Disorder and topology. The cases of Floquet and of chiral systems

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Topological Phases of Interacting Quantum Systems
Casa Matemática Oaxaca
2-7 June 2019

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based on joint work with J. Shapiro, C. Tauber

Outline

Topological insulators

Chiral systems

- An experiment
- A chiral Hamiltonian and its indices

Time periodic systems

- Definitions and results
- Some numerics

Topological insulators: definition stated

- Insulator in the Bulk: Excitation gap
  For independent electrons: spectral gap at Fermi energy μ

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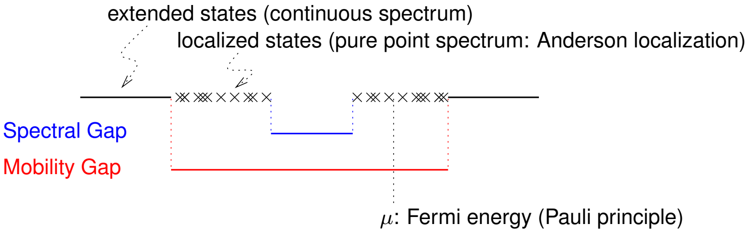
- Termination of bulk of a topological insulator implies edge states: Bulk index vs. edge index
- Refinement: The Hamiltonians enjoy a symmetry which is preserved under deformations.

The role of disorder

The spectrum of a single-particle Hamiltonian



- For a periodic (crystalline) medium:
  - Method of choice: Bloch theory and vector bundles (Thouless et al.)
  - Gap is spectral
- For a disordered medium:
  - Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
  - Fermi energy may lie in a spectral gap or (better, and more generally) in a mobility gap.

Spectral vs. Mobility gap, technically speaking

- Hamiltonian H on $\ell^2(\mathbb{Z}^d)$
- Fermi energy μ in gap
- $P_\mu = I_{(-\infty,\mu)}(H)$: Fermi projection with matrix elements $P_\mu(x, x')$, $(x, x' \in \mathbb{Z}^d)$

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Strong off-diagonal decay:

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$$\sup_{x' \in \mathbb{Z}^d} \mathrm{e}^{-\varepsilon|x'|} \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{\nu|x-x'|} |P_\mu(x, x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$).

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- Proven in (virtually) all cases where localization is known.
- Trivially false for extended states at $E = \mu$.

Periodic vs. non-periodic case

Difference illustrated for the conductance σ_{H} of (integer) quantum Hall effect (Kubo formula)

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- Periodic case. (Thouless et al., Avron)

$$\sigma_{\mathrm{H}} = -\frac{\mathrm{i}}{(2\pi)^2} \int_{\mathbb{T}} d^2k \, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$

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Remark.

$$2\pi\sigma_{\mathrm{H}} = \mathrm{ch}(P)$$

is the Chern number (index) of the vector bundle over $\mathbb{T}$ and fiber range $P(k)$

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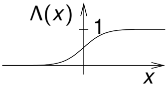
$$\sigma_{\mathrm{H}} = \mathrm{i}\, \mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

  where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ are switch functions



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- Alternative treatment of disorder (Thouless): Large, but finite system (square); $(k_1, k_2) \rightsquigarrow (\varphi_1, \varphi_2)$ phase slips in boundary conditions

An experiment: Amo et al.

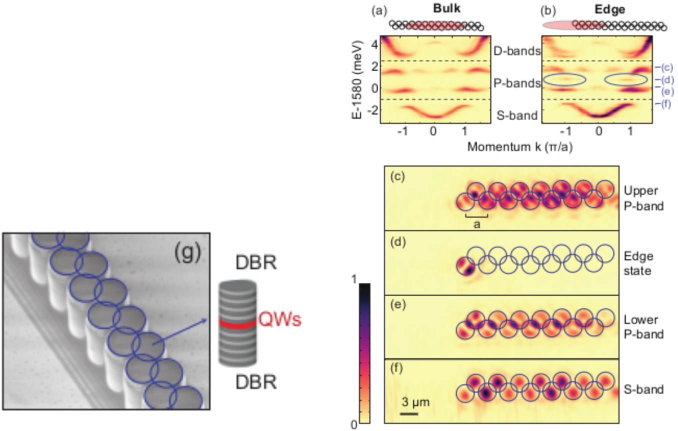

Figure: Zigzag chain of coupled micropillars and lasing modes (polaritons)

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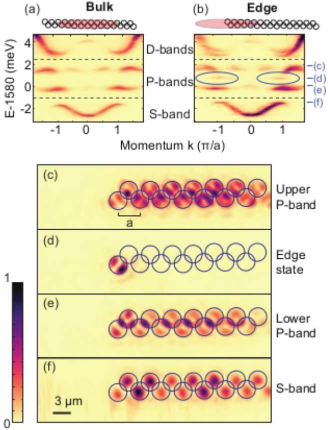


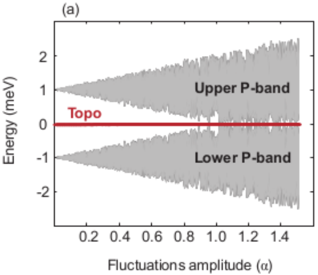
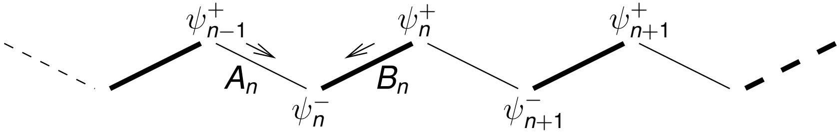


Figure: Lasing modes: bulk and edge

The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping



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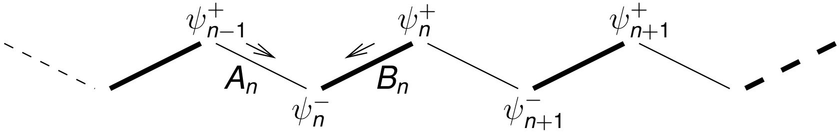

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n \in \mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n \psi^+_{n-1} + B_n \psi^+_n\,, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n \psi^-_n$$

(A_n random i.i.d. $\in \mathrm{GL}(N)$ almost surely, B_n too)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$H\psi = \lambda\psi \quad \Longrightarrow \quad H(\Pi\psi) = -\lambda(\Pi\psi)$$

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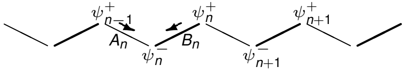


Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

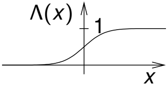
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

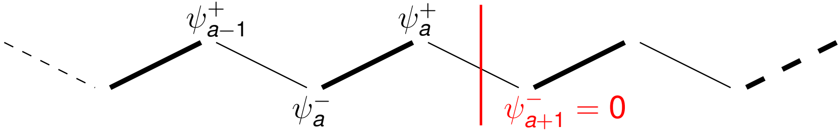
Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2}\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma])$$



with $\Lambda = \Lambda(n)$ a switch function (cf. Prodan et al.)

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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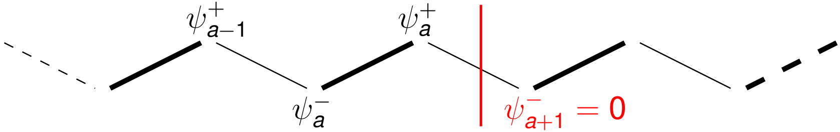

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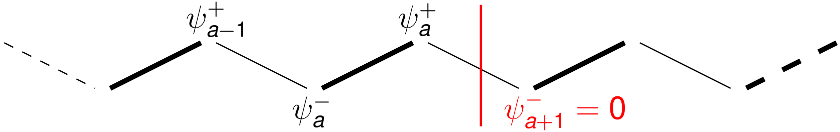

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$$\mathcal{N}_a^{\pm} := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

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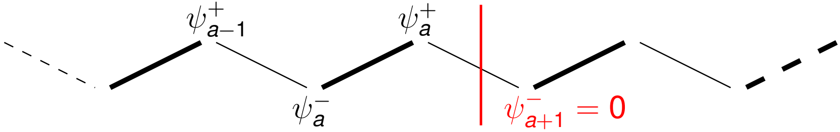

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Definition. The Edge index is

$$\mathcal{N}_a = \mathcal{N}^+_a - \mathcal{N}^-_a$$

and can be shown to be independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\sharp$$

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Remarks.

- Spectral gap case ($0 \notin \sigma_{\mathrm{ess}}(H) \supset \sigma_{\mathrm{ess}}(H_a)$)

$$H_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{N}_a^\sharp := \dim\ker S_a - \dim\ker S_a^* = \operatorname{ind} S_a \qquad \text{(Fredholm index)}$$

Bulk-edge duality by Schulz-Baldes. In mobility gap case, S_a is not Fredholm.

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- Periodic case

$$S = \int_{S^1}^{\oplus} S(k)$$

  Toeplitz index theorem:

$$\mathcal{N}^\sharp = -\mathrm{Wind}(k \mapsto \det S(k))$$

Bulk-edge duality: Lyapunov exponents

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Remark. Consider the dynamical system $A_n \psi^+_{n-1} + B_n \psi^+_n = 0$ with Lyaponov exponents

$$\gamma_1 \geq \dots \geq \gamma_N$$

The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\sharp = \sharp\{i \mid \gamma_i > 0\}$.

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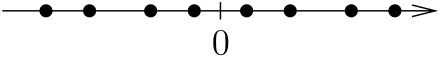
Lyapunov spectrum of the full chain has $2N$ exponents, spectrum is even (Example: $N = 4$)

- at energy $\lambda \neq 0$ (simple spectrum)



  - Spectrum is simple because measure on transfer matrices is irreducible
  - so $\gamma = 0$ is not in the spectrum; localization follows

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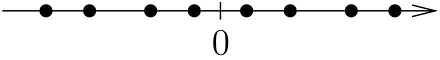

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces

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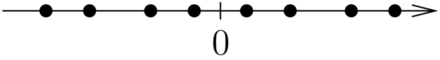
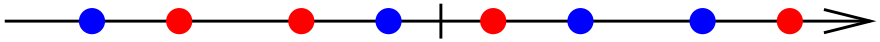
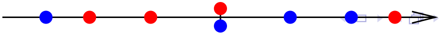


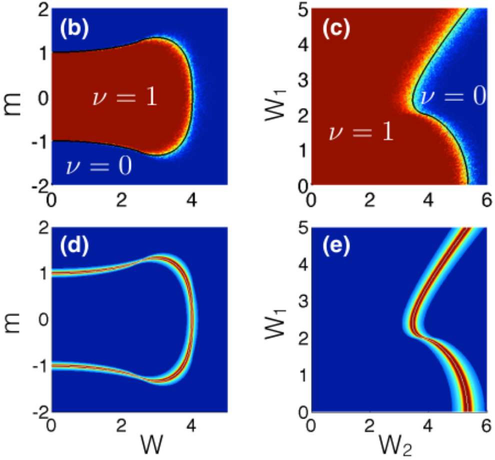
- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$



- at energy $\lambda = 0$ (phase boundary)



Some numerics



Left/right column: two parameterized chiral models ($N = 1$)
upper/lower row: index and Lyapunov exponent (from Prodan et al.)

Proof

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi P_{0,a})$, where

$$1 = P_{0,a} + P_{+,a} + P_{-,a}$$

is decomposition into states of energies $= 0, > 0, < 0$

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$$\mathcal{N}^\sharp = \lim_{a \to +\infty} \mathrm{tr}(\Pi \Lambda P_{0,a})$$

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\mathrm{tr}(\Pi \Lambda) = N\Big(\sum_{n \leq a} \Lambda(n)\Big)\, \mathrm{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi\Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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$$\begin{aligned} \mathrm{tr}(\Pi \Lambda P_{+,a}) &= \mathrm{tr}(P_{+,a} \Pi \Lambda P_{+,a}) = \mathrm{tr}(\Pi P_{-,a} \Lambda P_{+,a}) \\ &= \mathrm{tr}(\Pi P_{-,a} [\Lambda, P_{+,a}]) \end{aligned}$$

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$$\mathcal{N}^\sharp = \lim_{a \to +\infty} \operatorname{tr}(\Pi \Lambda P_{0,a})$$

Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi \Lambda) = 0$$

$$\operatorname{tr}(\Pi\Lambda) = \operatorname{tr}(\Pi\Lambda P_{0,a}) + \operatorname{tr}(\Pi\Lambda P_{+,a}) + \operatorname{tr}(\Pi\Lambda P_{-,a})$$

$$\begin{aligned} \operatorname{tr}(\Pi\Lambda P_{+,a}) &= \operatorname{tr}(P_{+,a}\Pi\Lambda P_{+,a}) = \operatorname{tr}(\Pi P_{-,a}\Lambda P_{+,a}) \\ &= \operatorname{tr}(\Pi P_{-,a}[\Lambda, P_{+,a}]) \to \operatorname{tr}(\Pi P_{-}[\Lambda, P_{+}]) \qquad (a \to +\infty) \end{aligned}$$

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In fact by $\Sigma = P_+ - P_-$ the last expression is

$$-(1/2)\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma]) = -\mathcal{N}$$

q.e.d.

Floquet topological insulators

$H = H(t)$ (bulk) Hamiltonian in the plane with period T

$$H(t + T) = H(t)$$

(disorder allowed, no adiabatic setting)

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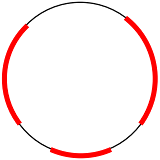
Assumption: Spectrum of $\widehat{U}$ has gaps:



spec $\widehat{U} \subset S^1$

Bulk index

Special case first: $U(t)$ periodic, i.e.

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with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$

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Remark. Extends the formula for the periodic case (Rudner et al.)

$$\mathcal{N}_{\mathrm{B}} = \frac{1}{8\pi^2} \int_0^T dt \int_{\mathbb{T}} d^2k \, \mathrm{tr}(U^* \partial_t U [U^* \partial_1 U, U^* \partial_2 U])$$

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Bulk index $\mathcal{N}_{\mathrm{B}}$ is degree of map.

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$H_{\mathrm{E}}(t)$ restriction of $H(t)$ to right half-space $x_1 > 0$

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Remarks.

- The trace is well-defined

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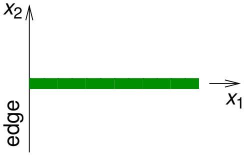

- $\mathcal{N}_{\mathrm{E}}$ is charge that crossed the line $x_2 = 0$ during a period.
- $\mathcal{N}_{\mathrm{E}}$ is independent of Λ_2 and an integer.

General case: Pair of Hamiltonians

$$\widehat{U} \neq 1$$

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Theorem (G., Tauber) $\mathcal{N} = \mathcal{N}_{\mathrm{E}}$

Duality in time and space

Let the interface Hamiltonian $H_{\mathrm{I}}(t)$ be a bulk Hamiltonian with

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Interface index

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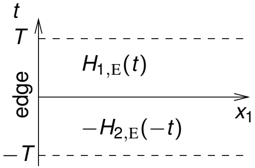

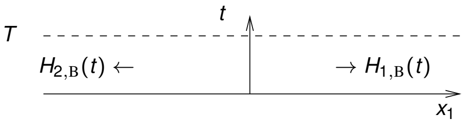

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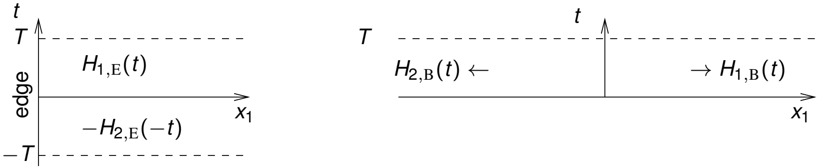
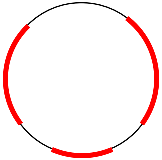


Theorem (G., Tauber) The indices for the two diagrams agree:

$$(\mathcal{N} =)\mathcal{N}_{\mathrm{E}} = \mathcal{N}_{\mathrm{I}}$$

Back to single Hamiltonian

$\widehat{U} \neq 1$



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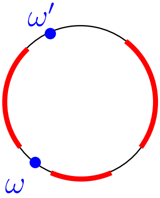

Let $\alpha \in \mathbb{R}$ and $\omega = \mathrm{e}^{\mathrm{i}\alpha}$. For $z \notin \omega\mathbb{R}_+$ (ray) define the branch

$$\log_\alpha z = \log|z| + \mathrm{i}\arg_\alpha z$$

by $\alpha - 2\pi < \arg_\alpha z < \alpha$.

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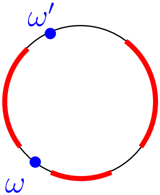

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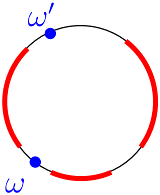

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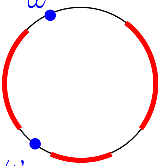

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- $U_{\alpha+2\pi}(t) = U_\alpha(t)\mathrm{e}^{2\pi\mathrm{i}t/T}$

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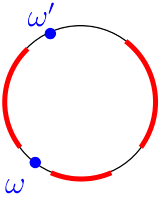

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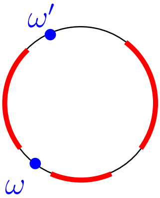

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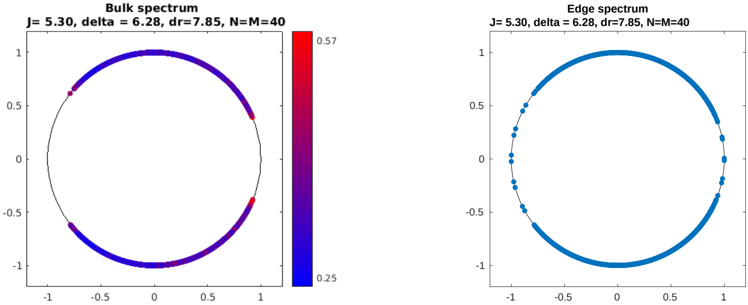
Theorem (Rudner et al.; G., Tauber) For ω, ω' in gaps

$$\mathcal{N}_{\omega'} - \mathcal{N}_{\omega} = \mathrm{i}\,\mathrm{tr}\, P\big[[P, \Lambda_1], [P, \Lambda_2]\big]$$

where $P = P_{\omega,\omega'}$ is the spectral projection associated with $\mathrm{spec}\,\widehat{U}$ between ω, ω' (counter-clockwise)

Bulk and Edge spectrum



Bulk (left) and Edge spectrum (right); color: participation ratio

Computing the edge index

Edge index $\mathcal{N}_{\mathrm{E},\alpha}$ based on the pair (H, H_α) (with $\alpha = \pi$)

$$\mathcal{N}_{\mathrm{E},\alpha} = \operatorname{tr} A \qquad A = \widehat{U}_{\mathrm{E}}^* \Lambda_2 \widehat{U}_{\mathrm{E}} - \widehat{U}_{\alpha,\mathrm{E}}^* \Lambda_2 \widehat{U}_{\alpha,\mathrm{E}}$$

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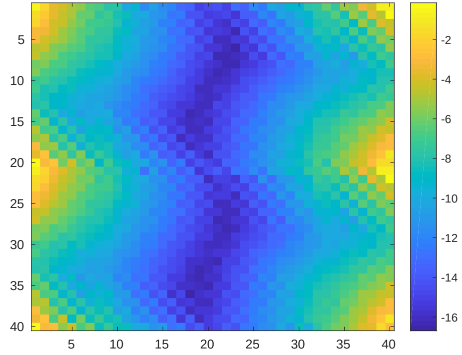
The diagonal integral kernel $A(x,x)$ as $\log|A(x,x)|$



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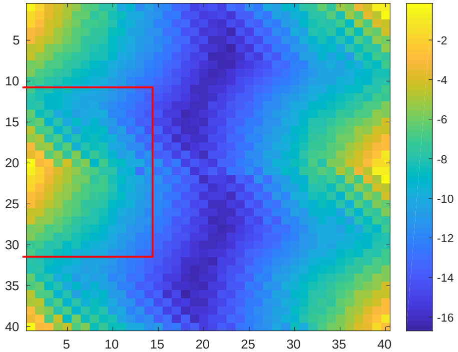
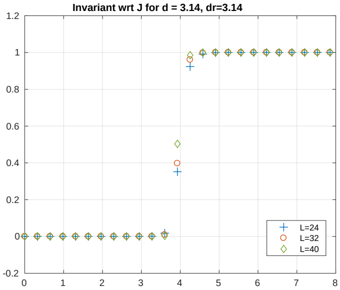
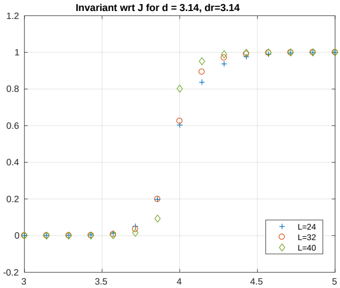


Boundary conditions:

- Vertical edges: Dirichlet
- Horizontal edges: Periodic

The transition





Edge index (left) and zoom (right)

Integer detected with 1 part in 10^{12}

Summary

- Chiral symmetry
- Floquet topological insulator

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Thank you for your attention!