Clique Is Hard for State-of-the-Art Algorithms

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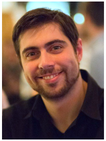
KTH Royal Institute of Technology

Oaxaca
August 2018

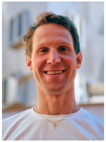
Talk based on joint work with:

A. Atserias

I. Bonacina



M. Lauria

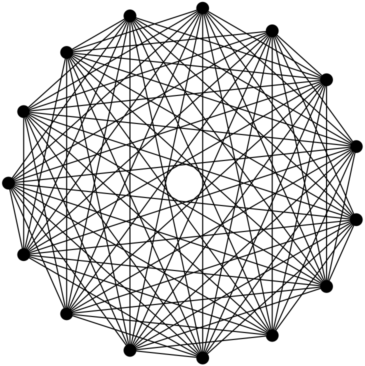


J. Nordström

A. Razborov

Maximum clique problem

- What is the size of a maximum clique in G?



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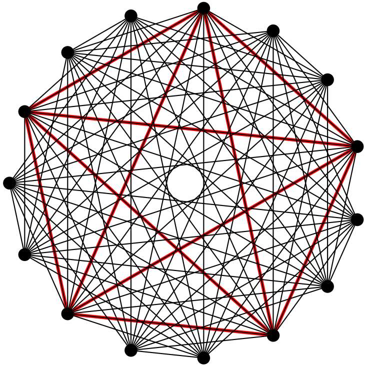

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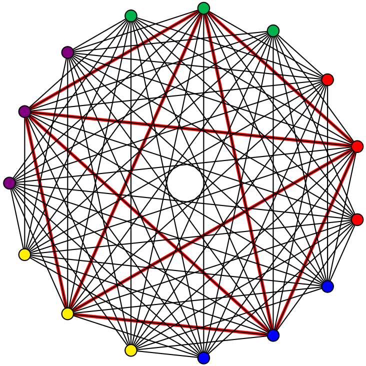

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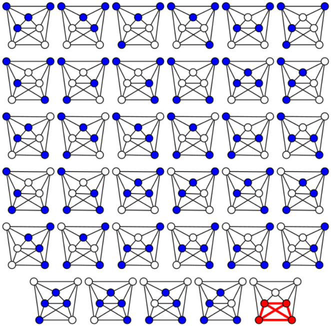

Credit: Thore Husfeldt

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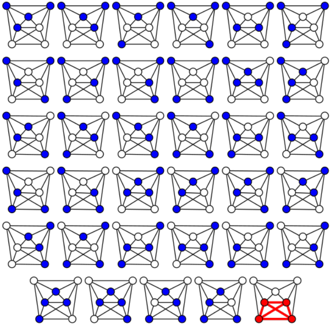

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Is this optimal?

Our main result

Theorem (informal)

State-of-the-art algorithms require time $n^{\Omega(k)}$ to determine that the maximum clique in a random graph is k.

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- If graph has no k-clique, trace of algorithm gives proof of this fact
- Lower bound on size of such proofs $\Rightarrow$ lower bound on running time

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- To analyse algorithms need to formalise method of reasoning used
- If graph has no k-clique, trace of algorithm gives proof of this fact
- Lower bound on size of such proofs $\Rightarrow$ lower bound on running time
- Brings us to topic of this talk: proof complexity

What is resolution?

Input: Unsatisfiable CNF formula, e.g.:

$$\neg x \wedge (\neg y \vee \neg z) \wedge (y \vee \neg w) \wedge (x \vee w) \wedge (\neg x \vee z) \wedge \neg y$$

Goal: Certify unsatisfiability using resolution rule

$$\frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

Resolution refutation

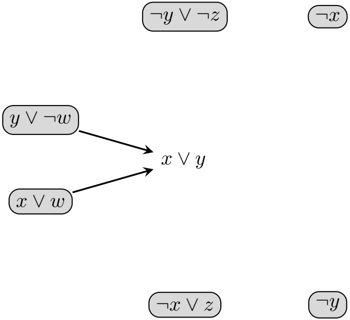
$\neg y \lor \neg z$ $\neg x$

$y \lor \neg w$

$x \lor w$

$\neg x \lor z$ $\neg y$

Resolution refutation



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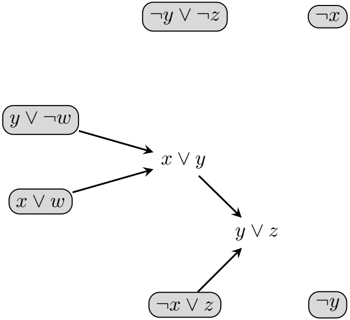

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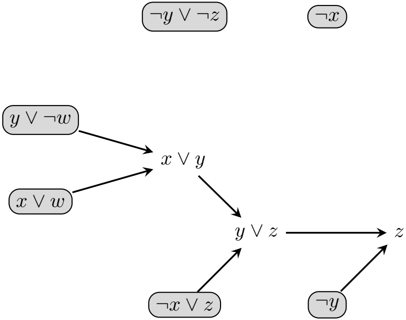

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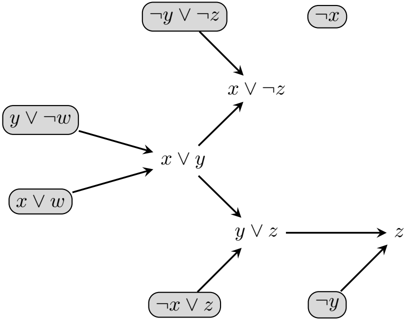

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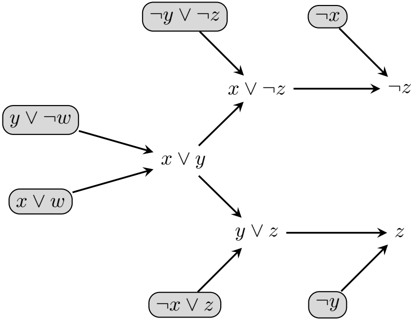

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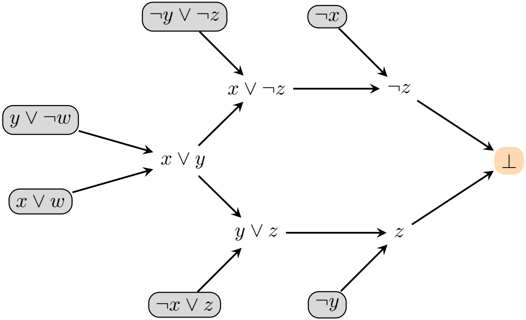

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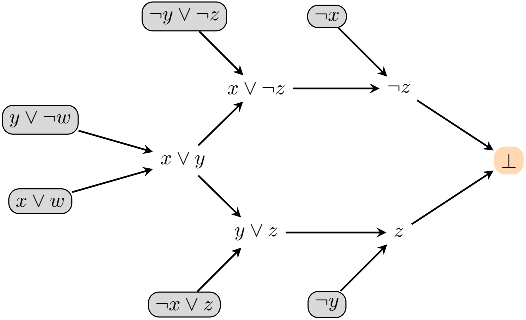
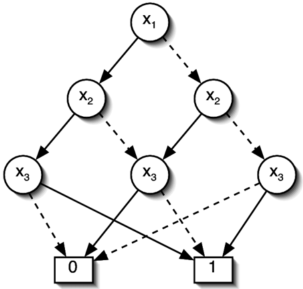


Tree-like = the proof DAG is a tree
Regular = no variable resolved twice in any source-to-sink path
Size = # of nodes in the proof DAG

Branching program



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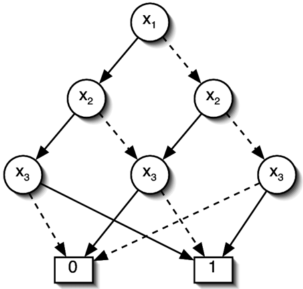

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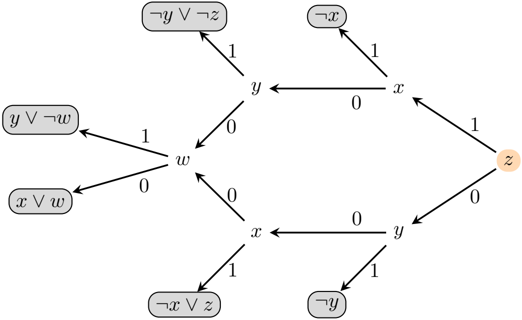
Interested in branching programs solving falsified clause search problem

Falsified clause search problem: given unsat formula and an assignment to variables, find a falsified clause

Branching program solving falsified clause problem



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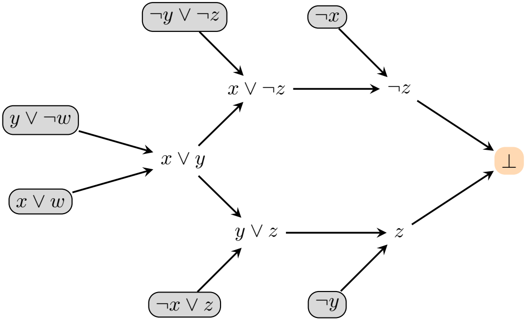

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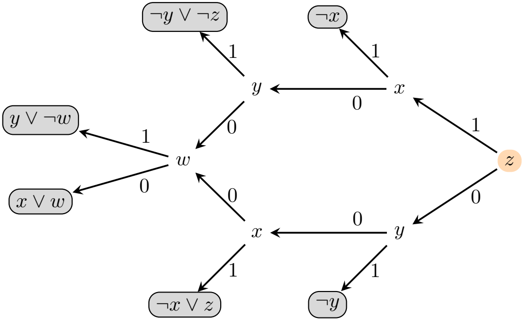

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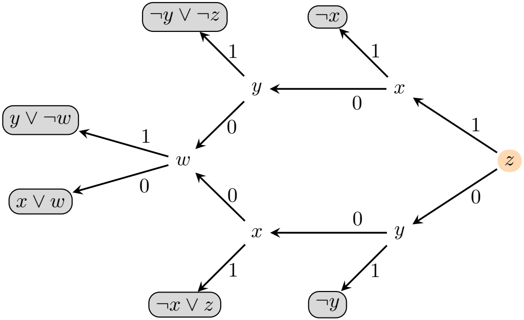

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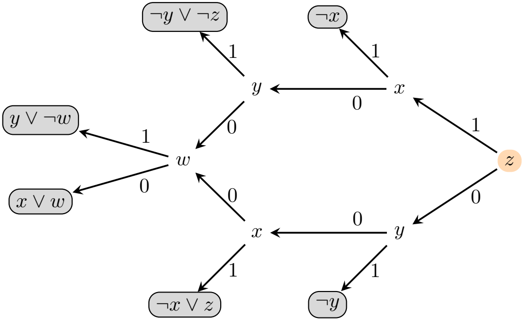

Tree-like = the DAG is a tree (a.k.a. decision tree)
Read-once = no variable queried twice in any source-to-sink path
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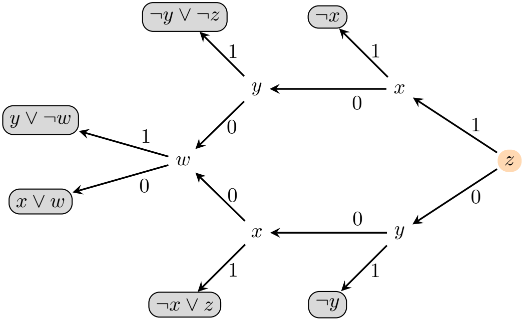
Branching program solving falsified clause problem



Decision tree
= tree-like resolution

Read-once branching program = regular resolution

General branching program stronger than general resolution

Tree-like = the DAG is a tree (a.k.a. decision tree)
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Encoding the k-clique problem in CNF

Graph $G = (V, E)$, $k \in \mathbb{N}$

Formula $\textbf{Clique}(\boldsymbol{G}, \boldsymbol{k})$:

$x_{v,i}$: "vertex v is i-th member of clique"

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non-neighbours are not both in clique	$\neg x_{v,i} \vee \neg x_{u,j}$	$(v, u) \notin E$

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Formula Clique(G, k) is satisfiable $\Leftrightarrow$ G has a k-clique

State-of-the-art algorithms

Östergård's algorithm using Russian doll search

- often used in practice
- has been available online since 2003
- main component of the Cliquer software
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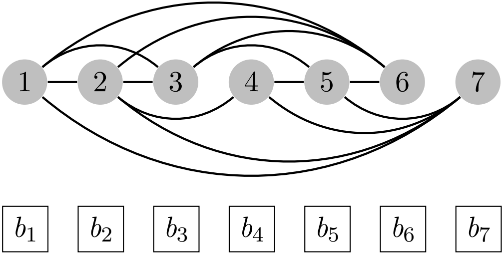
Colour-based branch-and-bound algorithms

- arguably the most successful in practice
- uses colouring as bounding (and often as branching) strategy
- basic idea: $(k-1)$-colourable graph cannot contain k-clique
- extended survey and computational analysis in [Prosser'12] and [McCreesh'17]

Östergård's algorithm

- $G = (V, E)$
- $V = [n] = \{1, 2, \ldots, n\}$



$b_i := \max\{\ell : \exists\ \ell\text{-clique among vertices } \{1, 2, \ldots, i\}\}$

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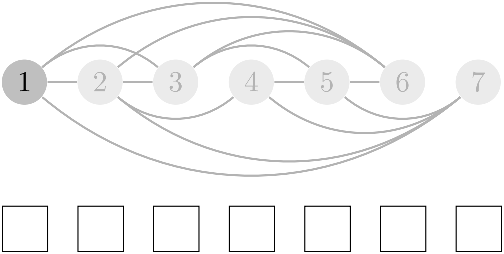

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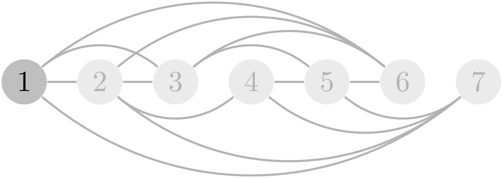

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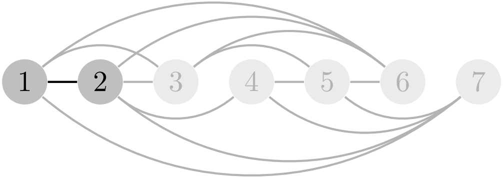

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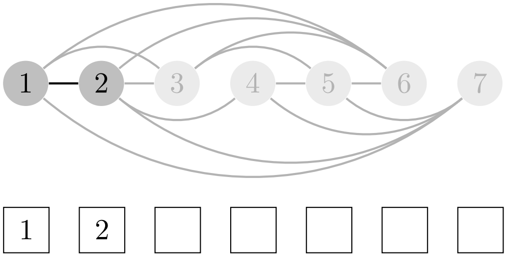

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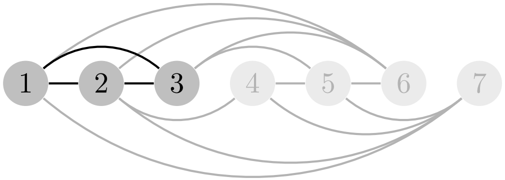

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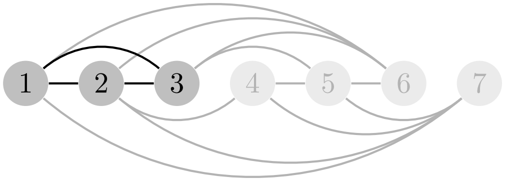

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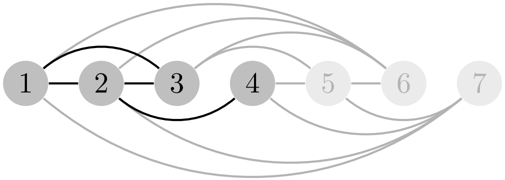

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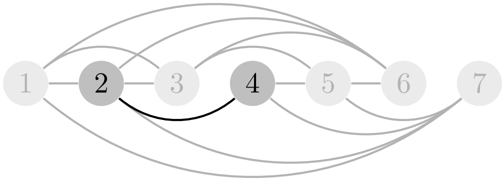

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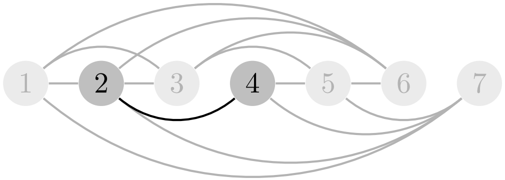

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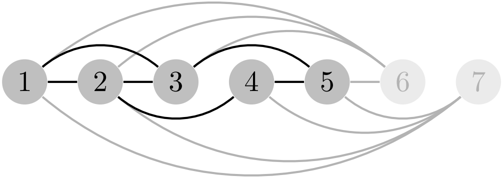

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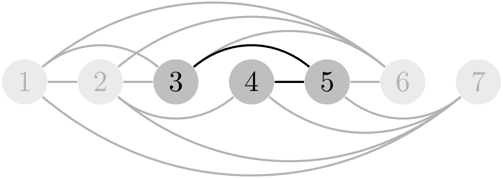

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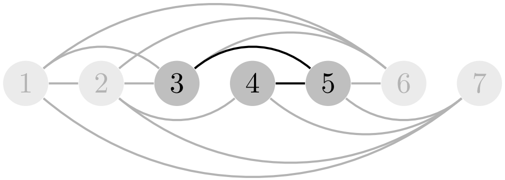

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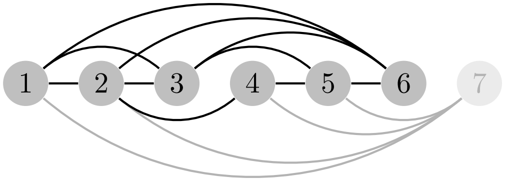

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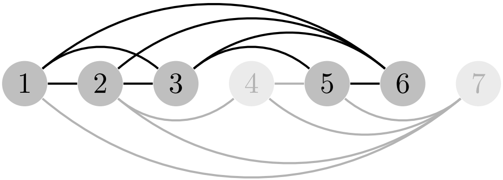

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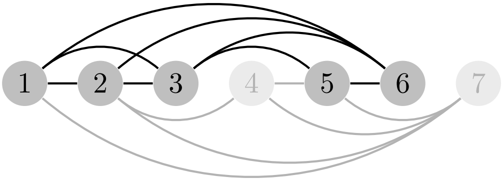

1	2	3	3	3	4	

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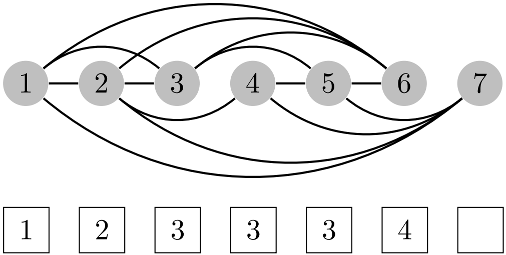

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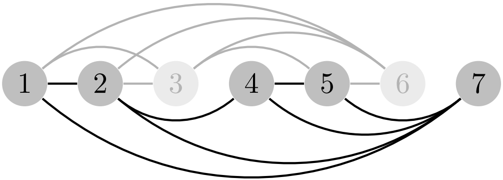

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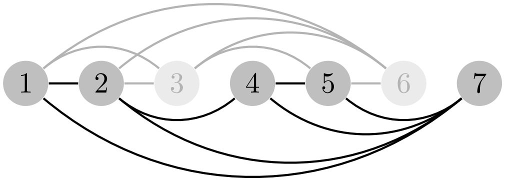

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Östergård's algorithm

$\texttt{Cliquer}(G)$:
begin
  $G \leftarrow \texttt{permute}(G)$
  $inc \leftarrow \emptyset$
  for $i = n$ ***down to*** 1 **do**
    $found \leftarrow$ false
    $\texttt{expand}(G[V_i \cap N(v_i)], \{v_i\})$
    $b[i] \leftarrow |inc|$
  return inc

$\texttt{expand}(H, sol)$:
begin
  while $V(H) \neq \emptyset$ **do**
    if $|sol| + |V(H)| \leq |inc|$ **then return**
    $i \leftarrow \min\{j \mid v_j \in V(H)\}$
    if $|sol| + b[i] \leq |inc|$ **then return**
    $sol' \leftarrow sol \cup \{v_i\}$
    $V' \leftarrow V(H) \cap N(v_i)$
    $\texttt{expand}(H[V'], sol')$
    if $found =$ true **then return**
    $H \leftarrow H \setminus \{v_i\}$
  if $|sol'| > |inc|$ **then**
    $inc \leftarrow sol'$, $found \leftarrow$ true
  return

Colour-based branch and bound

`MaxClique`(G):
begin
  global $inc \leftarrow \emptyset$
  `expand`$(G, \emptyset)$
  return inc

`expand`(H, sol):
begin
  $(order, b) \leftarrow$ `colourOrder`(H)
  while $V(H) \neq \emptyset$ **do**
    $i \leftarrow |V(H)|$
    if $|sol| + b[i] \leq |inc|$ **then return**
    $v \leftarrow order[i]$
    $sol' \leftarrow sol \cup \{v\}$
    $V' \leftarrow V(H) \cap N(v)$
    `expand`$(H[V'], sol')$
    $H \leftarrow H \setminus \{v\}$
  if $|sol'| > |inc|$ **then** $inc \leftarrow sol'$
  return

What kind of proofs do these algorithms generate?

Recall encoding of k-clique problem in CNF

$x_{v,i}$: "vertex v is i-th member of clique"

$\exists i$th clique-member	$\bigvee_{v \in V} x_{v,i}$	$i \in [k]$
non-neighbours are not both in clique	$\neg x_{v,i} \vee \neg x_{u,j}$	$(v,u) \notin E$

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G has no k-clique $\Rightarrow$ formula unsat (converse not necessarily true)

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Colouring-base branch-and-bound algorithm

- Branch = tree-like resolution

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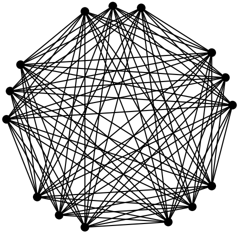

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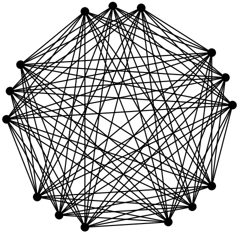

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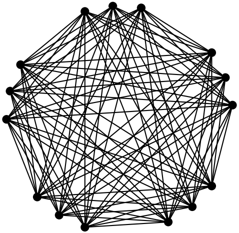

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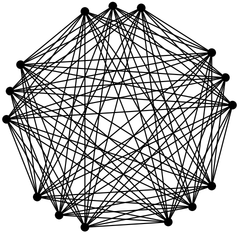

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Östergård's algorithm

- Branch = tree-like resolution
- Bound = regular resolution
- Reuse previous computations for bounding

Regular resolution captures any such algorithms, even for oracle access to optimal ordering of vertices and optimal colourings

Hardness of k-clique for resolution

Previous work

- $n^{\Omega(k)}$ for tree-like resolution [Beyersdorff, Galesi, Lauria '11]
- $n^{\Omega(k)}$ for general resolution for *binary encoding* [Lauria, Pudlák, Rödl, Thapen '13]
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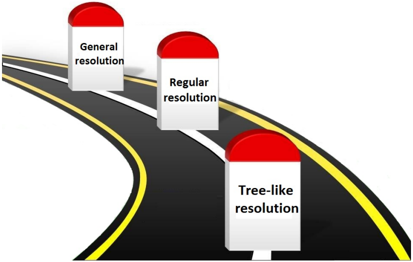

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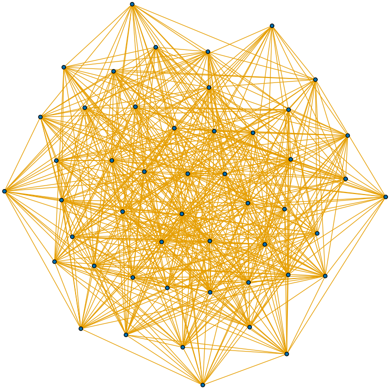
Usual proof complexity tool-box seems to fail:

- Random restrictions
- Interpolation techniques [Krajíček '97]
- Size-width lower bound [Ben-Sasson, Wigderson '99]

What are hard instance for regular resolution?

- Erdős-Rényi random graph: $G \sim \mathcal{G}(n, p)$
- n vertices
- include each possible edge with probability p



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What is an appropriate p?

$$E[\#\text{ of } k\text{-cliques}] = \binom{n}{k} p^{k(k-1)/2}$$

- Threshold value for having a k-clique $p \approx n^{-2/(k-1)}$
- Choose p slightly below so that w.h.p. no k-clique but still dense

Slightly more formal statement of main result

Theorem

Let $k \ll n^{1/4}$ and let $G \sim \mathcal{G}(n, p)$ for p slightly less than threshold. W.h.p. any regular resolution refutation of $\mathsf{Clique}(G, k)$ has length $n^{\Omega(k)}$.

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Theorem

Let $k \ll n^{1/4}$ *and let* $G \sim \mathcal{G}(n,p)$ *for* p *slightly less than threshold. W.h.p. any regular resolution refutation of* Clique(G,k) *has length* $n^{\Omega(k)}$.

- Tight: upper bound $n^{O(k)}$ (even for tree-like resolution)
- Lower bound degrades gracefully with smaller density

Take away

Summary

- k-clique fundamental problem
- Prove $n^{\Omega(k)}$ average case lower bound for regular resolution
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Open problems

- Prove hardness for explicit graphs
- Extend to *general* resolution
- Why are CDCL solvers slower than clique-solvers?
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Thanks!