Hamiltonian PDEs: KAM, Reducibility, Normal Forms and Applications

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1 General Informations and Structure

The workshop was attended by 37 scientists (some of them well established other young scientists) from 10 countries; 9 of them from Mexico.

The aim of the conference was to make the point on existing results and problems in Hamiltonian PDEs and their applications. One of the main focus was also to put together scientists from different communities in order to stimulate new interactions.

The communities present at the workshop can be summarized as follows: (1) Hamiltonian PDEs, namely KAM, normal form theory for PDEs, reducibility and growth of Sobolev norms; (2) Gross-Pitaevskii, namely justification of the GP equation and other effective equations and study of the their dynamical behavior, mainly in relation with experiments on cold atoms; (3) vortex filaments and fluidodynamics, namely evolution of the vortex filaments, dispersive and geometrical properties of the solution, interaction of vortex filaments, evolution of the free surface or of the interaction surface of a fluid; (4) other applications, namely models for suspended bridges and corresponding dynamics, soliton dynamics, ten martini problem, modulation equations.

Due to the presence of different communities, we decided to insert in the program some introductory talks, four, with the aim of presenting the well established (and updated) basic knowledge in each one of the communities. In particular we had a general talk on modulation equations, one on Gross-Pitaevskii equation, one on KAM theory, and one on the Ten Martini problem.

We also had the standard talks which were 45 minutes long, and sessions for young scientists who had 15 minutes each.

2 The introductory talks

Let me recall that modulation equations are a fundamental tool for the description in a simplified way of the dynamics of physical systems, like fluids. The problem of their deduction/justification is one of the most interesting in this domain. The connection with normal form theory is not yet completely clear and the possibility of exporting methods from the community of modulation equation to the community of normal form and back was one of the aim of the workshop. There have been interesting discussion on the subject, mainly in order to understand the relationship between the so called method of modified energies and normal form.

The general talks on GP equation and vortex dynamics (in many body systems) allowed to make the point on this problem. In particular it was particularly interesting to see that the analogy between the methods used in this domain and the methods of Hamiltonian perturbation theory. The domain has a history which is more than 50 years long and the actual recognition of the connection between the two fields requires more interactions. For this reason, some of the participants have already fixed further meetings. We think that this will lead to some new points of view on these problems.

Concerning KAM theory for PDEs, the aim of the general talk was to present the well established theory allowing to deal with the problems in one space dimension. Indeed there are quite a lot of cases in which more complicated systems reduce to one dimensional ones and it was important to make the point. We think that this stimulated people working in applied fields to use the methods that have been developed in this framework.

Finally we had an introductory talk on the Ten Martin problem, namely the problem of understanding the spectrum of a Schrödinger operators with a space quasiperiodic potential. This is a very important problem both from a mathematical and a physical point of view since it is a model for the understanding of the transition conductor/insulator in disordered systems. The methods used in this domain are very close to those used in Hamiltonian PDEs and furthermore they have been partially applied to some problems in the dynamics of infinite dimensional Hamiltonian systems. For this reason it was important to have a talk on this subject in this workshop.

3 Scientific discussions

We come now to some of the topics that have been the object of discussions in Oaxaca. Most of the discussion took place in small groups. No formal discussions were organized. Here we report on the discussions of which we are aware.

First, let us point out that, after the proposal was submitted, there has been some quite remarkable progress in the theory of Hamiltonian PDEs and in particular there are now more results, both on reducibility theory and long time existence, for quasilinear and for higher dimensional systems.

The talks were an occasion to make the point of some of the results on this problems, but there were quite a lot of discussions in which the participants had the occasion to share information on some works in progress.

3.1 Reducibility

We start with the problem of reducibility. The starting point is that it is well known that one can formulate the problem of existence of invariant tori as a fixed point problem. Such a fixed point problem is typically solved via the Newton algorithm. The main step for the application of the Newton algorithm consists in inverting a linear operator, which essentially is the operator obtained by linearizing the equation at an approximate quasiperiodic solution.

A particularly effective way in order to invert such an operator consists in conjugating it to a time independent diagonal operator. In turn the problem of conjugating a time dependent operator to a time independent one is called the reducibility problem.

The reducibility problem is also interesting in itself, since time dependent linear equations typically appear in stability problems and in quantum mechanics. As written in the application, in recent years a new method based on pseudodifferential calculus has been developed in order to attack this problem [1]. Again such method has shown to be very effective for problems in one space dimension. The main question is whether it can be used to study also higher dimensional problems.

This problem has been at the center of many discussions. In particular preliminary results are known, on the one side on the problem of reducibility of Schrödinger equation on Zoll Manifolds (the spectrally simplest possible higher dimensional manifolds) and on irrational tori (which are interesting mainly due to the fact that the spectrum of the Laplacian on such tori has no gaps).

On Zoll manifolds it was known how to conjugate the system to a block diagonal one, which however is still time dependent. Some results on the elimination of time were presented in informal discussions and the possibility of extending them to more general contexts, both from the point of view of the unperturbed system and for the perturbed system was largely discussed. In the future we will see if this was useful and will produce interesting new results.

Concerning irrational tori, some preliminary spectral results [2] were presented in one talk and then there were informal discussions in order to understand if it is possible to bypass the difficulties which are not yet solved. Furthermore, one of the central points is to understand if the methods developed in this context can be extended to more general manifolds. This probably requires more sophisticated methods of semiclassical analysis, so some interactions between KAM men and semiclassical men have started.

3.2 Growth of Sobolev Norms: upper bounds

A closely problem is that of understanding the rate of growth of Sobolev norms (if any) in linear time dependent systems or in nonlinear systems. The problem is also related to weak turbulence and might have interesting applications to the analysis of time-dependent Gross-Pitaevskii equation, on which we will come back later.

The point is to understand if along the dynamics there is transfer of energy from low frequency modes to high frequency modes. A powerful tool in order to get upper bounds on this phenomenon has been recently developed, again exploiting a combination of semiclassical methods (as in the reducibility problem) and functional analytic methods. The method works well for systems in one space dimensions or for example on Zoll manifolds. A very recent result (based on the extension of a clustering lemma by Bourgain) has been obtained for the Schrödinger equation on irrational tori. There were quite a lot of discussions in order to understand the connection between the two approaches and on the possibility of extending the methods to more general systems.

Very little is known on the existence of models in which there is growth of Sobolev norms. In linear time dependent systems there has been a method recently proposed which allowed to construct some examples in perturbations of quantum Harmonic oscillator. In the case of anharmonic oscillators, there are also some partial results which show some growth type behavior in the semiclassical limit of a time forced Schrödinger equation.

Concerning nonlinear systems, the stability part is well understood for semilinear systems in one space dimensions. Indeed some quite effective upper bounds on the growth of Sobolev norms are known in this case. The main two open questions concern the behavior of quasilinear systems and the behavior of systems in more then one space dimensions. Concerning quasilinear systems, recently a break through has been obtained by Berti and Delort [3] who, working on the water wave problem were able to obtain new estimates on the existence times. Their methods has been the object of several discussions. An extension has also been presented in a talk. The main point is that Berti and Delort's method is effective for reversible systems, but at present it does not allow to exploit the Hamiltonian structure of the system. During some discussions the focus was exactly on the possibility of exploiting the Hamiltonian structure in order to improve drastically the result.

Concerning equations in higher space dimensions, during informal discussions a new result was presented [4]: combining ideas from different authors (including Bourgain) and adding new ideas the long time stability of high Sobolev norms was obtained for Klein Gordon equation in arbitrary dimensions. The idea seems very powerful and probably it will be applicable to quite general systems.

3.3 Growth of Sobolev Norms: lower bounds

Concerning lower bounds on the evolution of Sobolev norms, as anticipated in the application, a break through was obtained in a work by Colliander, Keel, Staffilani, Takaoka and Tao in 2010 [7]; they showed how to construct in the cubic nonlinear Schroedinger equation on the 2 dimensional torus an orbit which starts with small Sobolev norm and reaches arbitrary high Sobolev norm. Afterwards some generalization were obtained, but some of the talks as well as the discussions that took place outside the talks, made it clear that the hot topic is now that of understanding how many orbits of this kind exist in NLS. More particularly, the main question is: are the orbits exhibiting large growth of Sobolev norms dense? Generic? Some partial steps in this direction have now been obtained, but at present it is quite difficult to guess what the final answer will be.

As I mentioned above a related problem is that of weak turbulence: several years ago Zakharov conjectured that the dynamics of waves in some nonlinear PDEs could be described through some kinetic equation, somehow in the same way as Boltzmann equation allows to describe the dynamics of a large number of interacting particles. Since that work, the problem of understanding (even heuristically) the validity of Zakharov equation (which also provides a description of how energy flows to high frequency modes) has been one of the most remarkable challenges for this Branch of mathematical physics (and analysis). At the workshop a very recent result which constitutes the first rigorous (very preliminary) justification of Zakharov equation was presented [5]. Of course there were many discussions on this subject and Zaher Hani (who also gave a talk on the subject) had many informal discussions on the subject.

3.4 Vortex Filaments and Fluidodynamics

Concerning the dynamics of vortex filaments, there were two types of results that were discussed at the workshop: the existence of quasiperiodic motions in weakly interacting filaments [8], and the dynamics of reconnecting vortex filaments. The first one represents an interesting application of KAM theory for PDEs, which shows that it has actually reached the point of a theory applicable to concrete models. The second one is really a topic of major interests in the theory of fluids. The result is that we start to have a good understanding of the phenomenon, both from the point of view of justification of effective models and from the point of vies of understanding the shape of the solutions that the equations predict. In particular I would like to mention a result that was at the center of many discussions, a result according to which any reconnection shape is actually realized in the solution of the Gross-Pitaevskii equation describing quantum vortexes [9].

Still on fluidodynamics: in the study of the evolution of the free surface of fluid over an infinitely depth bottom, there is a long standing conjecture according to which the dynamics is described for long times by an integrable equation. The conjecture originates from the fact that the formal computation of the resonant Hamiltonian normal form of the system has some vanishing coefficients. However the fact that the computation was only formal prevented a proof of the effective validity of the normal form. A remarkable result presented in the workshop is the proof of such a conjecture.

3.5 Gross-Pitaevskii Equation

Concerning the Gross-Pitaevskii equation and, more in general, the Nonlinear Schrödinger equation and their application to the physics of cold atoms, the introductory lecture was the opportunity to clarify the role of (stationary) Gross-Pitaevskii equation in the description of the ground state properties of a Bose-Einstein condensate. Furthermore, for rotating systems, the very same model allows to describe the superfluid features of condensates and, in particular, the emergence of point vortices [10]. The subsequent transitions (occurrence of a first vortex, emergence of a hole, transition to a giant vortex state) of a rotating condensate for increasing values of the angular velocities were then reviewed and the open questions mostly concerning the vortex distribution (Abrikosov lattice) were then discussed.

In this framework, a possible interplay between the mathematics of condensates and KAM theory was informally suggested concerning, for instance, the time-evolution of vortex states (e.g. with high degree) for short times and/or weak nonlinearity to observes the breaking of the rotational symmetry associated with the split of a single vortex into several singly-quantized vortices. The goal of developing a perturbation theory for the Gross-Pitaevskii equation in the semiclassical Thomas-Fermi regime, i.e., for very strong nonlinearity was also mentioned although it seemed definitely more ambitious. It was also pointed out how the propagation of Sobolev norms for the Gross-Pitaevskii equation in presence of a trapping potential would also play a relevant role in the rigorous derivation of the equation for the dynamics of a dilute Bose gas, which requires a control of the regularity of the solution to the Gross-Pitaevskii equation as time evolves.

Two talks were then devoted to the time-dependent (nonlinear) Scrödinger with concentrated point-like potential [6]. A model with time-dependent point interaction is often used as an effective model to describe the interaction of a particle with a strong electromagnetic field and it can be derived rigorously starting from a microscopic system of a particle interacting with a quantized radiation field in the semiclassical limit.

This set the stage for the discussion of a more involved model in which the singular potential is assumed to depend on the wave-functional itself, i.e., the nonlinear Schrödinger equation with concentrated nonlinearity: having established the local well-posed of the equation in dimension 2 (former results were available in dimension 1 and 3), it was natural to investigate the existence of global solution and the possible emergence of a blow-up phenomenon. Several informal discussions took place concerning this questions: the relation between the energy threshold for blow up in the focusing case and the energy of standing waves was studied, leading to a new result about the existence of two standing waves for given value of the mass, with opposite stability properties.

3.6 More on 1-d systems

Still KAM theory for PDEs is very far from being a complete theory. More generally, we can say that the theory of Integrable and close to integrable PDEs does not have a definite form. On the one hand, concerning the dynamics of integrable equations we known much. In particular during the last 20 years a quite general technique in order to introduce suitable action angle variables has been developed. The starting point being the Lax pair formulation of the problem. Accordingly it is known since several years that the gaps in the spectrum of the Lax operator provide the starting objects for the procedure. However there are some systems for which the spectrum of the Lax operator does not have objects clearly identifyed as gaps. For this reason it is not at all clear how to deal with these systems. The simplest equation which displays such a behaviour is the Benjamin Ono equation. Furthermore integrable equations in higher dimensions typically do not present spectral gaps.

During the workshop, a new result on the intagration of the Benjamin Ono equation has been presented. It is very innovative and it is based on the recognition that there is a quantity in the spectrum of the Lax operator, which plays the role of a spectral gap. The resulting theory is surprisingly simple and effective.

This also opens new perspective for the investigation of integrable equations, since it showes that the gaps are not the only relevant objects for the introduction of the action angle variables.

Concerning KAM theory for one dimensional systems, there is a still one major problem which is open: while in finite dimensional systems KAM theory allows to prove the persistence of the majority of solutions of an integrable system, in infinite dimensional systems, this is known only for finite dimensional invariant tori, objects which, althought very interesting, describe only solutions corresponding to a set of initial data which (in any sense), have measure zero.

The point is to try to extend the theory in order to be able to describe tori of infinite dimension. A break through in this problem was given by Bourgain several years ago: he proved that in a particular NLS equation there are lagrangian tori which persist under perturbation. Only in recent years, however, it has been understood that the ideas by Bourgain can be applied to much more general systems. During the workshop there was a talk on this subject and several discussions on the topic.

3.7 Other topics

At the worksho there were a few talks on subjects close to main topic of the conference. The schope was to stimulate discussions and interactions among people in order to create the background for the application of Hamiltonian techniques in different topics. We quote here the talks on suspende bridges, those on ground states, soliton stability and soliton dynamics.

Concerning suspended bridges, there was only one participant at the conference who actually works on the modellization of suspended bridges, but his presence was crucial, since he was able to interact with many people of the Hamiltonian PDEs community with whom he shared the problems that one encounter in this domain. I think that in the future this will give rise to some collaborations.

Concerning ground states, soliton stability and soliton dynamics, there were a few talks on the subject and the main point was to present new results on particular models which are very intersting either from a technical point of view or from the applicative point of view. In particular Pelinovsky presented a result on a nonlinear wave equation on higher dimensional spheres, which shows that in model, relevant for the study of general relativity, the normal forrm that occurs have particular properties and in particular it is integrable. As a consequence one can describe in a very precise way the ground state is has and its stability properties. Concerning the dynamics of "Ground states", we would like to mention also the talks by Gustaffson and the talk by Naumkin, who used techniques from the domain of dispersive equations to discuss the way these objects evolve under the dynamics.

4 Conclusion

The Workshop was an important occasion to put toghether people from different communities in order to stimulate the discussions. It is difficult to say now what will be the scientific progresses tht the meeting will stimulate, since any relevant scientific progress requires some years to take place. I would like to mention a series of meetings that were organized by the University of Nice starting from 2009 in order to put toghether the community of people dealing with paradifferential caculus and the community of Hamiltonian PDEs. The work [1] originated from this meeting and almost all the papers from the French and Italian School on the water wave problem originate from these meetings (we mean works from 2014 to 2018).

For this reason we think that it will take some time to evaluate in a concrete way the effectivness of the meeting, however, the contacts that took place were quite intense and, as we wrote in the previous part of the text, some new workshops are already scheduled for the meeting of people of these communities.

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