Some remarks on Nitsche's method

Roland Becker
University of Pau

Nitsche’s method
The elliptic interface problem

Alan Turing is reported as saying that **PDE’s are made by God**, the **boundary conditions by the Devil**! The situation has changed, Devil has changed places...We can say that the main challenges are in the **interfaces**, with Devil not far away from them…”
Jacques-Louis Lions

Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces
July/August 2018, Oaxaca

Nitsche's method

Instead of modification of the discrete spaces use an appropriate **discrete variational** formulation

This can be achieved with Lagrange multipliers (Mortaring, *Ivo Babuska*)

But the Lagrange multipliers can be avoided on the discrete level (*Joachim Nitsche*) !

Relation between Nitsche's method and multipliers (*Rolf Stenberg*)

Application to interface problems (*Anita and Peter Hansbo*)

Dirichlet

$$-\operatorname{div}(k\nabla u) = f \quad (\Omega)$$

$$\gamma(u - u^D) = 0 \qquad (\Gamma := \partial\Omega), \qquad u^D \in H^1(\Omega)$$

$$V_h = V_h^0 \oplus V_h^*$$

$$E(u) = \frac{1}{2}a(u,u) - l(u), \qquad a(u,v) = \int_\Omega k\nabla_h u \cdot \nabla_h v, \quad l(v) = \int_\Omega fv$$

$$(P) \quad u_h \in u_h^D + V_h^0 : \quad E'(u_h)(v) = 0 \quad \forall v \in V_h^0$$

$$\mathcal{L}(u,\lambda) = E(u) + \int_\Gamma (u - u_h^D)\lambda$$

$$(PL) \quad \mathcal{L}'(u_h,\lambda_h)(v,\lambda) = 0 \quad \Leftrightarrow \quad \begin{cases} E'(u_h)(v^0) & = l(v^0) \\ E'(u_h)(v^*) + \int_\Gamma v^*\lambda_h & = l(v^*) \\ \int_\Gamma u_h\mu & = \int_\Gamma u_h^D\mu \end{cases}$$

Nitsche

$$\int_\Omega k\nabla u \cdot \nabla v^* + \int_\Gamma v^*\lambda = \int_\Omega f v^* \quad \Rightarrow \quad \lambda = -k\frac{\partial u}{\partial n} = T(u)$$

$$\widetilde{E}(u) := \mathcal{L}_r(u, T(u)) = E(u) - \int_\Gamma (u - u^D)k\frac{\partial u}{\partial n} + \frac{r}{2}\int_\Gamma (u - u^D)^2$$

$$(\tilde{P}) \quad u_h \in V_h \qquad \widetilde{E}'(u_h)(v) = 0 \quad \forall v \in V_h$$

- ✦ No modification of FEM space
- ✦ Weighting of boundary conditions (...)

Weighting of boundary conditions

$$\begin{cases} -\operatorname{div}(k\nabla u) + \beta \cdot \nabla u = f & \text{in } \Omega, \\ u = u^D & \text{on } \Gamma^D, \\ k\frac{\partial u}{\partial n} = q & \text{on } \Gamma^N. \end{cases}$$

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\begin{cases} k\left(\int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\Gamma^D} \frac{\partial u_h}{\partial n} v_h - \int_{\Gamma^D} u_h \left(\frac{\partial v_h}{\partial n} - \frac{\gamma}{h} v_h\right)\right) \\ +\int_\Omega \beta \cdot \nabla u_h + s_h(u_h, v_h) - \int_{\partial\Omega^-} \beta_n^- uv \\ \\ \qquad =: a_h(u_h, v_h) \end{cases} = \begin{cases} -k\int_{\partial\Omega^-} u^D \left(\frac{\partial v_h}{\partial n} - \frac{\gamma}{h} v_h\right) \\ -\int_{\partial\Omega^-} \beta_n^- u^D v_h \\ \int_\Omega f v_h + \int_{\Gamma_N} q v_h. \\ \qquad =: l_h(v_h) \end{cases}$$

Alternative I

$\Lambda_h = \gamma(V_h^*)$

$$\begin{cases} E'(u_h)(v^0) & = 0 \\ E'(u_h)(v^*) + \int_\Gamma v^* \lambda_h & = 0 \\ \int_\Gamma u_h \mu & = \int_\Gamma u_h^D \mu \end{cases} \quad \Leftrightarrow \quad \begin{cases} \int_\Omega k \nabla u_h \cdot \nabla v^0 & = \int_\Omega f v^0 \\ \int_\Gamma u_h v^* & = \int_\Gamma u_h^D v^* \\ \int_\Gamma v^* \lambda_h & = \int_\Omega f v^* - \int_\Omega k \nabla u_h \cdot \nabla v^* \end{cases}$$

- Traditional FEM (with L2-projection of Dirichlet data)
- DMP if angle-condition and lumping
- Optional flux post-processing

J. W. Barrett and C. M. Elliott. *Total flux estimates for a finite-element approximation of elliptic equations.* IMA J. Numer. Anal., 7(2): 129–148, 1987.

M. Giles, M. Larson, M. Levenstam, and E. Süli. *Adaptive error control for finite element approximations of the lift and drag coefficients in viscous flow.* Technical Report NA-76/06, Oxford University Computing Laboratory, 1997.

$$\int_\Omega f = \int_\Gamma \lambda_h$$

Alternative II

$$\mathcal{L}(u,\lambda) = E(u) + \int_\Gamma (u - u_h^D)\lambda$$

$$\int_\Gamma v^*\lambda \approx \int_\Omega f v^* - \int_\Omega k\nabla u \cdot \nabla v^*$$

$$\widetilde{E}(u) := E(u) + \int_\Omega f(u^* - u_h^D) - \int_\Omega k\nabla u \cdot \nabla(u^* - u_h^D) + \frac{r}{2}\int_\Omega k\nabla(u^* - u_h^D)\cdot\nabla(u^* - u_h^D)$$

$r = 2$

$$\widetilde{E}(u) = E(u) + \int_\Omega f(u^* - u_h^D) - \int_\Omega k\nabla\left(u^0 + u_h^D\right)\cdot\nabla\left(u^* - u_h^D\right)$$

$$\Rightarrow \quad \begin{cases} \int_\Omega k\nabla_h u^0 \cdot \nabla_h v^0 = & \int_\Omega f v^0 - \int_\Omega k\nabla_h u_h^D \cdot \nabla_h v^0, \\ \int_\Omega k\nabla_h u^* \cdot \nabla_h v^* = & \int_\Omega k\nabla_h u_h^D \cdot \nabla_h v^*. \end{cases}$$

$$\widetilde{E}(u) = E(u) + \int_\Omega f(u^* - u_h^D) - \int_\Omega k\nabla\left(u^0 + u_h^D\right) \cdot \nabla\left(u^* - u_h^D\right)$$

$$\Rightarrow \quad \begin{cases} \int_\Omega k\nabla_h u^0 \cdot \nabla_h v^0 = & \int_\Omega f v^0 - \int_\Omega k\nabla_h u_h^D \cdot \nabla_h v^0, \\ \int_\Omega k\nabla_h u^* \cdot \nabla_h v^* = & \int_\Omega k\nabla_h u_h^D \cdot \nabla_h v^*. \end{cases}$$

- ✦ Traditional FEM
- ✦ DMP if angle-condition
- ✦ Global conservation
- ✦ Extension to conv-diff

$$\int_\Omega f = \int_\Omega f\chi_\Omega^* - \int_\Omega k\nabla u \cdot \nabla \chi_\Omega^*$$

$$u \in V_h : \quad \forall v \in V_h$$

$$\widetilde{E}'(u)(v) + \int_\Omega \beta \cdot \nabla u + s_h(u, v) - \int_{\partial\Omega} \beta_n^-(u - u_h^D)v = 0$$

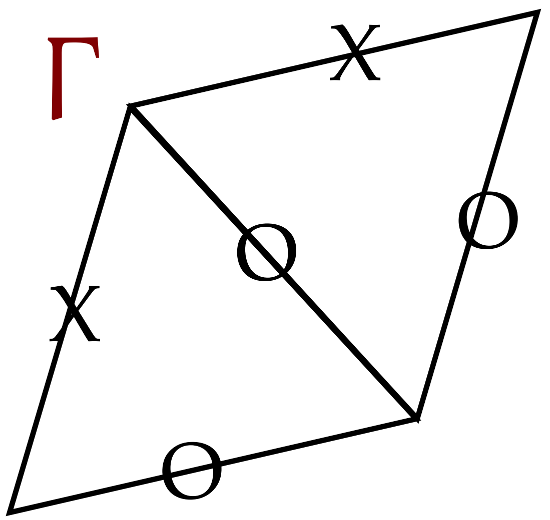
Crouzeix-Raviart

k constant

$$\widetilde{E}(u) := E(u) + \int_\Omega f(u^* - u_h^D) - \int_\Omega k\nabla u \cdot \nabla(u^* - u_h^D) + \frac{r}{2}\int_\Omega k\nabla(u^* - u_h^D)\cdot\nabla(u^* - u_h^D)$$

$$(1) \quad \int_\Omega \nabla_h u \cdot \nabla_h v^* = \int_\Gamma \frac{\partial u}{\partial n} v^* \qquad \forall u \in V_h, v^* \in V_h^*$$

$$(2) \quad \int_\Omega \nabla_h u^* \cdot \nabla_h v^* = \int_\Gamma \frac{1}{h} v^* u^* \qquad \forall u^*, v^* \in V_h^*$$



$$\widetilde{E}(u) = E(u) - \int_\Gamma k\frac{\partial u}{\partial n}\left(u^* - u_h^D\right) + \int_\Gamma \frac{rk}{2h}\left(u^*\right)^2 + \int_\Omega f(u^* - u_h^D)$$

$r = 0$

$$\widetilde{E}(u) = \frac{1}{2}\int_\Omega k|\nabla u^0|^2 - \frac{1}{2}\int_\Omega k|\nabla u^*|^2 - \int_\Omega f(u^0 + u_h^D) + \int_\Omega k\nabla u \cdot \nabla u_h^D$$

Robin-Fourier

$$-\operatorname{div}(k\nabla u) = f \quad (\Omega)$$
$$k\frac{\partial u}{\partial n} + \frac{1}{\varepsilon}(u - u^D) = q^D \quad (\Gamma := \partial\Omega)$$

$$a(u,v) = \int_\Omega k\nabla u\cdot\nabla v + \frac{1}{\varepsilon}\int_\Gamma uv, \quad l(v) = \int_\Omega fv + \int_\Gamma q^D v + \frac{1}{\varepsilon}\int_\Gamma u^D v$$

$$(u_h \in P_h^1) \quad \Rightarrow \quad |||u - u_h|||_\varepsilon \lesssim h\left(1 + \left(\frac{h}{\varepsilon}\right)^{\frac{1}{2}}\right)\|u\|_{H^2(\Omega)}$$

Mixed method $\lambda := -k\dfrac{\partial u}{\partial n} = Tu$

$$\mathcal{L}(u,\lambda) := \underbrace{\frac{1}{2}\int_\Omega k|\nabla u|^2 - \int_\Omega fu}_{=:\,E(u)} + \int_\Gamma (u-u^D)\lambda - \frac{\varepsilon}{2}\int_\Gamma (\lambda - 2q^D)\lambda$$

$LBB_h \quad \Rightarrow \quad$ good estimates

Nitsche-type

$(q^D = 0)$

$$\mathcal{L}_r(u,\lambda) := E(u) + \int_\Gamma (u-u^D)\lambda - \frac{\varepsilon}{2}\int_\Gamma \lambda^2 + \frac{r}{2}\int_\Gamma \left(\varepsilon k\frac{\partial u}{\partial n} + (u-u^D)\right)^2$$

Nitsche energy:

$$\widetilde{E}(u) := E(u) - \int_\Gamma (u-u^D)k\frac{\partial u}{\partial n} - \frac{\varepsilon}{2}\int_\Gamma \left(k\frac{\partial u}{\partial n}\right)^2 + \frac{r}{2}\int_\Gamma \left(\varepsilon k\frac{\partial u}{\partial n} + (u-u^D)\right)^2$$

$$\begin{aligned} 0 = \widetilde{E}'(u)(v) &= E'(u)(v) - \int_\Gamma (u-u^D)k\frac{\partial v}{\partial n} - \int_\Gamma vk\frac{\partial u}{\partial n} - \varepsilon\int_\Gamma k\frac{\partial u}{\partial n}k\frac{\partial v}{\partial n} + r\int_\Gamma \left(\varepsilon k\frac{\partial u}{\partial n} + (u-u^D)\right)\left(\varepsilon k\frac{\partial v}{\partial n} + v\right) \\ &= E'(u)(v) - (1-r\varepsilon)\int_\Gamma (u-u^D)k\frac{\partial v}{\partial n} - (1-r\varepsilon)\int_\Gamma vk\frac{\partial u}{\partial n} - \varepsilon(1-r\varepsilon)\int_\Gamma k\frac{\partial u}{\partial n}k\frac{\partial v}{\partial n} + r\int_\Gamma (u-u^D)\,v \end{aligned}$$

$$r = \frac{k}{h}\times(1-r\varepsilon)\times\gamma \quad \Rightarrow \quad r = \frac{k}{k\varepsilon + h/\gamma}, \quad 1-r\varepsilon = \frac{h/\gamma}{k\varepsilon + h/\gamma}$$

leads to robust estimates:

M. Juntunen and R. Stenberg. *Nitsche's method for general boundary conditions*. Math. Comp., 78(267):1353–1374, 2009.

Alternative

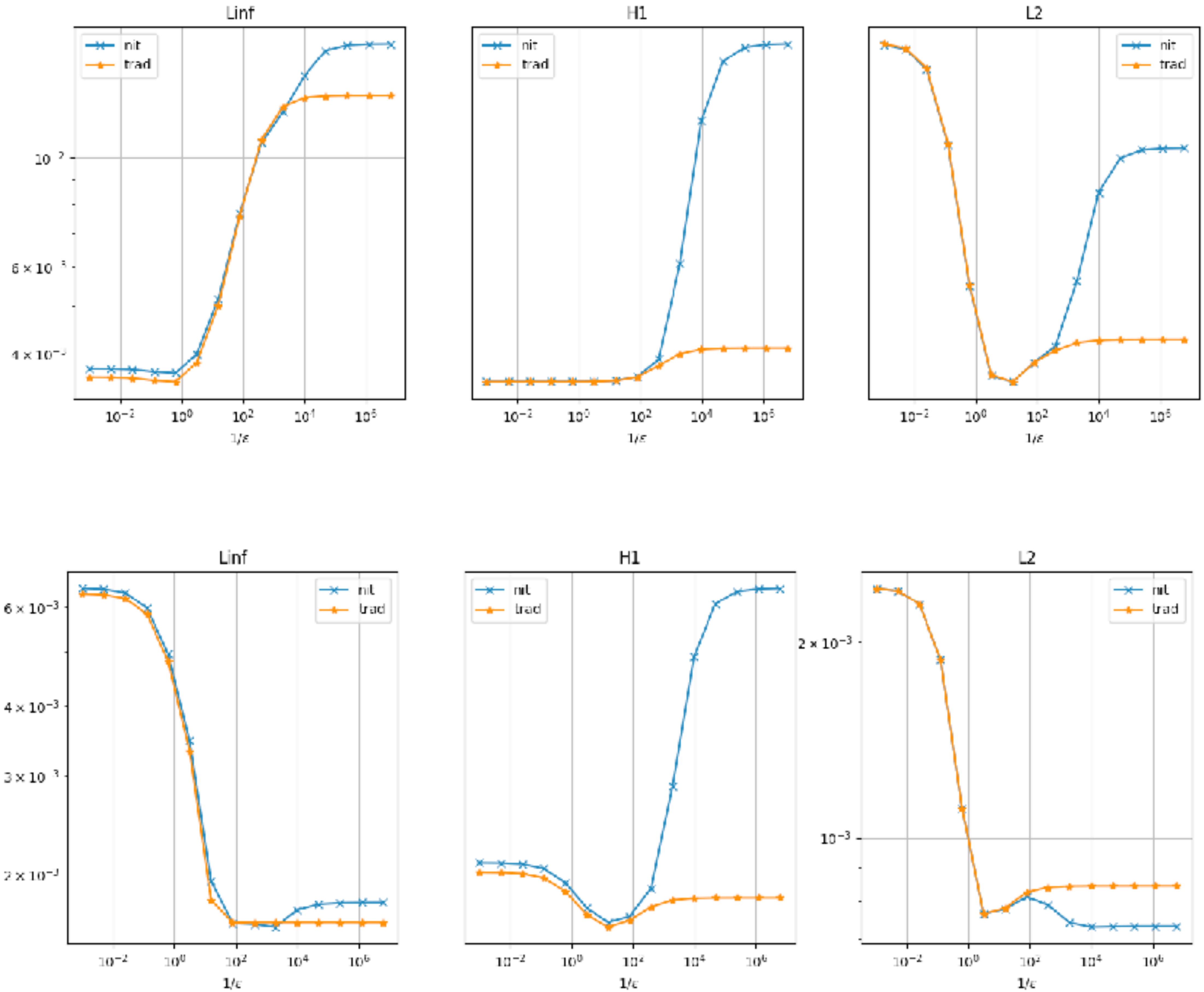
$$\mathcal{L}_r(u,\lambda) := E(u) + \int_\Gamma (u - u^D)\lambda - \frac{\varepsilon}{2}\int_\Gamma \lambda^2 + \frac{r}{2}\int_\Gamma \left(\varepsilon k \frac{\partial u}{\partial n} + (u - u^D)\right)^2$$

$(r = 0)$

$\Lambda_h = \gamma(V_h)$

$$\Rightarrow \begin{cases} E'(u)(v^0) = 0 \\ E'(u)(v^*) + \int_\Gamma v^*\lambda = 0 \\ \int_\Gamma (u - u^D)v^* - \varepsilon \int_\Gamma \lambda v^* = 0 \end{cases} \quad \Rightarrow \begin{cases} E'(u)(v^0) = 0 \\ \frac{1}{\varepsilon}\int_\Gamma (u - u^D)v^* + E'(u)(v^*) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} E'_\varepsilon(u)(v) = 0 \\ E_\varepsilon(u) := E(u) + \frac{1}{2\varepsilon}\int_\Gamma (u - u^D)^2 \end{cases}$$

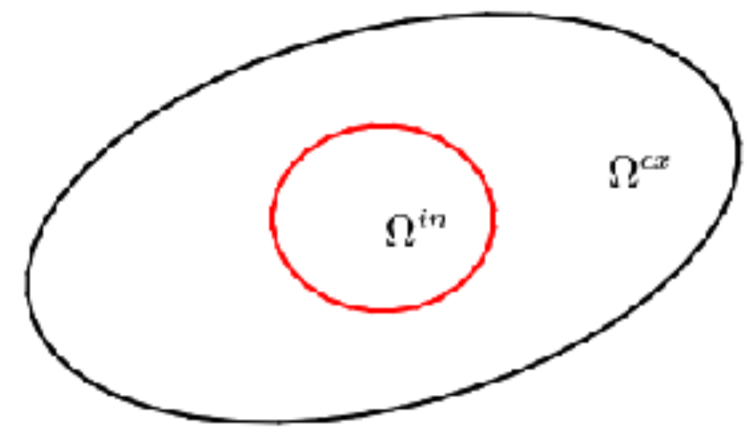


exact mass

lumped mass

The elliptic interface problem

$$\bar{\Omega}^{\text{in}} \cup \bar{\Omega}^{\text{ex}} = \bar{\Omega}, \quad \Gamma = \bar{\Omega}^{\text{in}} \cap \bar{\Omega}^{\text{ex}}, \quad \Gamma \cap \partial\Omega = \emptyset$$



$$\begin{cases} -\operatorname{div}(k\nabla u) = f \quad (\Omega), \qquad u = 0 \quad (\partial\Omega) \\ k = \begin{cases} k^{\text{in}} & \Omega^{\text{in}}, \\ k^{\text{ex}} & \Omega^{\text{ex}}. \end{cases} \end{cases}$$

$$\int_\Omega k\nabla u \cdot \nabla v = -\int_{\Omega^{\text{in}}} k^{\text{in}} \Delta u^{\text{in}} v^{\text{in}} - \int_{\Omega^{\text{ex}}} k^{\text{ex}} \Delta u^{\text{ex}} v^{\text{ex}} + \int_\Gamma k^{\text{in}} \frac{\partial u^{\text{in}}}{\partial n^{\text{in}}} v^{\text{in}} + \int_\Gamma k^{\text{ex}} \frac{\partial u^{\text{ex}}}{\partial n^{\text{ex}}} v^{\text{ex}}$$

$$\Rightarrow \quad \begin{cases} u^{\text{in}} = u^{\text{ex}} \quad \Gamma, \\ k^{\text{in}} \frac{\partial u^{\text{in}}}{\partial n^{\text{in}}} + k^{\text{ex}} \frac{\partial u^{\text{ex}}}{\partial n^{\text{ex}}} = 0. \end{cases} \iff \begin{cases} [u] = 0 \quad \Gamma, \\ [k \frac{\partial u}{\partial n}] = 0 \quad \Gamma \qquad (n = n^{\text{in}}). \end{cases}$$

Assumption: $\quad u \in \tilde{H}^2(\Omega) := \left\{ v \in H_0^1(\Omega) : \ v|_{\Omega^{\text{in/ex}}} \in H^2(\Omega^{\text{in/ex}}) \right\}$

Limits

$$\begin{cases} -k^{in}\Delta u^{in} = f^{in} \quad (\Omega^{in}) \\ -k^{ex}\Delta u^{ex} = f^{ex} \quad (\Omega^{ex}), \qquad u^{ex} = g \quad (\partial\Omega) \\ u^{in} = u^{ex} \quad \Gamma, \\ k^{in}\frac{\partial u^{in}}{\partial n^{in}} = k^{ex}\frac{\partial u^{ex}}{\partial n^{ex}} \quad \Gamma. \end{cases}$$

Important for **robustness** w.r.t parameters

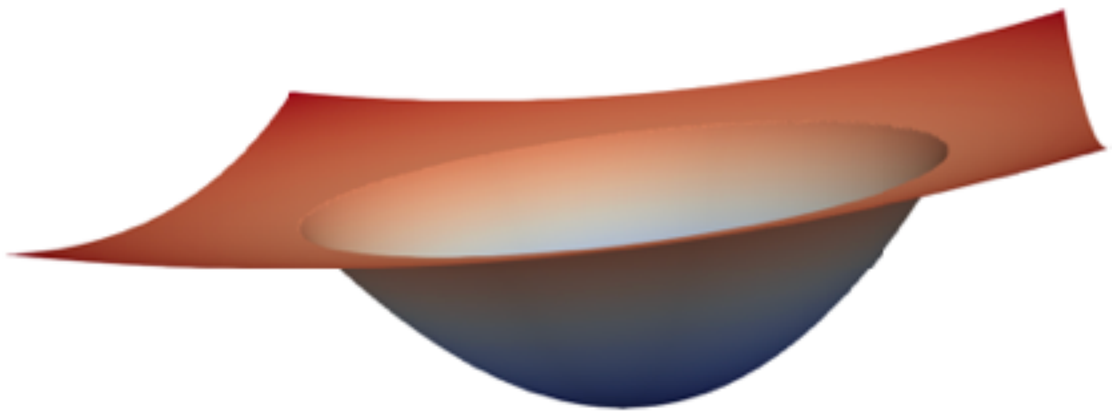
1. $k^{in} = 1 \quad k^{ex} \to +\infty$:

$$\Rightarrow \begin{cases} -\Delta u^{in} = f^{in} \quad (\Omega^{in}), \qquad u^{in} = u^{ex} \quad (\partial\Omega^{in} = \Gamma), \\ -\Delta u^{ex} = 0 \quad (\Omega^{ex}), \qquad u^{ex} = g \quad (\partial\Omega) \\ \frac{\partial u^{ex}}{\partial n^{ex}} = \frac{k^{in}}{k^{ex}}\frac{\partial u^{in}}{\partial n^{in}} \to 0 = \quad \Gamma. \end{cases} \Rightarrow \begin{cases} \text{Dirichlet for } u^{in} \\ \text{Dirichlet}(\partial\Omega)\text{-Neumann}(\Gamma) \text{ for } u^{ex} \end{cases}$$

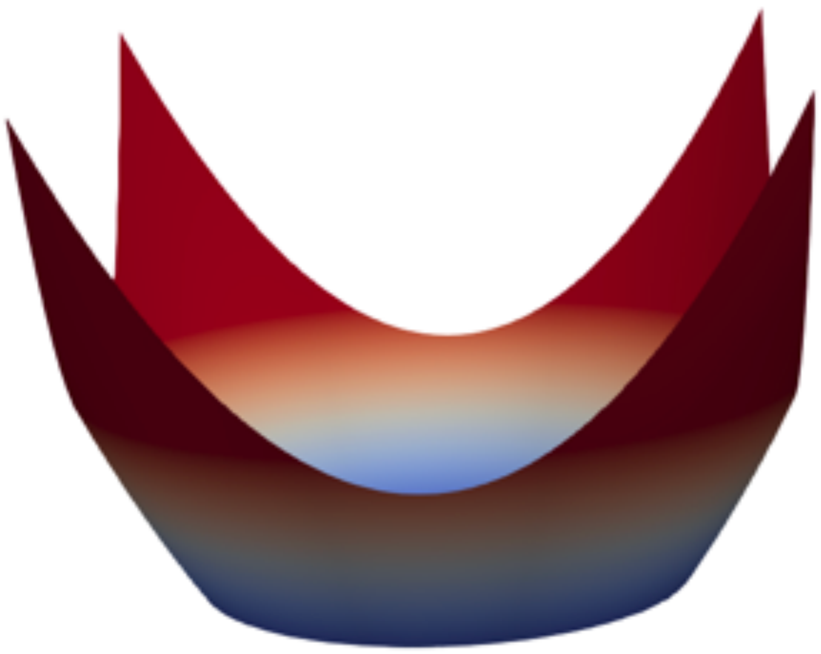
2. $k^{ex} = 1 \quad k^{in} \to +\infty$:

$$\Rightarrow \begin{cases} -\Delta u^{in} = 0 \quad (\Omega^{in}), \quad \frac{\partial u^{in}}{\partial n^{in}} = 0 \ (\partial\Omega^{in} = \Gamma), \\ -\Delta u^{ex} = f^{ex} \quad (\Omega^{ex}), \quad u^{ex} = g \quad (\partial\Omega),\ u^{ex} = u^{in} \ (\Gamma). \end{cases} \Rightarrow \begin{cases} \text{Neumann for } u^{in} \\ \text{Dirichlet for } u^{ex} \end{cases}$$

The standard test problem



$k^{in} = 1, \quad k^{ex} = 10$



$k^{in} = 10, \quad k^{ex} = 1$

Numerical Integration on cut cells

$$D = \left\{x \in \mathbb{R}^d : \; Ax \leqslant b \in \mathbb{R}^m\right\}$$ convex polytop

$$\partial D_i = \{x \in D : \; A_i \cdot x = b_i\}$$

$$f \in C^1(D, \mathbb{R}), \quad qf(x) = \nabla f(x) \cdot x$$ q-homogenuous

$$\int_D f(x)\,dx = \frac{1}{d+q} \sum_{i=1}^{m} \frac{b_i}{|A_i|} \int_{\partial D_i} f(s)\,ds$$

Stokes $$\int_D \underbrace{f(x)\,\mathrm{div}(x)}_{=d\times f}\,dx + \int_D \underbrace{\nabla f(x)\cdot x}_{q\times f}\,dx = \int_{\partial D} f(s) x \cdot n\,ds = \sum_{i=1}^{m} \int_{\partial D_i} f(s) \underbrace{x \cdot n}_{\frac{b_i}{|A_i|}}\,ds$$

Euler $I(b) := \int_{Ax\leqslant b} f(x)\,dx$ is homogenous $\Rightarrow$ $I(b) = \sum_{i=1}^{m} \frac{\partial I}{\partial b_i}(b) b_i \ldots ...$

J. B. Lasserre. *An analytical expression and an algorithm for the volume of a convex polyhedron in R^n*. J. Optim. Theory Appl., 39(3):363–377, 1983.

J. B. Lasserre. *Integration on a convex polytope*. Proc. Amer. Math. Soc., 126(8):2433–2441, 1998.

J. B. Lasserre. *Integration and homogeneous functions*. Proc. Amer. Math. Soc., 127(3):813–818, 1999.

E. B. Chin, J. B. Lasserre, and N. Sukumar. *Numerical integration of homogeneous functions on convex and nonconvex polygons and polyhedra*. Comput. Mech., 56(6):967–981, 2015.

S. E. Mousavi and N. Sukumar. *Numerical integration of polynomials and discontinuous functions on irregular convex polygons and polyhedrons*. Comput. Mech., 47(5):535–554, 2011.

Adapted FEM spaces: Nonconforming

Standard weak formulation for the interface problem with curved Γ:

$$u_h \in V_h : \qquad \int_\Omega k \nabla_h u_h \cdot \nabla_h v_h + s_h(u_h, v_h) = \int_\Omega v_h \qquad \forall v_h \in V_h$$

with a special FE-space V_h:

$$V_h = \left\{ v_h : \int_S [v_h] = 0 \; \forall S \in \mathcal{S}_h, \quad v_h|_K \in \begin{cases} P^1(K) & K \in \mathcal{K}_h \setminus \mathcal{K}_h^\Gamma, \\ P^1_\Gamma(K) & K \in \mathcal{K}_h \cap \mathcal{K}_h^\Gamma. \end{cases} \right\}$$

The space $P^1_\Gamma(K)$ is constructed such that for all $p \in P^1_\Gamma(K)$:

1. $\int_{\Gamma \cap K} [k \frac{\partial p}{\partial n}] = 0$,
2. $p \in P^1(K \cap \Omega^{in}) \times P^1(K \cap \Omega^{ex})$,
3. $p \in C(K)$,
4. we have a basis ψ_S such that $\int_{S'} \psi_S = \delta_{SS'}$ for all $S, S' \in \mathcal{S}_K$.

D. Y. Kwak, K. T. Wee, and K. S. Chang. *An analysis of a broken p1-nonconforming finite element method for interface problems.* SIAM Journal on Numerical Analysis, 48(6):2117–2134, 2010.

Adapted FEM spaces: « Conforming »

Standard weak formulation for the interface problem with curved Γ:

$$u_h \in V_h : \qquad \int_\Omega k\nabla_h u_h \cdot \nabla_h v_h + s_h(u_h, v_h) = \int_\Omega v_h \qquad \forall v_h \in V_h$$

with a special FE-space V_h:

$$V_h = \left\{ v_h : \text{ continuous in all nodes,} \quad v_h|_K \in \begin{cases} P^1(K) & K \in \mathcal{K}_h \setminus \mathcal{K}_h^\Gamma, \\ P^1_\Gamma(K) & K \in \mathcal{K}_h \cap \mathcal{K}_h^\Gamma. \end{cases} \right\}$$

The space $P^1_\Gamma(K)$ is constructed such that for all $p \in P^1_\Gamma(K)$:

1. $\int_{\Gamma \cap K} [k \frac{\partial p}{\partial n}] = 0$,
2. $p \in P^1(K \cap \Omega^{in}) \times P^1(K \cap \Omega^{ex})$,
3. $p \in C(K)$,
4. we have a basis $\hat{\lambda}_N$ such that $\hat{\lambda}_N(x_{N'}) = \delta_{NN'}$ for all $N, N' \in \mathcal{N}_K$.

Z. Li, T. Lin, and X. Wu. *New Cartesian grid methods for interface problems using the finite element formulation.* Numer. Math., 96(1):61–98, 2003.
Z. Li, T. Lin, Y. Lin, and R. C. Rogers. *An immersed finite element space and its approximation capability.* Numer. Methods Partial Differential Equations, 20(3):338–367, 2004.
S.-H. Chou, D. Y. Kwak, and K. T. Wee. *Optimal convergence analysis of an immersed interface finite element method.* Adv. Comput. Math., 33(2):149–168, 2010.

The space $P^1_\Gamma(K)$ is constructed such that for all $p \in P^1_\Gamma(K)$:

1. $\int_{\Gamma\cap K}[k\frac{\partial p}{\partial n}] = 0$,
2. $p \in P^1(K\cap\Omega^{in}) \times P^1(K\cap\Omega^{ex})$,
3. $p \in C(K)$,
4. we have a basis $\hat{\lambda}_N$ such that $\hat{\lambda}_N(x_{N'}) = \delta_{NN'}$ for all $N, N' \in \mathcal{N}_K$.

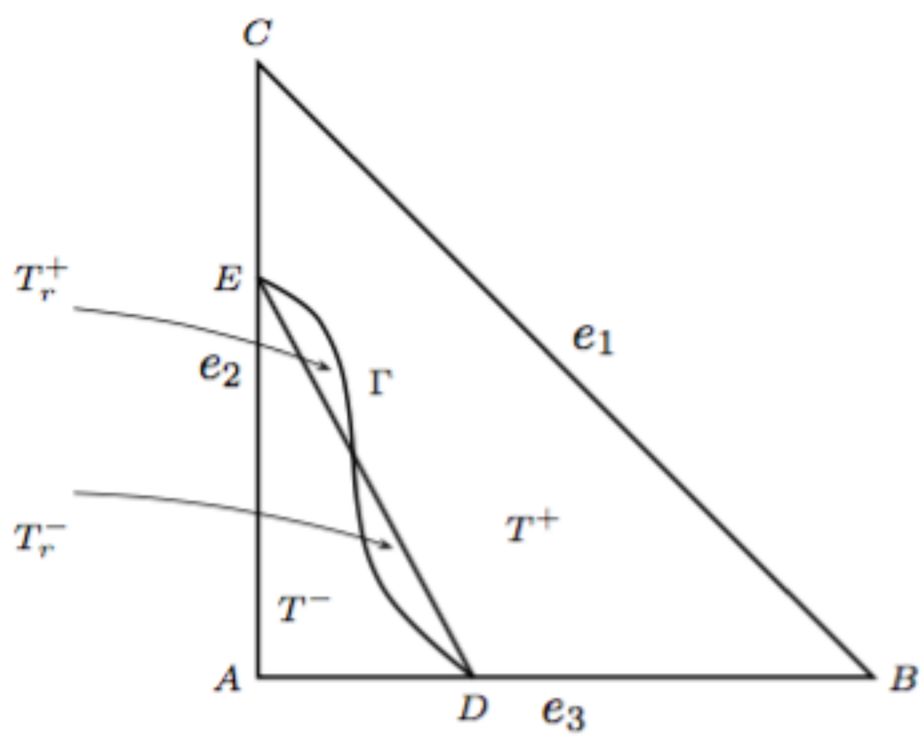


FIG. 2.2. *A typical reference interface triangle.*

The corresponding 6x6 system has a unique solution !

From **D. Y. Kwak, K. T. Wee, and K. S. Chang**, 2010.

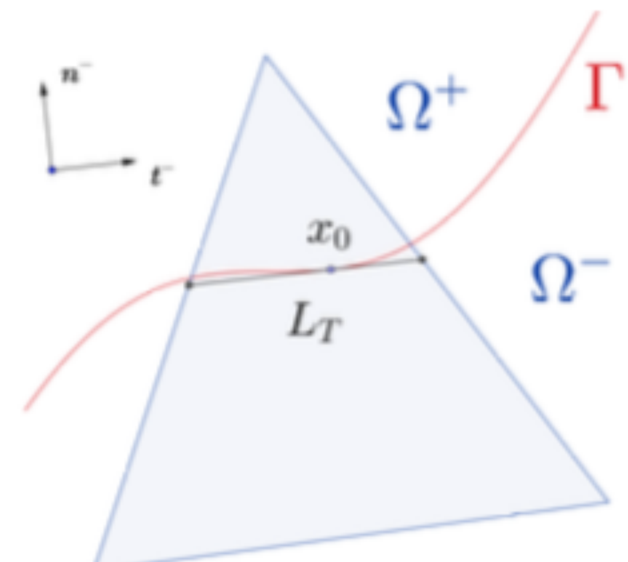
Alternative choice

Modification of P_h^1:

$$V_h := \left\{ v_h \in L^2(\Omega) \cap C(\bar{\Omega}^{in/ex}) : \; v_h \in P^1(K) \quad \forall K \in \mathcal{K}_h \setminus \mathcal{K}_h^\Gamma, \quad v_h \in P_\Gamma^1(K) \quad \forall K \in \mathcal{K}_h^\Gamma \right\}$$

The space $P_\Gamma^1(K) = P^1(K^{in}) \times P^1(K^{ex})$ such that for x_0 mean of $\Gamma_K = \Gamma \cap K$:

1. $p^{in}(x_0) = p^{ex}(x_0)$,
2. $\nabla p^{in}(x_0) \cdot n^\perp = \nabla p^{ex}(x_0) \cdot n^\perp$,
3. $k^{in} \nabla p^{in}(x_0) \cdot n = k^{ex} \nabla p^{ex}(x_0) \cdot n$,



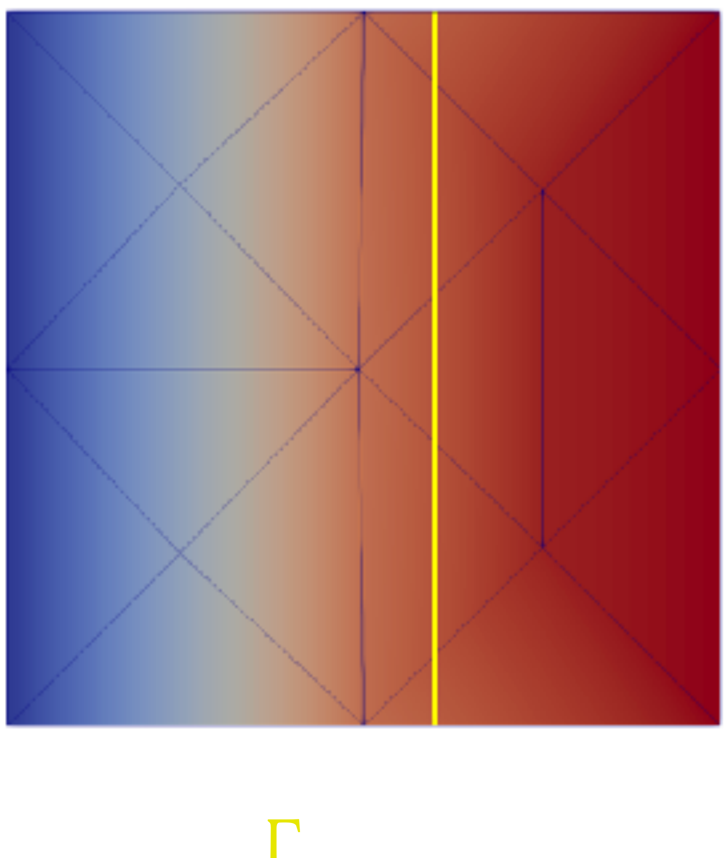
On this space

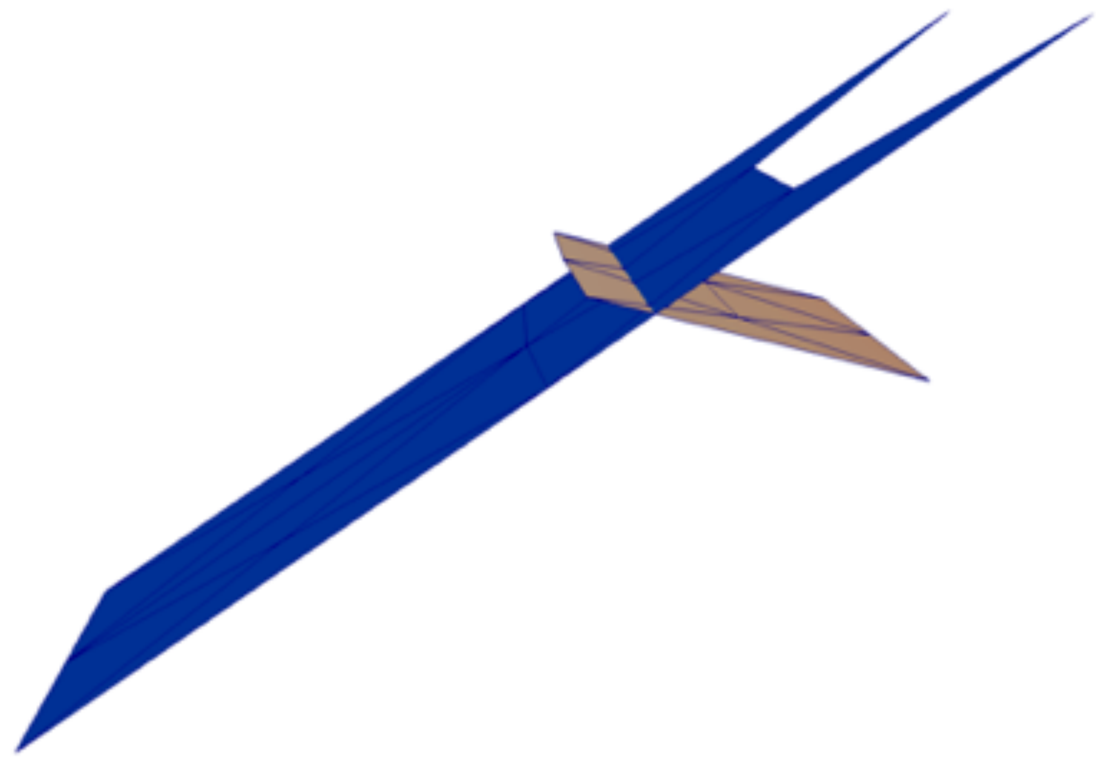
$$a_h(u_h, v_h) = a(u_h, v_h) + \underbrace{\int_{\mathcal{S}_h^{in/ex}} \frac{\gamma k}{|S|} [u_h][v_h]}_{s_h(u_h, v_h)} + \int_{\mathcal{S}_h^{in/ex}} \alpha k |S| [\nabla u_h][\nabla v_h]$$

J. Guzman, M. A. Sanchez, M. Sarkis. *A finite element method for high-contrast interface problems with error estimates independent of contrast,* **arXiv 2016**

Hansbos' idea for the interface problem

Doubling of unknowns on all interface cells





…interpolation error of subdomain-wise linear is zero !

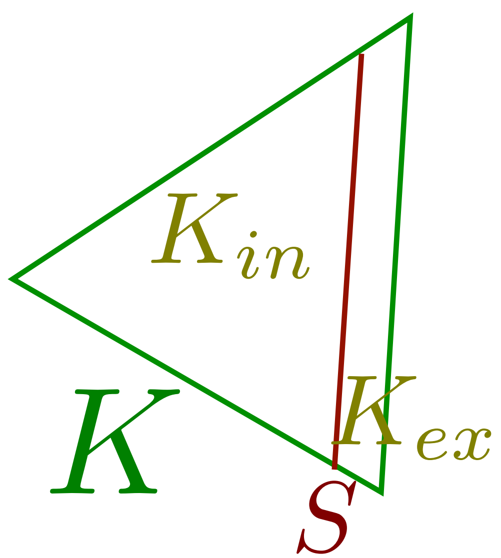
Discrete weak form on V_h:

$$\mathcal{K}_h^* := \mathcal{K}_h^{in} \cup \mathcal{K}_h^{ex}, \qquad \int_{\mathcal{K}_h^*} = \sum_{s \in \{in,ex\}} \int_{\Omega^s} .$$

$$a_h(u,v) := \int_{\mathcal{K}_h^*} k\nabla u \cdot \nabla v - \int_{\mathcal{S}_h^\Gamma} \left(\{k\frac{\partial u}{\partial n}\}_\alpha [v] + [u]\{k\frac{\partial v}{\partial n}\}_\alpha - \frac{\gamma}{h}[u][v] \right)$$

$$\{p\}_\alpha = \alpha p^{in} + (1-\alpha)p^{ex}, \qquad ([pq] = [p]\{q\}_\alpha + \{p\}_{1-\alpha}[q])$$

$$\alpha = \frac{k^{ex}|K^{in}|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}, \qquad \frac{\gamma}{h} = \gamma_0 \frac{k^{in}k^{ex}|S|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}.$$



Robust coercivity with respect to:

$$|||u_h|||_{h,*}^2 := \|k^{\frac{1}{2}}\nabla u_h\|^2 + \| \left(\frac{\gamma}{h}\right)^{\frac{1}{2}} [u_h]\|_\Gamma^2 + \| \left(\frac{\gamma}{h}\right)^{-\frac{1}{2}} \{k\frac{\partial u_h}{\partial n}\}_\alpha\|_\Gamma^2$$

$$a_h(u_h,u_h) \gtrsim |||u_h|||_{h,*}^2$$

C. Annavarapu, M. Hautefeuille, and J. E. Dolbow. *A robust Nitsche's formulation for interface problems.* Comput. Methods Appl. Mech. Engrg., 225/228:44–54, 2012.

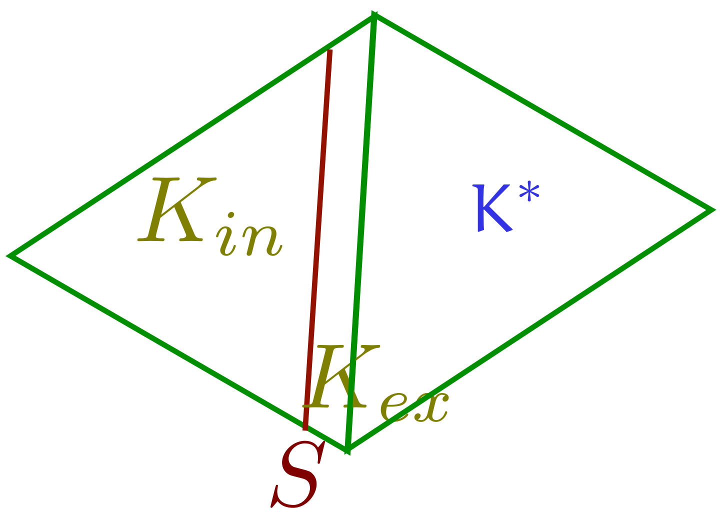
N. Barrau, R. Becker, E. Dubach, and R.Luce. *A robust variant of NXFEM for the interface problem.* C. R. Math. Acad. Sci. Paris, 350(15):789–792, 2012.

Modified discrete weak form:

$$a_h(u,v) := \sum_{s\in\{\text{in,ex}\}} \int_{\Omega^s} k^s \nabla u^s \cdot \nabla v^s - \int_{\mathcal{S}_h^\Gamma} \left(\{k\frac{\partial u}{\partial n}\}_\alpha [v] + [u]\{k\frac{\partial v}{\partial n}\}_\alpha - \frac{\gamma}{h}[u][v] \right)$$

$$+ \int_{\mathcal{S}_h^{in}\cup\mathcal{S}_h^{ex}} \delta_S [\frac{\partial u}{\partial n}][\frac{\partial v}{\partial n}]$$

Allows to shift the estimate (for P^1):



$$\int_S k^{ex}\frac{\partial u^{ex}}{\partial n} = \underbrace{\int_S k^{ex}\frac{\partial u^{ex}_{K^*}}{\partial n}}_{\text{inverse estimate ok!}} + \underbrace{\int_S k^{ex}\left(\frac{\partial u^{ex}}{\partial n} - \frac{\partial u^{ex}_{K^*}}{\partial n}\right)}_{\text{controlled by } \delta}$$

E. Burman. *La pénalisation fantôme.* C. R. Math. Acad. Sci. Paris, 348:1217–1220, 2010.

E. Burman and P. Zunino. *Numerical approximation of large contrast problems with the unfitted Nitsche method.* In Frontiers in numerical analysis—Durham 2010, volume 85 of Lect. Notes Comput. Sci. Eng., pages 227–282. Springer, Heidelberg, 2012.

E. Burman, J. Guzman, M. A. Sanchez, M. Sarkis. *Robust flux error estimation of an unfitted Nitsche method for high-contrast interface problems,* **arXiv 2016**

Lemma.

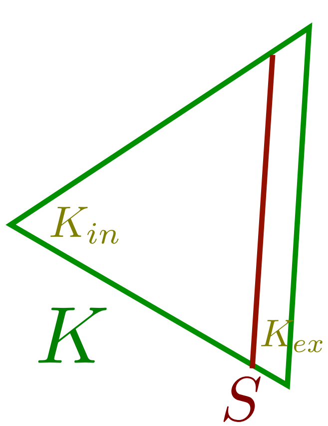
$$a_h(u_h,u_h) \gtrsim |||u_h|||^2_{h,*} \qquad \alpha^{in} = \frac{k^{ex}|K^{in}|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}, \qquad \frac{\gamma}{h} = \gamma_0 \frac{k^{in}k^{ex}|S|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}$$

Proof.

$$a_h(u_h,u_h) = \|k^{\frac{1}{2}}\nabla u_h\|^2 + \|\left(\frac{\gamma}{h}\right)^{\frac{1}{2}}[u_h]\|^2_\Gamma - 2\int_{\mathcal{S}_h^\Gamma}\{k\frac{\partial u_h}{\partial n}\}_\alpha[u_h]$$

Since (P^1)

$$\begin{aligned}
\frac{1}{2}\int_S\{k\frac{\partial u_h}{\partial n}\}^2_\alpha &\leqslant \int_S(\alpha^{in})^2(k^{in})^2\frac{\partial u_h^{in}}{\partial n}^2 + \int_S(\alpha^{ex})^2(k^{ex})^2\frac{\partial u_h^{ex}}{\partial n}^2 \\
&\leqslant \frac{(\alpha^{in})^2k^{in}|S|}{|K^{in}|}\int_{K^{in}}k^{in}|\nabla u_h^{in}|^2 + \frac{(\alpha^{ex})^2k^{ex}|S|}{|K^{ex}|}\int_{K^{ex}}k^{ex}|\nabla u_h^{ex}|^2 \\
&\leqslant \frac{k^{in}k^{ex}|S|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}\int_{K^{in}}k^{in}|\nabla u_h^{in}|^2 + \frac{k^{in}k^{ex}|S|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}\int_{K^{ex}}k^{ex}|\nabla u_h^{ex}|^2 \qquad (0 \leqslant \alpha \leqslant 1) \\
&= \frac{\gamma}{h\gamma_0}\int_{K^{in}}k^{in}|\nabla u_h^{in}|^2 + \frac{\gamma}{h\gamma_0}\int_{K^{ex}}k^{ex}|\nabla u_h^{ex}|^2 \\
&\leqslant \frac{\gamma}{h\gamma_0}\|k\nabla u_h\|^2.
\end{aligned}$$



Then

$$2\int_{\mathcal{S}_h^\Gamma}\{k\frac{\partial u_h}{\partial n}\}_\alpha[u_h] \leqslant \frac{1}{2}\|k\nabla u_h\|^2 + \frac{2}{\gamma_0}\|\left(\frac{\gamma}{h}\right)^{\frac{1}{2}}[u_h]\|$$

A Céa-type lemma

$$|||v|||_h^2 = \|k^{\frac{1}{2}}\nabla_h v\|^2 + \int_{\mathcal{S}_h^\Gamma} \left(\frac{\gamma}{h}\right)[v]^2 \qquad v \in V_h + V$$

$$|||v|||_{h,*}^2 = |||v|||_h^2 + \int_{\mathcal{S}_h^\Gamma} \left(\frac{\gamma}{h}\right)^{-1} \{k\frac{\partial v}{\partial n}\}_\alpha^2 \qquad v \in V_h$$

$$|||u - u_h|||_h \lesssim \inf_{w_h \in V_h} |||u - w_h|||_h + \mathrm{osc}_h$$

ok if:

$$|||u_h - w_h|||_h \lesssim |||u - w_h|||_h + \mathrm{osc}_h \qquad w_h \in V_h$$

T. Gudi. *A new error analysis for discontinuous finite element methods for linear elliptic problems.* Math. Comp., 79(272):2169–2189, 2010.

S. Mao and Z. Shi. *On the error bounds of nonconforming finite elements.* Science China Mathematics, 53(11):2917–2926, 2010.

$$|||u_h - w_h|||_h \lesssim |||u - w_h||| + \mathrm{osc}_h \qquad w_h \in V_h$$

$$E_h : V_h \to P_h^1 \quad E_h v_h|_K = \begin{cases} v_h|_K & K \notin \mathcal{K}_h^\Gamma \\ \sum_{i \in I_k^{in}} v_{in}^i \lambda_i + \sum_{i \in I_k^{ex}} v_{ex}^i \lambda_i & K \in \mathcal{K}_h^\Gamma \end{cases} \qquad (v_h|_K = \sum_i v_{in}^i \lambda_i \chi_{K_{in}} + v_{ex}^i \lambda_i \chi_{K_{ex}})$$

$$E_h v_h = v_h - \delta_h$$

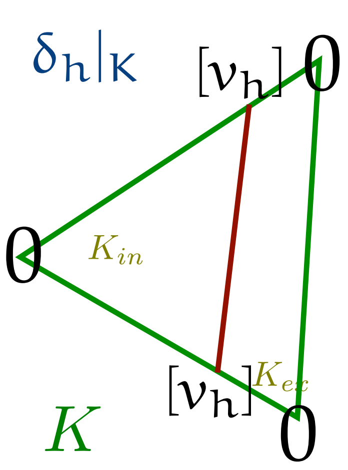
$$|||\underbrace{u_h - w_h}_{=:\, v_h}|||_{h,*}^2 \lesssim a_h(u_h - w_h, v_h) = l_h(v_h) - a_h(w_h, v_h) = \underbrace{l_h(E_h v_h) - a_h(w_h, E_h v_h)}_{=:A} + \underbrace{l_h(\delta_h) - a_h(w_h, \delta_h)}_{=:B}$$

$$A = \int_\Omega k\nabla(u - w_h) \cdot \nabla E_h v_h + \int_{\mathcal{S}_h^\Gamma} [w_h - u]\{k \frac{\partial E_h v_h}{\partial n}\}$$

$$\leqslant |||u - w_h)||| \times |||v_h||| \qquad (|||E_h v_h|||_h \lesssim |||v_h|||)_h$$

$$B = l_h(\delta_h) - a_h(w_h, \delta_h) = \int_{\mathcal{K}_h^\Gamma} (f + \mathrm{div}(k\nabla w_h))\delta_h + \int_{\mathcal{S}_h^\Gamma} [w_h](\{k \frac{\partial \delta_h}{\partial n}\} - \gamma_h[\delta_h])$$

$$(|||u - w_h|||_h + \mathrm{osc}_h) \times |||\delta_h|||_{h,*} \qquad \left(|||\delta_h|||_{h,*} \lesssim \|h^{-\frac{1}{2}}[v_h]\|\right)$$



Theorem. *If* $u \in H^2_*$

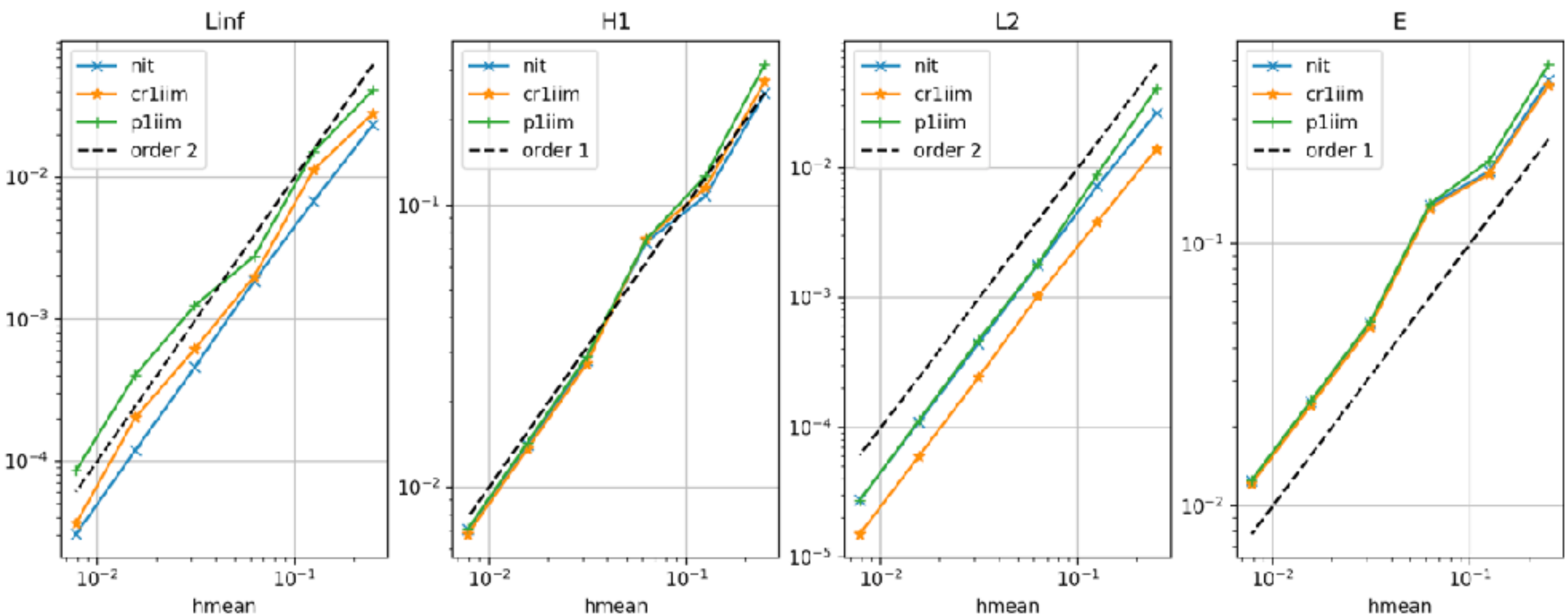
$$|||u - I_h u|||_h \lesssim h\|u\|_{H^2_*}.$$

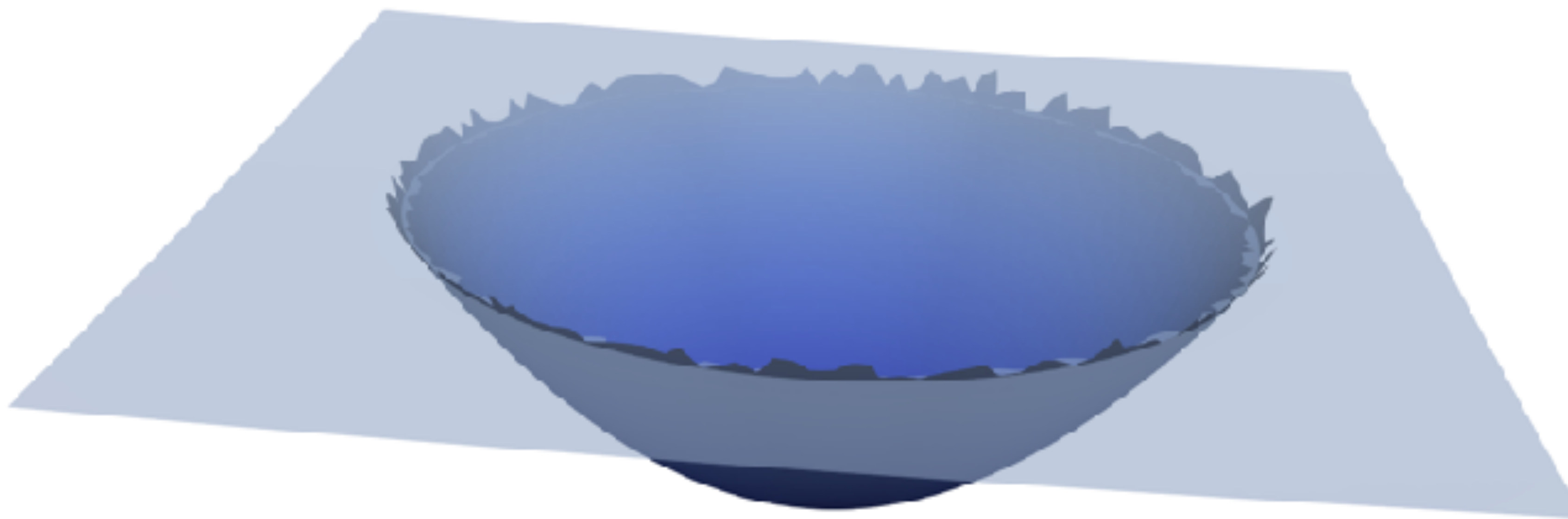
Interpolation based on extension operators

$$E^{in/ex} : H^2(\Omega^{in/ex}) \to H^2(\Omega)$$

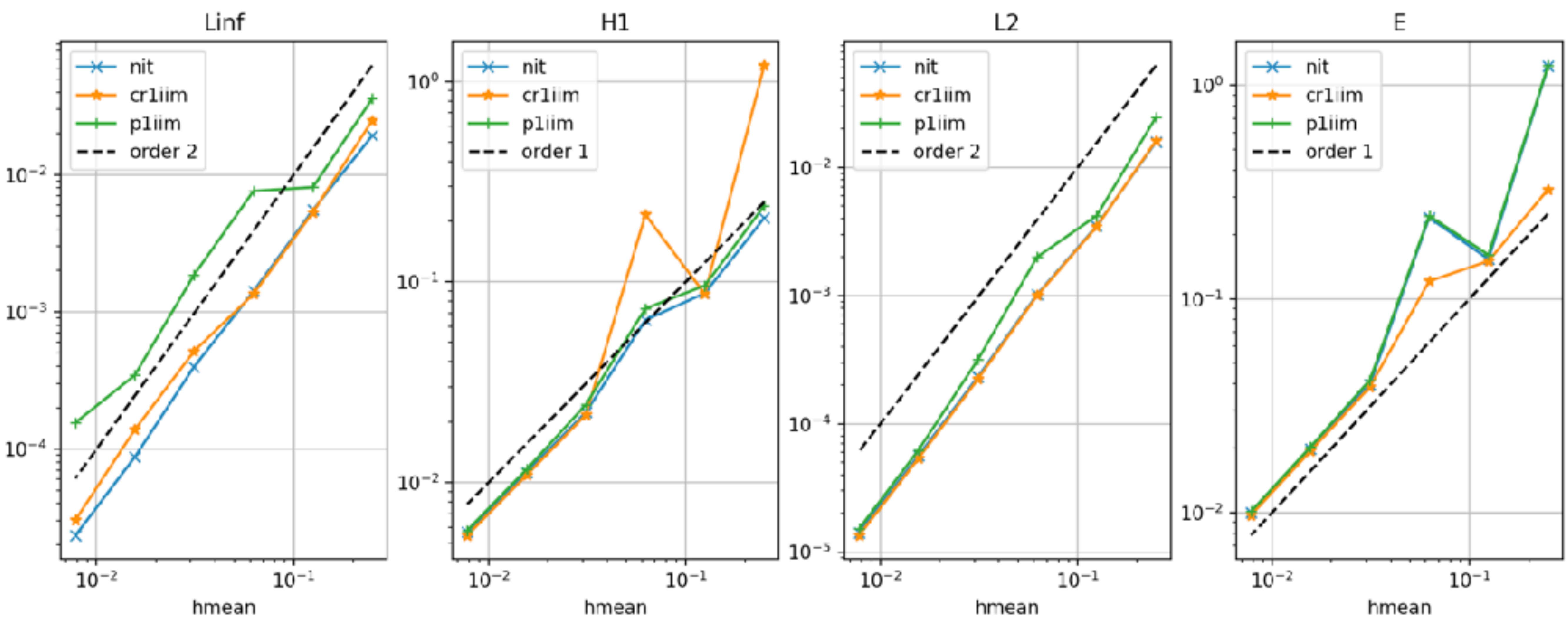
Numerical test

$$k^{in} = 100, \quad k^{ex} = 1$$

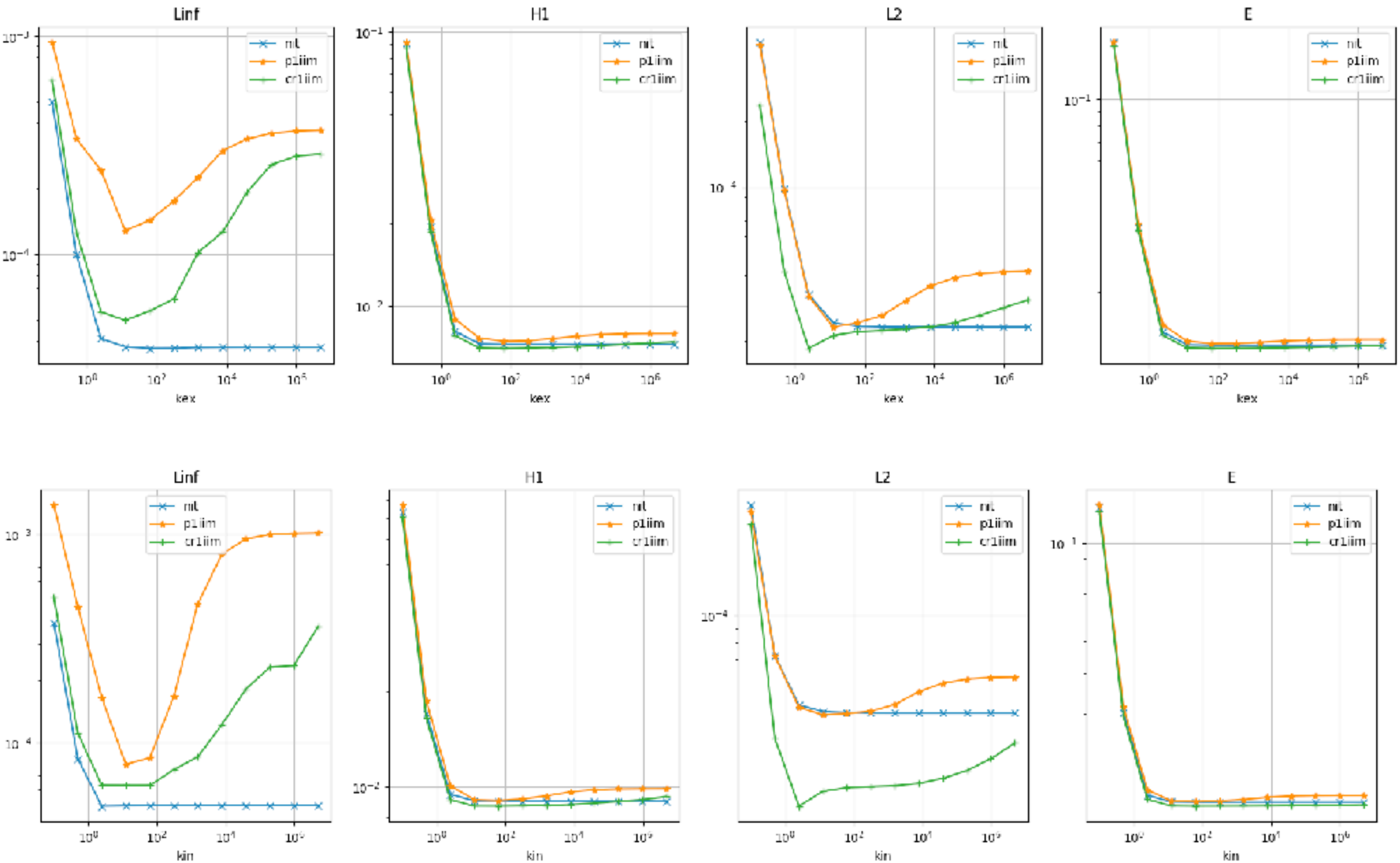




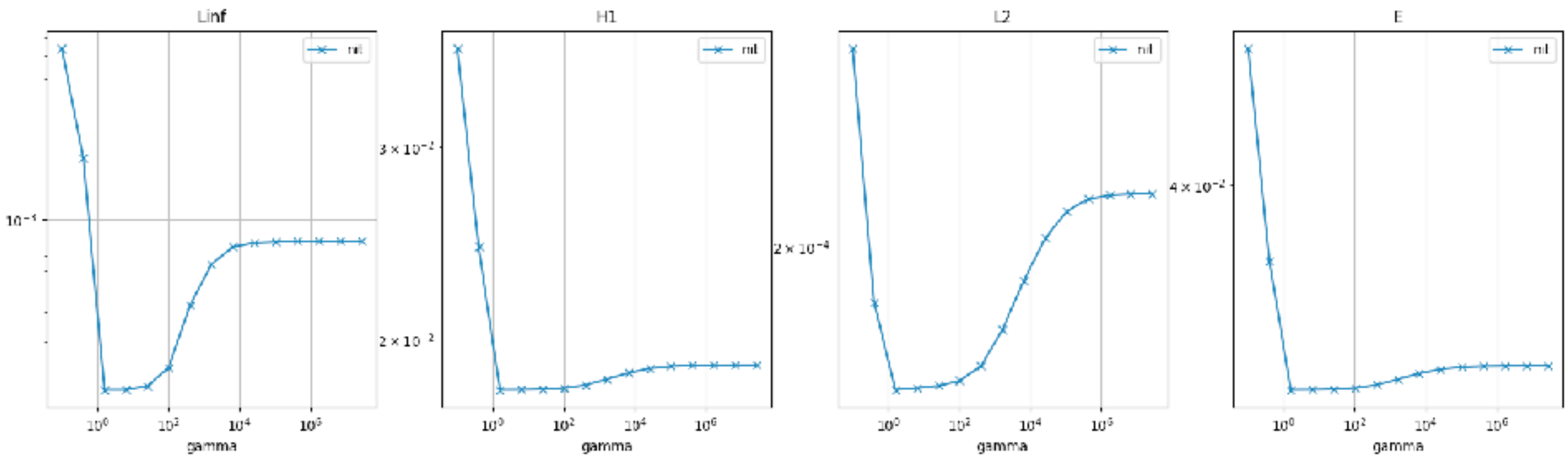
$$k^{in} = 1, \quad k^{ex} = 100$$



Robustness



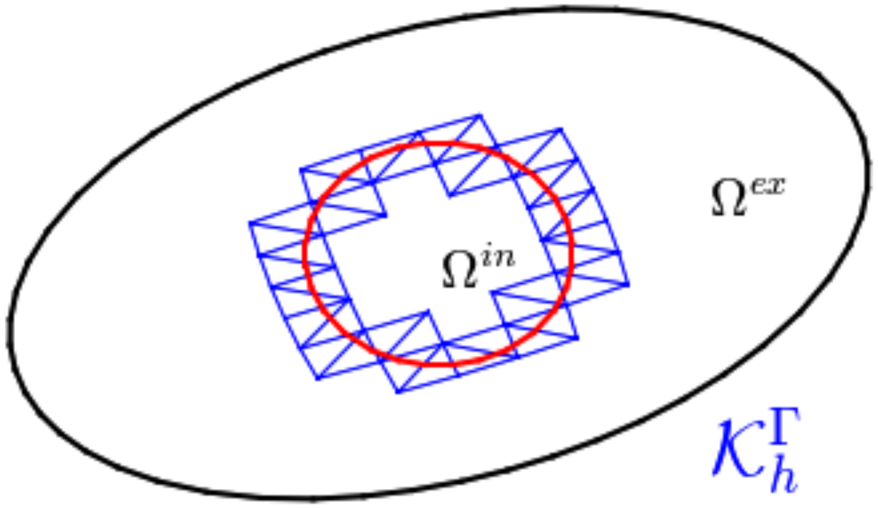
Stabilization



Alternative

Subspace-splitting global

$V_h = V^0 \oplus V^*$



First approach: local projections

Second approach : Lagrange multipliers

Local projections

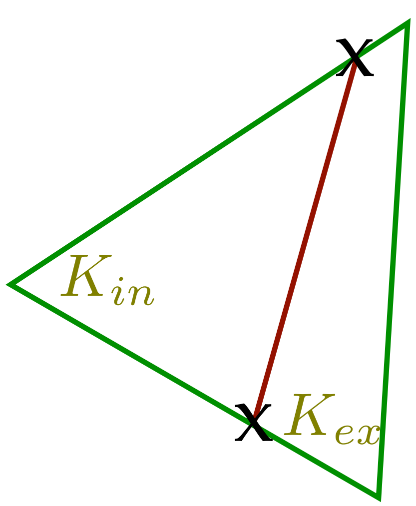
Alternative for Dirichlet ($u^D = 0$))

$V_h = V^0 \oplus V^*, \qquad P : V \to V^0, \quad Pu = u^0, \qquad Q = I - P$

$$\widetilde{E}(u) = \frac{1}{2}\int_\Omega k|\nabla Pu|^2 + \frac{1}{2}\int_\Omega k|\nabla Qu|^2 - \int_\Omega fu$$

Ideally we would like to have:

$P(u^{in}, u^{ex}) = (u^{in} - R[u], u^{ex} + R[u]) \quad [u] = u^{in} - u^{ex}$



Locally

$P_K(u_K^{in}, u_K^{ex}) = (u_K^{in} - R[u]_K, u_K^{ex} + R[u]_K) \quad [u]_K = u_K^{in} - u_K^{ex}$

$$\widetilde{E}(u) = E(u) + \sum_{K \in \mathcal{K}_h^\Gamma} (l_K(Q_K u) - a_K(P_K u, Q_K u))$$

Lagrange multipliers

$$V^0 = \left\{ v \in V_h | \quad [v](x_E^*) = 0 \quad \forall E \in \mathcal{E}_h^\Gamma \right\}$$

$$u \in V^0 : \qquad a(u,v) = l(v) \quad v \in V^0$$

Stabilized multiplier method: (*Hughes,Stenberg*)

$$\Lambda_h := D^1(\Gamma_h)$$

$$(u,\lambda) \in V \times \Lambda_h : \qquad a(u,v) + \int_\Gamma [v]\lambda + \int_\Gamma [u]\mu + s_h(u,\lambda,v,\mu) = l(v) \quad (v,\mu) \in V \times \Lambda_h$$

$$s_h(u,\lambda,v,\mu) := -r \int_\Gamma \frac{h^{in}}{k^{in}} \left(k^{in} \frac{\partial u^{in}}{\partial n} - \lambda \right) \left(k^{in} \frac{\partial v^{in}}{\partial n} - \mu \right) - r \int_\Gamma \frac{h^{ex}}{k^{ex}} \left(k^{ex} \frac{\partial u^{ex}}{\partial n} - \lambda \right) \left(k^{ex} \frac{\partial v^{ex}}{\partial n} - \mu \right)$$

$$\Rightarrow r \left(\frac{h^{in}}{k^{in}} + \frac{h^{ex}}{k^{ex}} \right) \lambda = [u] - r \frac{h^{in}}{k^{in}} k^{in} \frac{\partial u^{in}}{\partial n} - r \frac{h^{ex}}{k^{ex}} k^{ex} \frac{\partial u^{in}}{\partial n}$$

$$\Rightarrow \lambda = \frac{\gamma}{h}[u] - \{k \frac{\partial v}{\partial n}\}_\alpha, \qquad \alpha = \frac{k^{ex}|K^{in}|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}, \qquad \frac{\gamma}{h} = \gamma_0 \frac{k^{in}k^{ex}|S|}{k^{ex}|K^{in}| + k^{in}|K^{ex}|}.$$

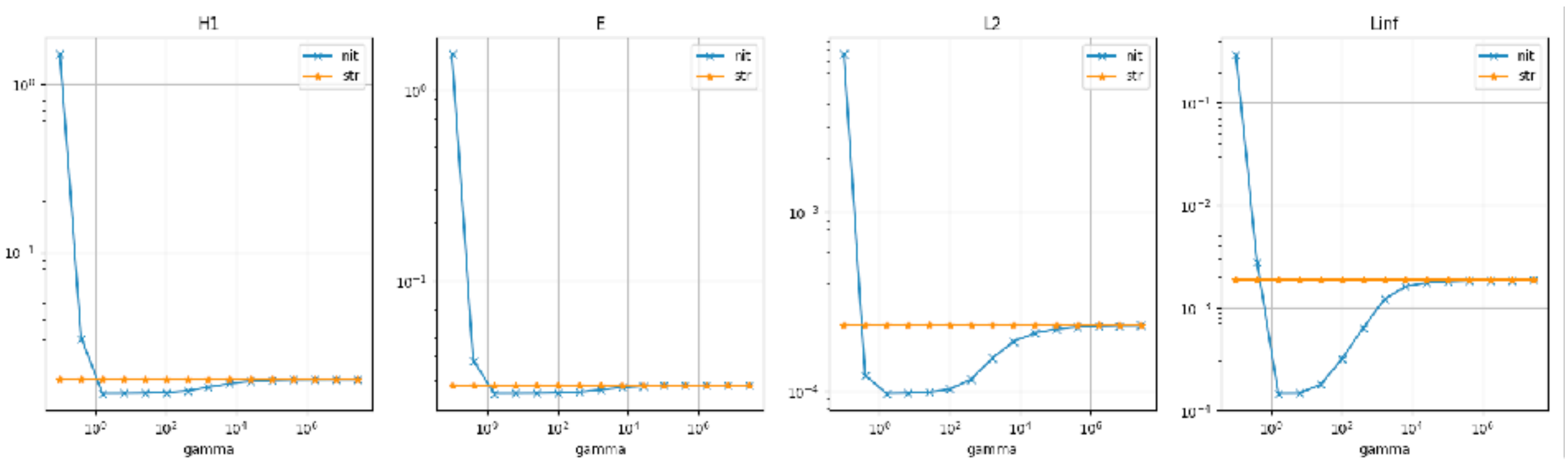
$$a_h^{Nit}(u,v) := a(u,v) - \int_{\mathcal{S}_h^\Gamma} \left(\{k \frac{\partial u}{\partial n}\}_\alpha [v] + [u]\{k \frac{\partial v}{\partial n}\}_\alpha - \frac{\gamma}{h}[u][v] + \frac{k^{in}|K^{in}|}{\gamma|S|} \frac{\partial u^{in}}{\partial n} \frac{\partial v^{in}}{\partial n} + \frac{k^{ex}|K^{ex}|}{\gamma|S|} \frac{\partial u^{ex}}{\partial n} \frac{\partial v^{ex}}{\partial n} \right)$$

Another multiplier method:

$$\tilde{\Lambda}_h = \gamma^{in}({V_h}^{in}) + \gamma^{ex}({V_h}^{ex}) = P^1(\Gamma_h)$$

$$(u_r, \lambda) \in V \times \tilde{\Lambda}_h : \qquad a_h^{Nit}(u_r, v) + \int_\Gamma [v]\theta + \int_\Gamma [u_r]\mu - r\int_\Gamma \theta\mu = l(v) \quad (v, \mu) \in V \times \tilde{\Lambda}_h$$

$$\Rightarrow \quad \theta = \frac{1}{r}[u_r] \quad \Rightarrow \quad \text{Equivalent to previuous}$$



$$u_h^r \to u_h^{str} \quad (r \to 0)$$

DMP

$$\int_{\Omega^{in}} k^{in} \nabla u^{in} \cdot \nabla v^{in} - \int_{\Gamma} \lambda v^{in} = \int_{\Omega^{in}} f v^{in}$$

$$\int_{\Omega^{ex}} k^{ex} \nabla u^{ex} \cdot \nabla v^{ex} + \int_{\Gamma} \lambda v^{ex} = \int_{\Omega^{ex}} f v^{ex}$$

$$u_h = \sum_i u_i \lambda_i \quad \Rightarrow u_h^- := \sum_i u_i^- \lambda_i \quad (x^- = \min\{x, 0\})$$

$$\int_{\Omega^{in}} k^{in} |\nabla (u^{in})^-|^2 \underbrace{\leqslant}_{\int_{\Omega^{in}} k^{in} \nabla (u^{in})^+ \cdot \nabla (u^{in})^- \geqslant 0} \int_{\Omega^{in}} f (u^{in})^- \int_{\Gamma} + \lambda (u^{in})^-$$

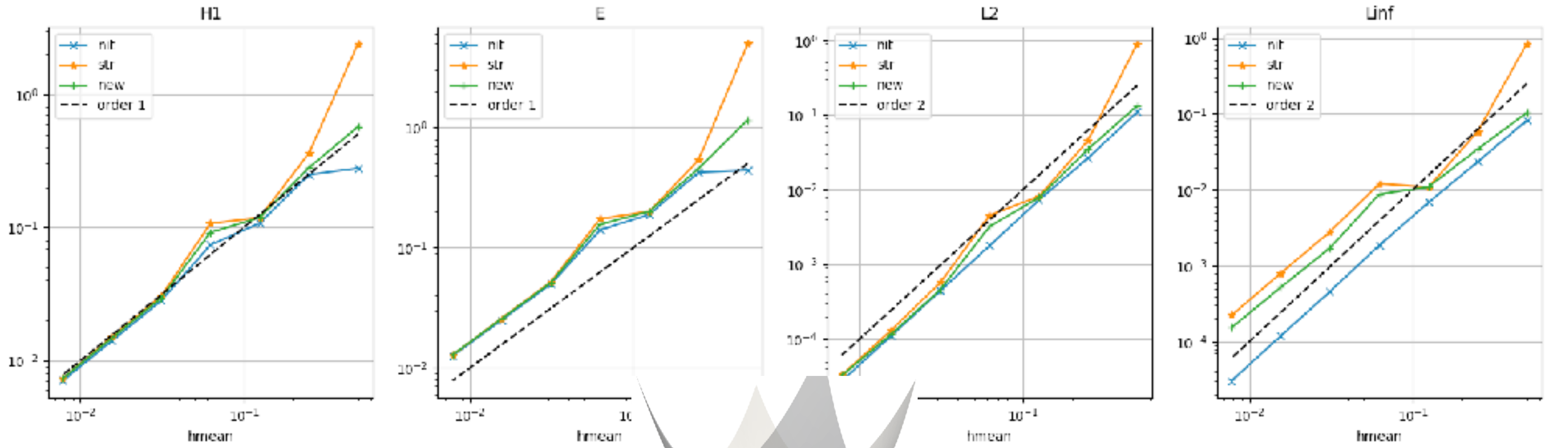
angle condition

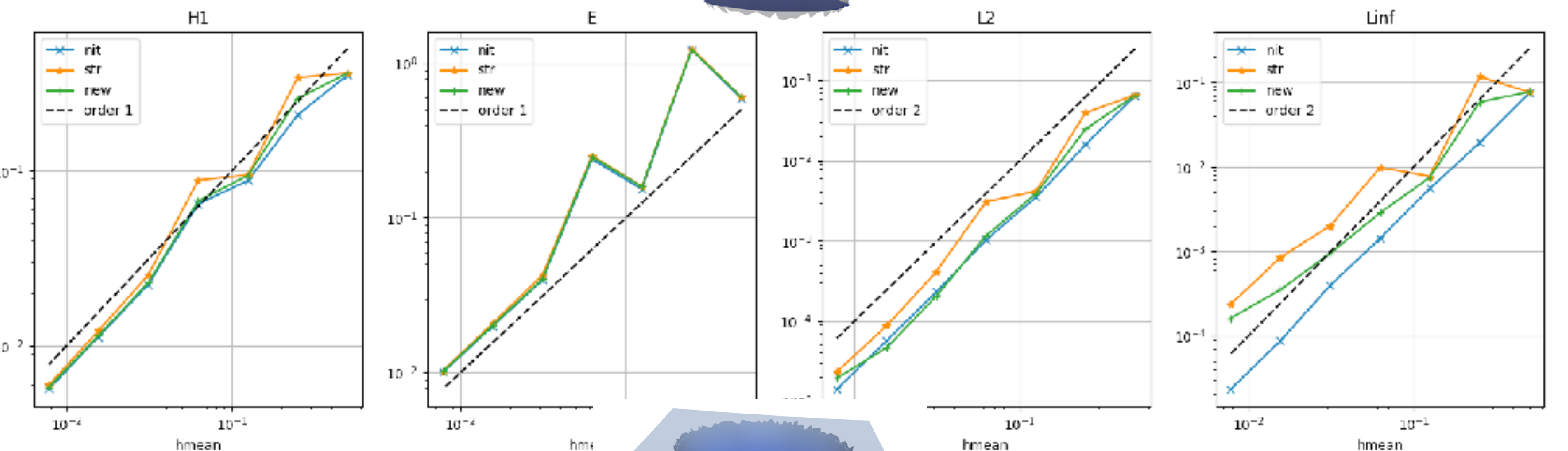
$$\int_{\Omega^{ex}} k^{in} |\nabla (u^{ex})^-|^2 \leqslant \int_{\Omega^{ex}} f (u^{ex})^- \int_{\Gamma} - \lambda (u^{ex})^-$$

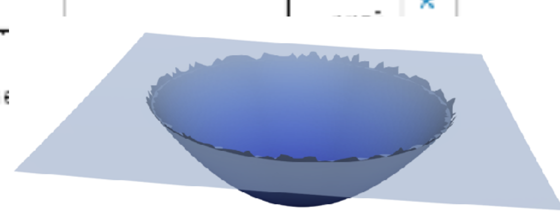
$$u^{in}|_{\Gamma} = u^{ex}|_{\Gamma} \quad \Rightarrow \quad (u^{in})^-|_{\Gamma} = (u^{ex})^-|_{\Gamma}$$

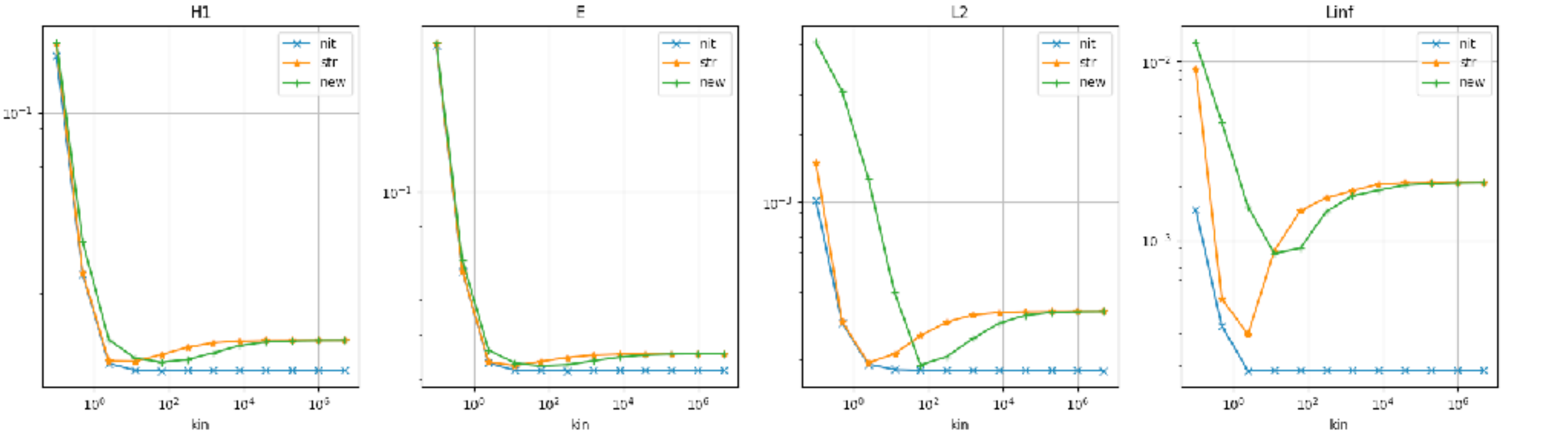
$$\Rightarrow \quad (f \geqslant 0 \quad \Rightarrow \quad u_h \geqslant 0)$$

Numerical test









Robustness