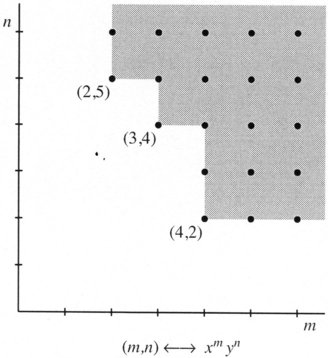
Paul Gordan (1837-1912), Leonard Eugene Dickson (1874-1954)

- A sequence $\{p_1, p_2, \ldots\}$ of vectors in $\mathbb{Z}_+^n$ such that $p_i \not\leq p_j$ for all $i < j$ is called an **ANTICHAIN.**

  In $\left(\mathbb{Z}_+^n, \leq\right)$ there are no antichains of infinite cardinality.



$(m,n) \longleftrightarrow x^m y^n$



- Every infinite set in $\mathbb{Z}_+^n$ has only finitely many $\leq$-minimal points.

Augmentation - Test Set = The Promise:

A set $\mathcal{T}_c \subseteq \mathbb{Z}^n$ is called a test set
for the family of integer linear programs

$$(IP)_{c,b} \qquad \min\{c^\intercal z : Az = b,\, z \in \mathbb{Z}_+^n\}$$

as $b \in \mathbb{R}^l$ varies if

1. $c^\intercal t > 0$ for all $t \in \mathcal{T}_c$, and
2. for every $b \in \mathbb{R}^l$ and for every non-optimal feasible solution $z_0 \in \mathbb{Z}_+^n$ to $Az = b$, there exists an improving vector $t \in \mathcal{T}_c$ such that $z_0 - t$ is feasible.

Obviously, $\mathcal{T}_c$ must be a subset of the kernel of A.

- **Jack Graver:** Let A an $m \times n$ integer matrix. The set of all $\sqsubseteq$-minimal points of $\ker_{\mathbb{Z}^n}(A) \setminus \{0\}$ is called Graver Basis $\mathcal{G} = \mathcal{G}(A)$

- **Jack Graver:** Let A an $m \times n$ integer matrix. The set of all $\sqsubseteq$-minimal points of $\ker_{\mathbb{Z}^n}(A) \setminus \{0\}$ is called Graver Basis $\mathcal{G} = \mathcal{G}(A)$

$\mathcal{G}(A)$ is a test set and can be computed by a finite algorithm.

- **normalForm Procedure:** While there is some g in a set G such that $g \sqsubseteq s$ do $s := s - g$ Division with Remainder
- **Completion Algorithm:** Yields a set G which contains $\mathcal{G}(A)$.

Algorithm (Computing IP Graver Sets via Completion Procedures)

Input: $F = \bigcup_{f \in F(A)} \{f, -f\}$, where $F(A)$ is a set of vectors generating $\ker(A)$ over $\mathbb{Z}$

Output: a set G which contains the IP Graver set $\mathcal{G}(A)$.

$G := F$

$C := \bigcup_{f,g \in G} \{f + g\}$ (forming S-vectors)

while $C \neq \emptyset$ do

  $s :=$ an element in C

  $C := C \setminus \{s\}$

  $f :=$ normalForm(s, G)

  if $f \neq 0$ then

    $C := C \cup \bigcup_{g \in G} \{f + g\}$ (adding S-vectors)

    $G := G \cup \{f\}$

return G.

Proposition

The above algorithm terminates with a set G containing the IP Graver Set $\mathcal{G}(A)$ for $(IP)_{c,b}$

$$(IP)_{c,b} \qquad \min\{c^\intercal z : Az = b,\ z \in \mathbb{Z}_+^n\}$$

Proof (termination) : If f =normalForm(s, G), then there is no $g \in G$ such that

$$(g^+, g^-) \leq (f^+, f^-).$$

Hence $(g^+, g^-) \not\leq (f^+, f^-)$ for any $g \in G$.

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$$(IP)_{c,b} \qquad \min\{c^\intercal z : Az = b,\, z \in \mathbb{Z}^n_+\}$$

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In case the algorithm does not terminate, an infinite number of normalForm computations occurs.

In other words, there exists an infinite sequence in $\mathbb{N}^{2n}$ such that $a_i \not\leq a_j$ for any $i \neq j$. This contradicts te Gordan-Dickson Lemma, hence the algorithm terminates.

Two-Stage Stochastic Integer Programs

$$\min\{c^\intercal z : A_N z = b, z \in \mathbb{Z}_+^d\}$$

$$A_N := \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ T & W & 0 & \cdots & 0 \\ T & 0 & W & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T & 0 & 0 & \cdots & W \end{pmatrix}$$

with

N denoting the number of scenarios, $d = m + Nn$,

$c = (c_0, c_1, \ldots, c_N)^\intercal := (h, \pi_1 q, \ldots, \pi_N q)^\intercal$

$b = (a, \xi^1, \ldots, \xi^N)^\intercal$.

A Detail

Lemma

$(u, v_1, \ldots, v_N) \in \mathsf{ker}(A_N)$ if and only if $(u, v_1), \ldots, (u, v_N) \in \mathsf{ker}(A_1)$.

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Conclusions:

- By permuting the v_i we do not leave $\ker(A_N)$.

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- Thus, a Graver test set vector is transformed into a Graver test set vector by such a permutation. This leads us to the following definition:

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Definition

Let $z = (u, v_1, \ldots, v_N) \in \ker(A_N)$ and call the vectors $u, v_1, \ldots, v_N$ the building blocks of z. Denote by $\mathcal{G}_N$ the Graver test set associated with A_N and collect into $\mathcal{H}_N$ all those vectors arising as building blocks of some $z \in \mathcal{G}_N$ By $\mathcal{H}_\infty$ denote the set $\bigcup_{N=1}^{\infty} \mathcal{H}_N$.

The set $\mathcal{H}_\infty$ contains both m-dimensional vectors u associated with the first-stage and n-dimensional vectors v related to the second-stage in the stochastic program. For convenience, we will arrange the vectors in $\mathcal{H}_\infty$ into **pairs** (u, V_u).

Definition

For fixed $u \in \mathcal{H}_\infty$, all those vectors $v \in \mathcal{H}_\infty$ are collected into V_u for which $(u, v) \in \ker(A_1)$.

Towards Finiteness of $\mathcal{H}_\infty$

Reduction at pair level:

Definition

We say that $(u', V_{u'})$ reduces (u, V_u), or $(u', V_{u'}) \sqsubseteq (u, V_u)$ for short, if the following conditions are satisfied:

- $u' \sqsubseteq u$,
- for every $v \in V_u$ there exists a $v' \in V_{u'}$ with $v' \sqsubseteq v$,
- $u' \neq 0$ or there exist vectors $v \in V_u$ and $v' \in V_{u'}$ with $0 \neq v' \sqsubseteq v$.

Monomials Enter

Definition

We associate with (u, V_u), $u \neq 0$,and with $(0, V_0)$ the monomial ideals

$$I(u, V_u) \in Q[x_1, \ldots, x_{2m+2n}] \qquad \text{and} \qquad I(0, V_0) \in Q[x_1, \ldots, x_{2n}]$$

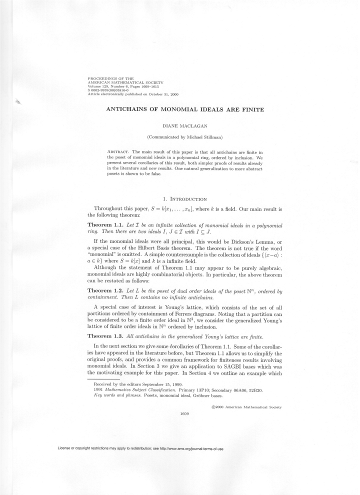
generated by all the monomials $x^{(u^+, u^-, v^+, v^-)}$ with $v \in V_u$, and by all the monomials $x^{(v^+, v^-)}$ with $v \neq 0$ and $v \in V_0$, respectively.

Ideal:
$\mathcal{I} \subseteq k[x_1, \ldots, x_n]$ is an ideal, if (i) $0 \in \mathcal{I}$; (ii) If $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$;
(iii) If $f \in \mathcal{I}$ and $h \in k[x]$, then $hf \in \mathcal{I}$.

Theorem (Maclagan 2001)
Let $\mathcal{I}$ be an infinite collection of monomial ideals in a polynomial ring. Then there are two ideals $I, , J \in \mathcal{I}$ with $I \subseteq J$.

Antichains of monomial ideals are finite.

Diane Maglagan



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ANTICHAINS OF MONOMIAL IDEALS ARE FINITE

DIANE MACLAGAN

(Communicated by Michael Stillman)

ABSTRACT. The main result of this paper is that all antichains are finite in the poset of monomial ideals in a polynomial ring, ordered by inclusion. We present several corollaries of this result, both simpler proofs of results already in the literature and new results. One natural generalization to more abstract posets is shown to be false.

1. INTRODUCTION

Throughout this paper, $S = k[x_1, \dots, x_n]$, where k is a field. Our main result is the following theorem:

Theorem 1.1. *Let $\mathcal{I}$ be an infinite collection of monomial ideals in a polynomial ring. Then there are two ideals I, $J \in \mathcal{I}$ with $I \subseteq J$.*

If the monomial ideals were all principal, this would be Dickson's Lemma, or a special case of the Hilbert Basis theorem. The theorem is not true if the word "monomial" is omitted. A simple counterexample is the collection of ideals $\{\langle x-a \rangle : a \in k\}$ where $S = k[x]$ and k is a infinite field.

Although the statement of Theorem 1.1 may appear to be purely algebraic, monomial ideals are highly combinatorial objects. In particular, the above theorem can be restated as follows:

Theorem 1.2. *Let L be the poset of dual order ideals of the poset $\mathbb{N}^n$, ordered by containment. Then L contains no infinite antichains.*

A special case of interest is Young's lattice, which consists of the set of all partitions ordered by containment of Ferrers diagrams. Noting that a partition can be considered to be a finite order ideal in $\mathbb{N}^2$, we consider the generalized Young's lattice of finite order ideals in $\mathbb{N}^n$ ordered by inclusion.

Theorem 1.3. *All antichains in the generalized Young's lattice are finite.*

In the next section we give some corollaries of Theorem 1.1. Some of the corollaries have appeared in the literature before, but Theorem 1.1 allows us to simplify the original proofs, and provides a common framework for finiteness results involving monomial ideals. In Section 3 we give an application to SAGBI bases which was the motivating example for this paper. In Section 4 we outline an example which

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Computation of $\mathcal{H}_\infty$

Idea:

- Retain the completion pattern of Graver set computation, but work with pairs (u, V_u) instead.
- Define the two main ingredients, S-vectors and normalForm, that means the operations $\oplus$ and $\ominus$, appropriately.
- Now, the objects f, g, and s all are pairs of the form (u, V_u).

> **Algorithm** (Extended normal form algorithm)
>
> Input: a pair s, a set G of pairs
> Output: a normal form of s with respect to G
>
> while there is some $g \in G$ such that $g \sqsubseteq s$ do $\quad s := s \ominus g$
> return s

Algorithm (Compute $\mathcal{H}_\infty$)

Input: a generating set F of $\ker(A_1)$ in (u, V_u)-notation to be specified below

Output: a set G which contains $\mathcal{H}_\infty$

$G := F$

$C := \bigcup_{f,g \in G} \{f \oplus g\}$ (forming S-vectors)

while $C \neq \emptyset$ do

  $s :=$ an element in C

  $C := C \setminus \{s\}$

  $f :=$ normalForm(s, G)

  if $f \neq (0, \{0\})$ then

    $C := C \cup \bigcup_{g \in G \cup \{f\}} \{f \oplus g\}$ (adding S-vectors)

    $G := G \cup \{f\}$

return G.

Early Activities in Test-Set Methods for Stochastic Integer Programs

Bibliographical Background

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- Rekha Thomas (now UW Seattle): An Interior-Point Method for Integer Programs, Lecture at ZIB Berlin, 1995.
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- Graver sets of linear **multistage** stochastic integer programs,
- Decomposition into finitely many building blocks, independently on number of scenarios,
- Completion-type of algorithm for computing Graver sets,
- "Theory of Better-Quasi-Orderings" (Nash-Williams) used for termination proof.

Matthias Aschenbrenner, Raymond Hemmecke:
Finiteness Theorems in Stochastic Integer Programming
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Antichains of collections of monomial ideals are finite.

FINITENESS THEOREMS IN STOCHASTIC INTEGER PROGRAMMING

MATTHIAS ASCHENBRENNER AND RAYMOND HEMMECKE

Dedicated to the memory of C. St. J. A. Nash-Williams, 1932–2001.

ABSTRACT. We study Graver test sets for families of linear multi-stage stochastic integer programs with varying number of scenarios. We show that these test sets can be decomposed into finitely many "building blocks", independent of the number of scenarios, and we give an effective procedure to compute these building blocks. The paper includes an introduction to Nash-Williams' theory of better-quasi-orderings, which is used to show termination of our algorithm. We also apply this theory to finiteness results for Hilbert functions.

CONTENTS

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Key words and phrases. Test sets, Graver bases, multi-stage stochastic integer programming, decomposition methods, well-quasi-orderings.
The work of the first author was partially supported by NSF grant DMS 03-03618. The work on this paper was begun in 2001, while the first author was a Charles B. Morrey Jr. Visiting Research Professor at UC Berkeley and the second author was a Visiting Research Assistant Professor at UC Davis. The support of these institutions is gratefully acknowledged. The authors would also like to thank Bernd Sturmfels for his encouragement to write this paper.

1

Theorem.

Let $\mathcal{S}$ be a collection of monomial ideals in a polynomial ring, and let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an infinite sequence of collections of monomial ideals from $\mathcal{S}$ where each $\mathcal{M}_i$ is closed under inclusion,

(if $I \subseteq \mathcal{M}_i$ and $J \subseteq \mathcal{S}$ is a monomial ideal such that $J \subseteq I$, then $J \in \mathcal{M}_j$)

Then $\mathcal{M}_i \subseteq \mathcal{M}_j$ for some indices $i \neq j$.

i.e. no infinite antichain of $\mathcal{M}_i$.