THE SYMMETRY RULE FOR QUANTIFIED BOOLEAN FORMULAS

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Quantified Boolean Formulas

- **Syntax**

$$\underbrace{\exists x \exists y \forall u \exists z.}_{\text{prefix}}\underbrace{(\overset{\text{literals}}{\overset{\downarrow}{\neg u}} \lor \overset{\downarrow}{z}) \land \overbrace{(y \lor u \lor \neg z)}^{\text{clause}} \land (x \lor \neg u \lor \neg z)}_{\text{propositional matrix in CNF}}$$

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- **Semantics**

  - $\forall x \mathcal{Q}.\varphi$ true $\Leftrightarrow$ $\mathcal{Q}.\varphi[x]$ **and** $\mathcal{Q}.\varphi[\neg x]$ true
  - $\exists x \mathcal{Q}.\varphi$ true $\Leftrightarrow$ $\mathcal{Q}.\varphi[x]$ **or** $\mathcal{Q}.\varphi[\neg x]$ true

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Note: prefix imposes ordering $x, y < u < z$

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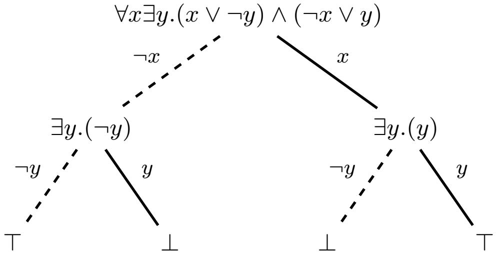

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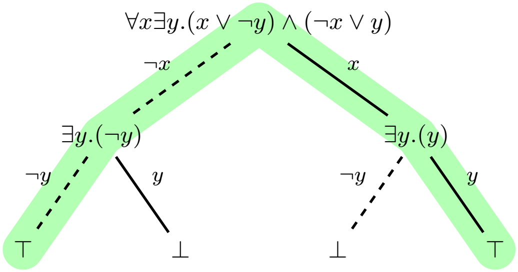
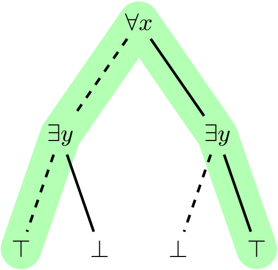


QBF Models

Tree model of **true** formula:

$$\forall x \exists y.(x \vee \bar{y}) \wedge (\bar{x} \vee y)$$



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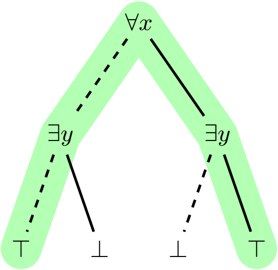

Tree refutation of **false** formula:

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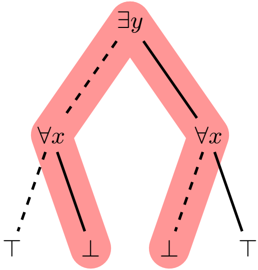

QBF Models

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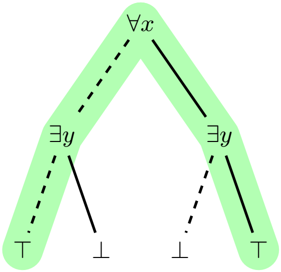

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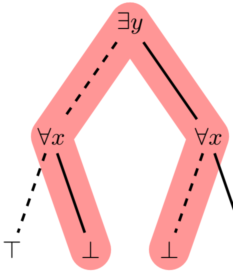

Skolem-functions of ∃-variables:
$f_y(x) = x$

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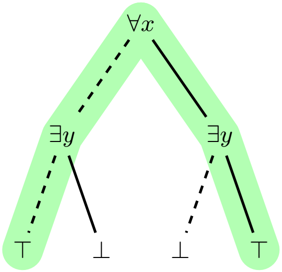

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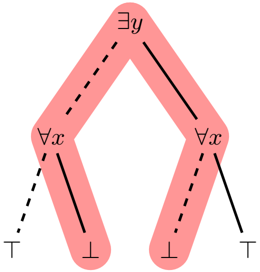

Herbrand-functions of ∀-variables:
$f_x(y) = \bar{y}$

Symmetries

A **symmetry** σ for a QBF $Q_1 x_1 \dots Q_n x_n.\varphi$ has the following properties:

- σ is a bijective function mapping literals of φ to literals of φ
- $\overline{\sigma(l)} = \sigma(\bar{l})$
- if applied on all literal occurrences of φ, then $\varphi = \sigma(\varphi)$
- the variables of l, $\sigma(l)$ belong to the same quantifier block

Example

$$\mathcal{Q}.\varphi = \forall x \exists y.((x \vee y) \wedge (\bar{x} \vee \bar{y}))$$

has the symmetry

$$\sigma(x) = \bar{x}, \sigma(\bar{x}) = x, \sigma(y) = \bar{y}, \sigma(\bar{y}) = y$$

$$\varphi = \sigma(\varphi) = ((\sigma(x) \vee \sigma(y)) \wedge (\sigma(\bar{x}) \vee \sigma(\bar{y}))) = ((\bar{x} \vee \bar{y}) \wedge (x \vee y))$$

Symmetry Example: KBFK-Formulas

For $n \in \mathbb{N}$, the formula KBKF_n is defined by the prefix

$$\exists x_1 y_1 \forall a_1 \exists x_2 y_2 \forall a_2 \ldots \exists x_n y_n \forall a_n \exists z_1 \ldots z_n$$

and the following clauses:

- $C_1 = (\bar{x}_1 \vee \bar{y}_1)$
- for $j = 1, \ldots, n-1$:
  $C_{2j} = (x_j \vee \bar{a}_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1})$
  $C_{2j+1} = (y_j \vee a_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1})$.
- $C_{2n} = (x_n \vee \bar{a}_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n)$,
  $C_{2n+1} = (y_n \vee a_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n)$
- for $j = 1, \ldots, n$:
  $B_{2j-1} = (a_j \vee z_j)$ and $B_{2j} = (\bar{a}_j \vee z_j)$.

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Symmetry:
$\sigma_i = (x_i \; y_i) \; (\bar{x}_i \; \bar{y}_i) \; (a_i \; \bar{a}_i)$

Symmetry Example: Parity Formulas

For $n \in \mathbb{N}$ with $n > 1$, the formula $\mathrm{QPARITY}_n$ is defined by the prefix

$$\exists x_1 \dots x_n \forall a \exists y_2 \dots y_n$$

and the following clauses:

- $A_2 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2)$
  $B_2 = (\bar{x}_1 \vee x_2 \vee y_2)$
  $C_2 = (x_1 \vee \bar{x}_2 \vee y_2)$
  $D_2 = (x_1 \vee x_2 \vee \bar{y}_2)$
- for $j = 3, \dots, n$:
  $A_j = (\bar{y}_{j-1} \vee \bar{x}_j \vee \bar{y}_j)$
  $B_j = (\bar{y}_{j-1} \vee x_j \vee y_j)$
  $C_j = (y_{j-1} \vee \bar{x}_j \vee y_j)$
  $D_j = (y_{j-1} \vee x_j \vee \bar{y}_j)$
- $E_1 = (a \vee y_n)$ and $E_2 = (\bar{a} \vee \bar{y}_n)$

$$A_2 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2)$$
$$B_2 = (\bar{x}_1 \vee x_2 \vee y_2)$$
$$C_2 = (x_1 \vee \bar{x}_2 \vee y_2)$$
$$D_2 = (x_1 \vee x_2 \vee \bar{y}_2)$$
$$A_3 = (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3)$$
$$B_3 = (\bar{y}_2 \vee x_3 \vee y_3)$$
$$C_3 = (y_2 \vee \bar{x}_3 \vee y_3)$$
$$D_3 = (y_2 \vee x_3 \vee \bar{y}_3)$$
$$E_1 = (a \vee y_3)$$
$$E_2 = (\bar{a} \vee \bar{y}_3)$$

Symmetries: $\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$ and
$\sigma_i = (x_i\ \bar{x}_i)(a\ \bar{a})(y_i\ \bar{y}_i) \cdots (y_n\ \bar{y}_n)$ for every $i = 2, \dots, n$.

Resolution for QBF

Let $\mathcal{Q}.\varphi$ be a QBF. The Q-Res calculus consists of the following rules:

Resolution Rule (R):

$C_1 \lor x,\ \ C_2 \lor \bar{x} \ \xrightarrow{\text{R}}\ C_1 \lor C_2$ — $C_1 \lor x, C_2 \lor \bar{x}$ already derived; x is existential

Universal Reduction (U):

$D \lor l \ \xrightarrow{\text{U}}\ D$ — $D \lor l$ already derived; l is universal; for all existential literals $k \in D$: $k < l$

QBF $\mathcal{Q}.\varphi$ is false $\Leftrightarrow$ The empty clause can be derived by the rules of Q-Res

Example: $\exists x \forall y.((x \lor y) \land (\bar{x} \lor \bar{y}))$

$$x \lor y \ \xrightarrow{\text{U}}\ x$$
$$\bar{x} \lor \bar{y} \ \xrightarrow{\text{U}}\ \bar{x}$$
$$x,\ \ \bar{x} \ \xrightarrow{\text{R}}\ \epsilon$$

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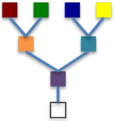
Overview: Solving Approaches for QBF

Q-Resolution

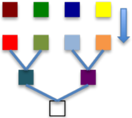
[Kleine Büning, Karpinski, Flögel, 95]



- QCDCL

Expansion

[Beyersdorff, Chew, Janota, 14]



- CEGAR

Interference

[Heule, Seidl, Biere, 14]

- Preprocessing

- no short proofs for QPARITY and KBKF in Q-Res
- short proofs for KBKF in extensions of Q-Res
- short proofs for QPARITY in $\forall$-Exp+Res

The Symmetry Rule: Q-Res+S

Let $\mathcal{Q}.\varphi$ be a QBF. The Q-Res calculus consists of the following rules:

Resolution Rule (R):

$C_1 \vee x,\ \ C_2 \vee \bar{x} \ \stackrel{\text{R}}{\longrightarrow}\ C_1 \vee C_2$

$C_1 \vee x, C_2 \vee \bar{x}$ already derived
x is existential

Universal Reduction (U):

$D \vee l \ \stackrel{\text{U}}{\longrightarrow}\ D$

$D \vee l$ already derived
l is universal
for all existential literals $k \in D$: $k < l$

Symmetry Rule (S):

$E,\ \sigma \ \stackrel{\text{S}}{\longrightarrow}\ \sigma(E)$

C already derived
σ is a symmetry

Properties of the Symmetry Rule

- QBF $\mathcal{Q}.\varphi$ is false $\Leftrightarrow$
  The empty clause can be derived by the rules of Q-Res+S
  - Observation 1: if clause C can be derived by $C_1,\ C_2 \xrightarrow{\text{R}} C$, then $\sigma(C)$ can be derived by $\sigma(C_1),\ \sigma(C_2) \xrightarrow{\text{R}} \sigma(C)$
  - Observation 2: if clause D can be derived by $D_1 \xrightarrow{\text{U}} D$, then $\sigma(D)$ can be derived by $\sigma(D_1) \xrightarrow{\text{U}} \sigma(D)$
  - Observation 3: if $E \in \varphi$, then $\sigma(E) \in \varphi$

  Example $\exists x \forall y.((x \vee y) \wedge (\bar{x} \vee \bar{y}))$
  with symmetry $\sigma(x) = \bar{x}, \sigma(\bar{x}) = x, \sigma(y) = \bar{y}, \sigma(\bar{y}) = yx$

$$x \vee y \xrightarrow{\text{U}} x$$
$$x,\ \sigma \xrightarrow{\text{S}} \bar{x} \quad \text{instead of} \quad \sigma(x \vee y) \xrightarrow{\text{U}} \sigma(x)x$$
$$x,\ \bar{x} \xrightarrow{\text{R}} \epsilon$$

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$$
\begin{array}{lll}
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x, \; \sigma \;\xrightarrow{\text{S}}\; \bar{x} & \text{instead of} & \sigma(x \vee y) \;\xrightarrow{\text{U}}\; \sigma(x)x \\
x, \; \bar{x} \;\xrightarrow{\text{R}}\; \epsilon & &
\end{array}
$$

- shorter proofs!

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Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \dots y_3$

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$E_1 = (a \vee y_3)$
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Symmetries:
$\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$
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Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \ldots y_3$

$$a \vee y_3, \quad D_3 \quad \xrightarrow{\mathrm{R}} \quad a \vee y_2 \vee x_3$$

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$$a \vee y_3,\ D_3 \xrightarrow{\mathrm{R}} a \vee y_2 \vee x_3$$
$$a \vee y_2 \vee x_3,\ D_2 \xrightarrow{\mathrm{R}} a \vee x_1 \vee x_2 \vee x_3$$
$$a \vee x_1 \vee x_2 \vee x_3 \xrightarrow{\mathrm{U}} x_1 \vee x_2 \vee x_3$$

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C_2 &= (x_1 \vee \bar{x}_2 \vee y_2)\\
D_2 &= (x_1 \vee x_2 \vee \bar{y}_2)\\
A_3 &= (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3)\\
B_3 &= (\bar{y}_2 \vee x_3 \vee y_3)\\
C_3 &= (y_2 \vee \bar{x}_3 \vee y_3)\\
D_3 &= (y_2 \vee x_3 \vee \bar{y}_3)\\
E_1 &= (a \vee y_3)\\
E_2 &= (\bar{a} \vee \bar{y}_3)
\end{aligned}
$$

$$
\begin{aligned}
a \vee y_3,\ D_3 &\xrightarrow{\mathrm{R}} a \vee y_2 \vee x_3\\
a \vee y_2 \vee x_3,\ D_2 &\xrightarrow{\mathrm{R}} a \vee x_1 \vee x_2 \vee x_3\\
a \vee x_1 \vee x_2 \vee x_3 &\xrightarrow{\mathrm{U}} x_1 \vee x_2 \vee x_3\\
x_1 \vee x_2 \vee x_3,\ \sigma_3 &\xrightarrow{\mathrm{S}} x_1 \vee x_2 \vee \bar{x}_3\\
x_1 \vee x_2 \vee x_3,\ x_1 \vee x_2 \vee \bar{x}_3 &\xrightarrow{\mathrm{R}} x_1 \vee x_2
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
\sigma_1 &= (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)\\
\sigma_2 &= (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)\\
\sigma_3 &= (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)
\end{aligned}
$$

Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \dots y_3$

$$A_2 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2)$$
$$B_2 = (\bar{x}_1 \vee x_2 \vee y_2)$$
$$C_2 = (x_1 \vee \bar{x}_2 \vee y_2)$$
$$D_2 = (x_1 \vee x_2 \vee \bar{y}_2)$$
$$A_3 = (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3)$$
$$B_3 = (\bar{y}_2 \vee x_3 \vee y_3)$$
$$C_3 = (y_2 \vee \bar{x}_3 \vee y_3)$$
$$D_3 = (y_2 \vee x_3 \vee \bar{y}_3)$$
$$E_1 = (a \vee y_3)$$
$$E_2 = (\bar{a} \vee \bar{y}_3)$$

$$a \vee y_3,\ \ D_3 \ \xrightarrow{\mathrm{R}}\ a \vee y_2 \vee x_3$$
$$a \vee y_2 \vee x_3,\ \ D_2 \ \xrightarrow{\mathrm{R}}\ a \vee x_1 \vee x_2 \vee x_3$$
$$a \vee x_1 \vee x_2 \vee x_3 \ \xrightarrow{\mathrm{U}}\ x_1 \vee x_2 \vee x_3$$
$$x_1 \vee x_2 \vee x_3,\ \ \sigma_3 \ \xrightarrow{\mathrm{S}}\ x_1 \vee x_2 \vee \bar{x}_3$$
$$x_1 \vee x_2 \vee x_3,\ \ x_1 \vee x_2 \vee \bar{x}_3 \ \xrightarrow{\mathrm{R}}\ x_1 \vee x_2$$
$$x_1 \vee x_2,\ \ \sigma_2 \ \xrightarrow{\mathrm{S}}\ x_1 \vee \bar{x}_2$$

Symmetries:

$$\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$$
$$\sigma_2 = (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)$$
$$\sigma_3 = (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)$$

Example: QPARITY_3

$\exists x_1 x_2 x_3 \forall a \exists y_2 \ldots y_3$

$$A_2 = (\bar{x}_1 \lor \bar{x}_2 \lor \bar{y}_2)$$
$$B_2 = (\bar{x}_1 \lor x_2 \lor y_2)$$
$$C_2 = (x_1 \lor \bar{x}_2 \lor y_2)$$
$$D_2 = (x_1 \lor x_2 \lor \bar{y}_2)$$
$$A_3 = (\bar{y}_2 \lor \bar{x}_3 \lor \bar{y}_3)$$
$$B_3 = (\bar{y}_2 \lor x_3 \lor y_3)$$
$$C_3 = (y_2 \lor \bar{x}_3 \lor y_3)$$
$$D_3 = (y_2 \lor x_3 \lor \bar{y}_3)$$
$$E_1 = (a \lor y_3)$$
$$E_2 = (\bar{a} \lor \bar{y}_3)$$

$$a \lor y_3,\ \ D_3 \xrightarrow{\text{R}} a \lor y_2 \lor x_3$$
$$a \lor y_2 \lor x_3,\ \ D_2 \xrightarrow{\text{R}} a \lor x_1 \lor x_2 \lor x_3$$
$$a \lor x_1 \lor x_2 \lor x_3 \xrightarrow{\text{U}} x_1 \lor x_2 \lor x_3$$
$$x_1 \lor x_2 \lor x_3,\ \ \sigma_3 \xrightarrow{\text{S}} x_1 \lor x_2 \lor \bar{x}_3$$
$$x_1 \lor x_2 \lor x_3,\ \ x_1 \lor x_2 \lor \bar{x}_3 \xrightarrow{\text{R}} x_1 \lor x_2$$
$$x_1 \lor x_2,\ \ \sigma_2 \xrightarrow{\text{S}} x_1 \lor \bar{x}_2$$
$$x_1 \lor x_2,\ \ x_1 \lor \bar{x}_2 \xrightarrow{\text{R}} x_1$$

Symmetries:

$$\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$$
$$\sigma_2 = (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)$$
$$\sigma_3 = (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)$$

Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \dots y_3$

$$
\begin{aligned}
A_2 &= (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2) \\
B_2 &= (\bar{x}_1 \vee x_2 \vee y_2) \\
C_2 &= (x_1 \vee \bar{x}_2 \vee y_2) \\
D_2 &= (x_1 \vee x_2 \vee \bar{y}_2) \\
A_3 &= (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3) \\
B_3 &= (\bar{y}_2 \vee x_3 \vee y_3) \\
C_3 &= (y_2 \vee \bar{x}_3 \vee y_3) \\
D_3 &= (y_2 \vee x_3 \vee \bar{y}_3) \\
E_1 &= (a \vee y_3) \\
E_2 &= (\bar{a} \vee \bar{y}_3)
\end{aligned}
$$

$$
\begin{aligned}
&a \vee y_3,\ D_3 \xrightarrow{\mathrm{R}} a \vee y_2 \vee x_3 \\
&a \vee y_2 \vee x_3,\ D_2 \xrightarrow{\mathrm{R}} a \vee x_1 \vee x_2 \vee x_3 \\
&a \vee x_1 \vee x_2 \vee x_3 \xrightarrow{\mathrm{U}} x_1 \vee x_2 \vee x_3 \\
&x_1 \vee x_2 \vee x_3,\ \sigma_3 \xrightarrow{\mathrm{S}} x_1 \vee x_2 \vee \bar{x}_3 \\
&x_1 \vee x_2 \vee x_3,\ x_1 \vee x_2 \vee \bar{x}_3 \xrightarrow{\mathrm{R}} x_1 \vee x_2 \\
&x_1 \vee x_2,\ \sigma_2 \xrightarrow{\mathrm{S}} x_1 \vee \bar{x}_2 \\
&x_1 \vee x_2,\ x_1 \vee \bar{x}_2 \xrightarrow{\mathrm{R}} x_1 \\
&x_1,\ \sigma_1 \xrightarrow{\mathrm{S}} x_2
\end{aligned}
$$

Symmetries:

$\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$

$\sigma_2 = (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)$

$\sigma_3 = (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)$

Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \ldots y_3$

$$
\begin{aligned}
A_2 &= (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2)\\
B_2 &= (\bar{x}_1 \vee x_2 \vee y_2)\\
C_2 &= (x_1 \vee \bar{x}_2 \vee y_2)\\
D_2 &= (x_1 \vee x_2 \vee \bar{y}_2)\\
A_3 &= (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3)\\
B_3 &= (\bar{y}_2 \vee x_3 \vee y_3)\\
C_3 &= (y_2 \vee \bar{x}_3 \vee y_3)\\
D_3 &= (y_2 \vee x_3 \vee \bar{y}_3)\\
E_1 &= (a \vee y_3)\\
E_2 &= (\bar{a} \vee \bar{y}_3)
\end{aligned}
$$

$$
\begin{aligned}
&a \vee y_3,\ D_3 \xrightarrow{\mathrm{R}} a \vee y_2 \vee x_3\\
&a \vee y_2 \vee x_3,\ D_2 \xrightarrow{\mathrm{R}} a \vee x_1 \vee x_2 \vee x_3\\
&a \vee x_1 \vee x_2 \vee x_3 \xrightarrow{\mathrm{U}} x_1 \vee x_2 \vee x_3\\
&x_1 \vee x_2 \vee x_3,\ \sigma_3 \xrightarrow{\mathrm{S}} x_1 \vee x_2 \vee \bar{x}_3\\
&x_1 \vee x_2 \vee x_3,\ x_1 \vee x_2 \vee \bar{x}_3 \xrightarrow{\mathrm{R}} x_1 \vee x_2\\
&x_1 \vee x_2,\ \sigma_2 \xrightarrow{\mathrm{S}} x_1 \vee \bar{x}_2\\
&x_1 \vee x_2,\ x_1 \vee \bar{x}_2 \xrightarrow{\mathrm{R}} x_1\\
&x_1,\ \sigma_1 \xrightarrow{\mathrm{S}} x_2\\
&x_2,\ \sigma_2 \xrightarrow{\mathrm{S}} \bar{x}_2
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
\sigma_1 &= (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)\\
\sigma_2 &= (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)\\
\sigma_3 &= (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)
\end{aligned}
$$

Example: $\mathrm{QPARITY}_3$

$\exists x_1 x_2 x_3 \forall a \exists y_2 \ldots y_3$

$$
\begin{aligned}
A_2 &= (\bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_2) \\
B_2 &= (\bar{x}_1 \vee x_2 \vee y_2) \\
C_2 &= (x_1 \vee \bar{x}_2 \vee y_2) \\
D_2 &= (x_1 \vee x_2 \vee \bar{y}_2) \\
A_3 &= (\bar{y}_2 \vee \bar{x}_3 \vee \bar{y}_3) \\
B_3 &= (\bar{y}_2 \vee x_3 \vee y_3) \\
C_3 &= (y_2 \vee \bar{x}_3 \vee y_3) \\
D_3 &= (y_2 \vee x_3 \vee \bar{y}_3) \\
E_1 &= (a \vee y_3) \\
E_2 &= (\bar{a} \vee \bar{y}_3)
\end{aligned}
$$

$$
\begin{aligned}
& a \vee y_3,\ \ D_3 \xrightarrow{\mathrm{R}} a \vee y_2 \vee x_3 \\
& a \vee y_2 \vee x_3,\ \ D_2 \xrightarrow{\mathrm{R}} a \vee x_1 \vee x_2 \vee x_3 \\
& a \vee x_1 \vee x_2 \vee x_3 \xrightarrow{\mathrm{U}} x_1 \vee x_2 \vee x_3 \\
& x_1 \vee x_2 \vee x_3,\ \ \sigma_3 \xrightarrow{\mathrm{S}} x_1 \vee x_2 \vee \bar{x}_3 \\
& x_1 \vee x_2 \vee x_3,\ \ x_1 \vee x_2 \vee \bar{x}_3 \xrightarrow{\mathrm{R}} x_1 \vee x_2 \\
& x_1 \vee x_2,\ \ \sigma_2 \xrightarrow{\mathrm{S}} x_1 \vee \bar{x}_2 \\
& x_1 \vee x_2,\ \ x_1 \vee \bar{x}_2 \xrightarrow{\mathrm{R}} x_1 \\
& x_1,\ \ \sigma_1 \xrightarrow{\mathrm{S}} x_2 \\
& x_2,\ \ \sigma_2 \xrightarrow{\mathrm{S}} \bar{x}_2 \\
& x_2,\ \ \bar{x}_2 \xrightarrow{\mathrm{R}} \epsilon
\end{aligned}
$$

Symmetries:

$\sigma_1 = (x_1\ x_2)(\bar{x}_1\ \bar{x}_2)$

$\sigma_2 = (x_2\ \bar{x}_2)(a\ \bar{a})(y_2\ \bar{y}_2)(y_3\ \bar{y}_3)$

$\sigma_3 = (x_3\ \bar{x}_3)(a\ \bar{a})(y_3\ \bar{y}_3)$

Hindering the Symmetry Rule

For $n \in \mathbb{N}$, the formula KBKF'_n is defined by the prefix

$$\exists x_1 \forall u_1 \exists y_1 \forall a_1 \exists x_2 \forall u_2 \exists y_2 \forall a_2 \ldots \exists x_n \forall u_n \forall y_n \forall a_n \exists z_1 \ldots z_n$$

and the following clauses:

- $C_1 = (\bar{x}_1 \vee \bar{y}_1)$
- for $j = 1, \ldots, n-1$:
  $C_{2j} = (x_j \vee \bar{a}_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1} \vee u_j)$
  $C_{2j+1} = (y_j \vee a_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1})$.
- $C_{2n} = (x_n \vee \bar{a}_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n \vee u_n)$,
  $C_{2n+1} = (y_n \vee a_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n)$
- for $j = 1, \ldots, n$:
  $B_{2j-1} = (a_j \vee z_j)$ and $B_{2j} = (\bar{a}_j \vee z_j)$.

Hindering the Symmetry Rule

For $n \in \mathbb{N}$, the formula KBKF'_n is defined by the prefix

$$\exists x_1 \forall u_1 \exists y_1 \forall a_1 \exists x_2 \forall u_2 \exists y_2 \forall a_2 \ldots \exists x_n \forall u_n \forall y_n \forall a_n \exists z_1 \ldots z_n$$

and the following clauses:

- $C_1 = (\bar{x}_1 \vee \bar{y}_1)$
- for $j = 1, \ldots, n-1$:
  $C_{2j} = (x_j \vee \bar{a}_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1} \vee u_j)$
  $C_{2j+1} = (y_j \vee a_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1})$.
- $C_{2n} = (x_n \vee \bar{a}_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n \vee u_n)$,
  $C_{2n+1} = (y_n \vee a_n \vee \bar{z}_1 \vee \ldots \vee \bar{z}_n)$
- for $j = 1, \ldots, n$:
  $B_{2j-1} = (a_j \vee z_j)$ and $B_{2j} = (\bar{a}_j \vee z_j)$.

$\sigma_i = (x_i \ y_i)(\bar{x}_i \ \bar{y}_i)(a_i \ \bar{a}_i)$
are no symmetries

Hindering the Symmetry Rule

For $n \in \mathbb{N}$, the formula KBKF'_n is defined by the prefix

$$\exists x_1 \forall u_1 \exists y_1 \forall a_1 \exists x_2 \forall u_2 \exists y_2 \forall a_2 \dots \exists x_n \forall u_n \forall y_n \forall a_n \exists z_1 \dots z_n$$

and the following clauses:

- $C_1 = (\bar{x}_1 \vee \bar{y}_1)$
- for $j = 1, \dots, n-1$:
  $C_{2j} = (x_j \vee \bar{a}_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1} \vee u_j)$
  $C_{2j+1} = (y_j \vee a_j \vee \bar{x}_{j+1} \vee \bar{y}_{j+1})$.
- $C_{2n} = (x_n \vee \bar{a}_n \vee \bar{z}_1 \vee \dots \vee \bar{z}_n \vee u_n)$,
  $C_{2n+1} = (y_n \vee a_n \vee \bar{z}_1 \vee \dots \vee \bar{z}_n)$
- for $j = 1, \dots, n$:
  $B_{2j-1} = (a_j \vee z_j)$ and $B_{2j} = (\bar{a}_j \vee z_j)$.

$\sigma_i = (x_i \ y_i)(\bar{x}_i \ \bar{y}_i)(a_i \ \bar{a}_i)$
are no symmetries

$\Rightarrow$ **S is not applicable**

Hindering the Symmetry Rule

For $n \in \mathbb{N}$, the formula KBKF'_n is defined by the prefix

$$\exists x_1 \forall u_1 \exists y_1 \forall a_1 \exists x_2 \forall u_2 \exists y_2 \forall a_2 \ldots \exists x_n \forall u_n \forall y_n \forall a_n \exists z_1 \ldots z_n$$

and the following clauses:

- $C_1 = (\bar{x}_1 \lor \bar{y}_1)$
- for $j = 1, \ldots, n-1$:
  $C_{2j} = (x_j \lor \bar{a}_j \lor \bar{x}_{j+1} \lor \bar{y}_{j+1} \lor u_j)$
  $C_{2j+1} = (y_j \lor a_j \lor \bar{x}_{j+1} \lor \bar{y}_{j+1})$.
- $C_{2n} = (x_n \lor \bar{a}_n \lor \bar{z}_1 \lor \ldots \lor \bar{z}_n \lor u_n)$,
  $C_{2n+1} = (y_n \lor a_n \lor \bar{z}_1 \lor \ldots \lor \bar{z}_n)$
- for $j = 1, \ldots, n$:
  $B_{2j-1} = (a_j \lor z_j)$ and $B_{2j} = (\bar{a}_j \lor z_j)$.

$\sigma_i = (x_i \; y_i)(\bar{x}_i \; \bar{y}_i)(a_i \; \bar{a}_i)$
are no symmetries

$\Rightarrow$ **S is not applicable**

$\Rightarrow$ **no short proof in Q-Res+S**

Conclusion

- Q-Res+S extends classical QBF resolution with the symmetry rule
- Q-Res+S has short proofs on some hard formulas
- S is very sensitive to formula structure

Conclusion

- Q-Res+S extends classical QBF resolution with the symmetry rule
- Q-Res+S has short proofs on some hard formulas
- S is very sensitive to formula structure

Future Work

- practical application in solving
- consider independence of variables
- add S to other proof systems than Q-Res