A phase conservative, monolithic level-set method (with built-in redistancing) for multiphase flow

Manuel Quezada de Luna [1]
Dmitri Kuzmin [2]   Chris E. Kees [1]

[1]US Army ERDC-CHL

[2]TU Dortmund

Outline

Introduction to our group

- Goal, motivations and introduction to Proteus.
- Collaborators.

A phase conservative level set method

- Motivation, applications and goals.
- Conservation within level-set like methods.
- Background.
- Two monolithic models.
- Details on the numerical discretization.
- Numerical examples.
- Conclusion.

Introduction to our group

Introduction to our group

- U.S. Army Engineer Research and Development Center.
- Solve problems for the Army, DoD, civilian agencies and others [1].
- Focus of work [1]:
    - Civil and military engineering.
    - Geospatial sciences.
    - Environmental sciences.
    - Water resources: **Coastal and Hydraulics Lab (CHL)**.

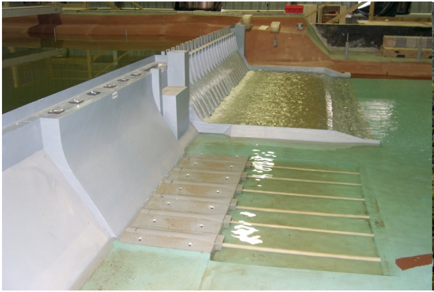
[1] https://www.erdc.usace.army.mil

Goal of our group

- The goal of our group is to simulate fluid-solid interaction (FSI) problems with one, two and potentially three fluid phases.
- The third phase is composed by granular media.

Riverine Structures: Bluestone Dam

- Constructed in 1949, primarily to reduce flood damage along the New, Kanawha, Ohio, and Mississippi rivers
- Designed for 430,000 cfs but now needs to meet probable maximum flood of 1,000,000 cfs
- Studied with multiple physical models: 1:65-scale (complete), 1:36-scale and 1:25-scale (sections)
- Downstream *scour* is a major concern




Coastal Structures: Azores Breakwater

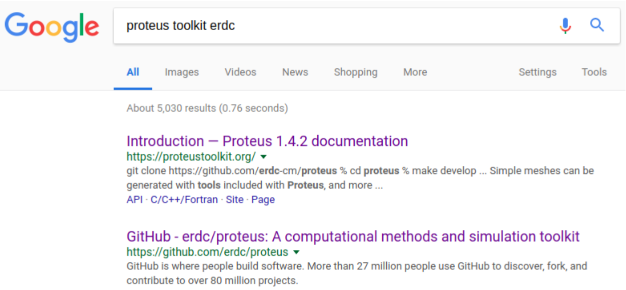
- Protects critical military refueling station
- Retrofit of concrete armoring designed by CHL to withstand larger storms
- Experiences waves and flow with with significant viscous, air, and porosity effects that can *move* armoring units if not designed properly

Navigation

- Need to understand vessel dynamics, particularly in coastal channels and ports
- Need to understand bank *erosion* due to vessel wakes
- Need to understand *deposition* and *erosion* in navigation channels


Introduction to Proteus

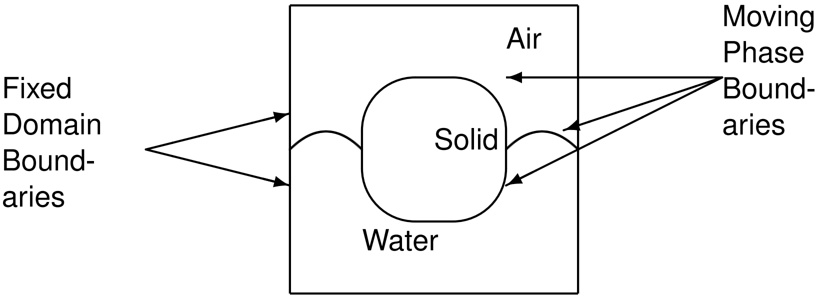
- Proteus is a Python package but uses C/C++/Fortran as needed.
- Parallel, unstructured, higher-order, *adaptive*,...
- Includes hybrid LS/VOF formulation with ALE or immersed/embedded methods for moving solid boundaries.
- Released under the MIT license following guidance from US Defense Digital Service and ERDC counsel.



Introduction to Proteus: current capabilities and development

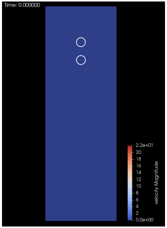
- **Existing equation sets:** multiphase Navier-Stokes and RANS formulations (air/water), shallow water equations, diffusive wave equation, Richards' equations, two-phase flow in porous mdiea, elasticity, plasticity,...
- **Existing methods:** Lagrange/Bernstein finite elemenents, Variational multiscale methods, Discontinuous Galerkin methods, non-conforming methods.
- **In development:** three-phase RANS, Serre-Green-Naghdi equations, Entropy Viscosity methods, Schur complement preconditioners, anisotropic adaptivity,. . .

Introduction to Proteus: Domain with Three Mobile Phases



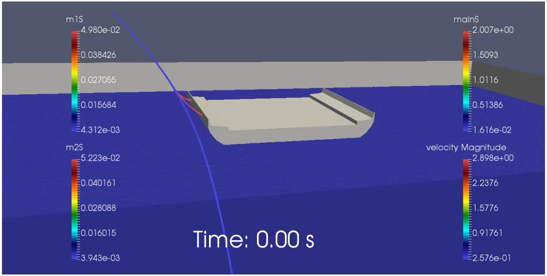
Collaborators
Immersed and shifted boundary method

- Yong Yang leads this development within our lab.
- Guillermo Scovazzi and Leo Nouveau from Duke University.
- Arnold Song from U.S. Army ERDC Cold Regions Research and Eng. Lab. (CRREL).



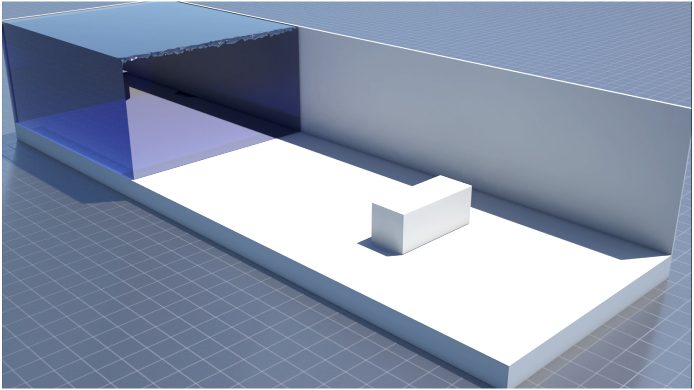
HR Wallingford (Aggelos Dimakopoulos, Tristan de Lataillade, Branoc Richards, Pedro Otinar Morillas, Jonathan Simm and Giovanni Cuomo).

Wave generation, dynamic mooring, wave structure interaction,hydraulic structures, moving floating structures, turbulence modelling, sediment transport and other real world applications.



Collaborators
Level-sets/volume of fluids

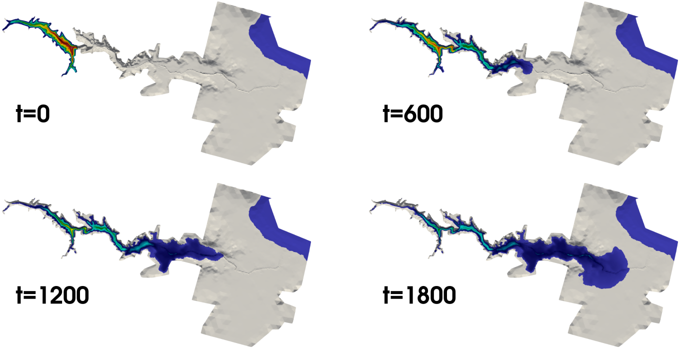
- Dmitri Kuzmin from TU Dortmund, Germany.



Collaborators

Shallow water equations and dispersive corrections

- Jean-Luc Guermond, Bojan Popov and Eric Tovar from Texas A& M University.
- Matthew W. Farthing, U.S. Army ERDC-CHL.



Collaborators

h-adaptivity

- Onkar Sahni and Alvin Zhang from Rensselaer Polytechnic Institute.

Preconditioners

- Andrew Wathen and Nial Bootland from Oxford University.
- Vince Ervin, Alistair Bentley from Clemson University.

Absorbing boundary conditions

- Jon Chapman and Helen Fletcher from Oxford University.

Vegetation effects on water waves

- Jon Chapman and Clint Wong from Oxford University.

Others

- Dmitri Kuzmin (TU Dortmund). $\mathcal{C}^0$ blending FE spaces and time discretizations.
- Haydel J. Collins (U.S. Army Corps of Eng., NOLA District). Applications.

A phase conservative level set method

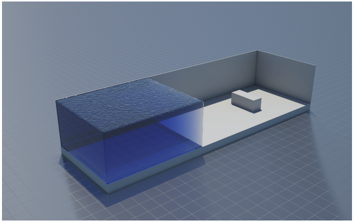
Motivation

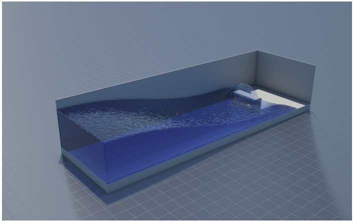
Model and simulate the interface evolution during incompressible multiphase flows.

Applications

Some applicaitons are:

- Water-air interaction.
- Multiple phase fluid solid interaction.





Goals

- Reformulate/propose a level set and/or volume of fluid like method for moving interfaces with incompressible flows.
- We aim to obtain:
  - Phase/volume conservation.
  - Natural/easy representation of the interfaces.
  - A monolithic model.
  - Robust and with few (or no) parameters.
  - Suitable for standard finite element discretizations in space.

Conservation

Assume $\mathbf{v} \cdot \mathbf{n} = 0$.

$$\partial_t V^+ = \partial_t \int_\Omega H_\epsilon(\phi) d\mathbf{x} = 0$$

Background: level-set (LS) vs volume of fluid (VOF)

Level-set (LS) method

- Proposed by [Osher and Sethian(1988),Sussman et all.(1994)].
- Interface is the isosurface of an auxiliary function.
- Transport the level-set function.
- Easy and natural representation of the interface.
- Loss of phase/volume conservation.

Volume of fluid (VOF) method

- Proposed by [Hirt and Nichols(1981)].
- Characteristic function to identify phases.
- Transport the characteristic function.
- Requires interface reconstruction.
- Phase/volume conservative.

Background: hybrid methods

Hybrid methods: [Enright et al.(2002), Ianniello and Di Mascio(2010), Sussman and Puckett(2000)].

Conservative level-set method by [Kees et al.(2011)]

- Goal: correct a level-set solution to obtain conservation.
- Level-set: distance function to the interface.

Stage 1: level-set

$$\partial_t \phi + \mathbf{v} \cdot \nabla \phi = 0$$

Stage 2: redistancing

$$\left| \nabla \hat{\phi} \right| = 1$$

Stage 3: volume of fluid

$$\partial_t \tilde{H} + \nabla \cdot \left[\mathbf{v} H_\epsilon \left(\hat{\phi} \right) \right] = 0$$

Stage 4: mass correction

$$H_\epsilon \left(\hat{\phi} + \phi' \right) - \tilde{H} - \kappa h \Delta \phi' = 0$$

Towards a monolithic level-set method

Consider stages 3 and 4 from [Kees et al.(2011)] (via forward Euler).

Stage 3: volume of fluid

$$\frac{\tilde{H}-H_\epsilon(\phi^n)}{\Delta t} + \nabla \cdot [\mathbf{v} H_\epsilon(\phi^n)] = 0$$

Stage 4: mass correction

$$H_\epsilon\left(\hat{\phi} + \phi'\right) - \tilde{H} - \kappa h \Delta \phi' = 0$$

$$\phi^{n+1} = \hat{\phi} + \phi'$$

⬇

$$\frac{H_\epsilon(\phi^{n+1}) - H_\epsilon(\phi^n)}{\Delta t} + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi^n) - \frac{\kappa h}{\Delta t} \nabla \left(\phi^{n+1} - \hat{\phi}\right)\right] = 0$$

Towards a monolithic level-set method

Consider stages 3 and 4 from [Kees et al.(2011)] (via forward Euler).

Stage 3: volume of fluid

$$\frac{\tilde{H}-H_\epsilon(\phi^n)}{\Delta t} + \nabla \cdot [\mathbf{v} H_\epsilon(\phi^n)] = 0$$

Stage 4: mass correction

$$H_\epsilon\left(\hat{\phi}+\phi'\right) - \tilde{H} - \kappa h \Delta \phi' = 0$$
$$\phi^{n+1} = \hat{\phi} + \phi'$$

$$\partial_t H_\epsilon(\phi) + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi) - \frac{\kappa h}{\Delta t} \nabla\left(\phi - \hat{\phi}\right)\right] = 0$$

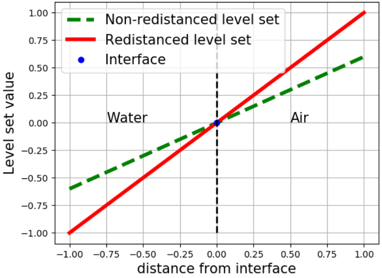
First 'ingredient'!

Redistancing

We still need a process to redistance the level set.

Some alternatives are:

- Geometric approaches.
- PDE based methods. Impose $|\nabla\phi| = 1$ and keep $\{\mathbf{x} \in \Omega \mid \phi(\mathbf{x}) = 0\}$ unchanged.
    - Hyperbolic.
    - Convective.
    - Parabolic.
    - **Elliptic**.



Elliptic redistancing

- We consider the elliptic redistancing by [Basting and Kuzmin(2013)].
- Based on the parabolic redistancing by [Chunming Li et al(2013)].

$$\min \frac{\alpha}{2} \int_{\Gamma(\tilde{\phi})} \hat{\phi}^2 ds + \frac{1}{2} \int_\Omega \left(|\nabla \hat{\phi}| - 1 \right)^2 d\mathbf{x}$$

$$\alpha \int_{\Gamma(\tilde{\phi})} \hat{\phi} w ds + \int_\Omega \left(\nabla \hat{\phi} - \frac{\nabla \hat{\phi}}{|\nabla \hat{\phi}|} \right) \cdot \nabla w d\mathbf{x} = 0$$

Second ‘ingredient’!

First monolithic model

Reformulated conservative level-set method

$$\partial_t H_\epsilon(\phi) + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi) - \lambda \left(\nabla \phi - \nabla \hat{\phi} \right) \right] = 0$$

Elliptic redistancing

$$\hat{\phi} \delta \left(\Gamma(\tilde{\phi}) \right) - \nabla \cdot \frac{1}{\alpha} \left(\nabla \hat{\phi} - \frac{\nabla \hat{\phi}}{|\nabla \hat{\phi}|} \right) = 0$$

Monolithic conservative level-set model 1

$$\partial_t H_\epsilon(\phi) + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi) - \lambda \left(\nabla \phi - \frac{\nabla \phi}{|\nabla \phi|} \right) \right] = 0$$

First monolithic model: benefits

$$\partial_t H_\epsilon(\phi) + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi) - \lambda \left(\nabla\phi - \frac{\nabla\phi}{|\nabla\phi|} \right) \right] = 0$$

- **Monolithicity.**
- **Transport:** the transport of the interface is embedded.
- **Conservation:** conservation is achieved provided BCs:

$$\mathbf{n} \cdot \left(\nabla\phi - \frac{\nabla\phi}{|\nabla\phi|} \right) = 0, \quad \forall \mathbf{x} \in \partial\Omega$$

- **Regularization and redistancing:** the term $\lambda \left(\nabla\phi - \frac{\nabla\phi}{|\nabla\phi|} \right)$ regularizes the equation and penalizes deviations from the distance function property.

First monolithic model: drawbacks

$$\partial_t H_\epsilon(\phi) + \nabla \cdot \left[\mathbf{v} H_\epsilon(\phi) - \lambda \left(\nabla \phi - \frac{\nabla \phi}{|\nabla \phi|} \right) \right] = 0$$

- Two main sources of non-linearity with different behavior.
  - Due to smoothed Heavisides. Well behaved, we need small tolerances.
  - Due to redistancing. Not well behaved, we don't need small tolerances.
- Redistancing problems near peaks.
- Not well behaved Jacobian.
- Slow convergence of (pseudo) Newton method.
- Bad quality at the interface.

Second monolithic model

- We borrow an idea from [Chan, Golub and Mulet(1999)].
- Do a $\mathcal{C}^0$ reconstruction of $\frac{\nabla\phi}{|\nabla\phi|}$.
- Their motivation is to accelerate the nonlinear solver by easing the nonlinearity due to redistancing.

Monolithic conservative level-set model 2

$$\begin{aligned} \partial_t H_\epsilon(\phi) + \nabla\cdot\left[\mathbf{v}H_\epsilon(\phi) - \lambda\left(\nabla\phi - \mathbf{q}\right)\right] &= 0, && \forall\mathbf{x}\in\Omega \\ \sqrt{|\nabla\phi| + \delta}\,\mathbf{q} &= \nabla\phi, && \forall\mathbf{x}\in\Omega \\ (\nabla\phi - \mathbf{q})\cdot\mathbf{n} &= 0 && \forall\mathbf{x}\in\partial\Omega \end{aligned}$$

Second monolithic model... about parameter λ

The parameter λ scales like speed.

Towards a dimensionless parameter

To make the parameter dimensionless we consider

$$\lambda \sim \left(\frac{h(\mathbf{x})}{\Delta t}\right)\tilde{\lambda}, \qquad \frac{h(\mathbf{x})}{\Delta t} = \mathcal{O}(1), \quad \text{via a CFL restriction}$$

For consistency

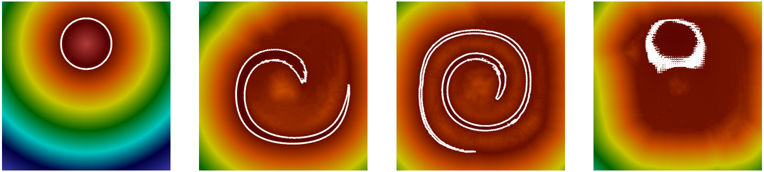
For consistency we further scale the parameter λ as follows:

$$\lambda \sim \tilde{\lambda}\left(\frac{h(\mathbf{x})}{\Delta t}\right)\frac{h(\mathbf{x})}{||\phi - \bar{\phi}||_{L^\infty(\Omega)}}$$

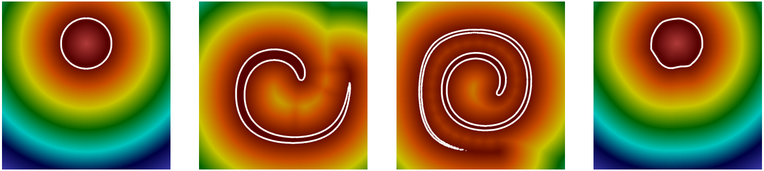
Monolithic conservative level-set model 1 vs model 2

We consider the 2D vortex problem proposed by [Rider and Kothe(1995)].

Model 1



Model 2



Solution at $t = 0, 2, 4, 8$.

Some details on the numerical discretization

Time discretization

2nd order IMEX prediction correction scheme by [Hundsdorfer and Verwer(2013)].

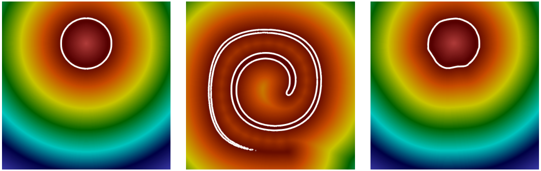
Spatial discretization

- Standard CG-FEM.
- No need of numerical artifacts like: extra stabilization, flux limiting, artificial compression, mass correction, post redistancing, etc.
- We integrate by parts the flux term.
- $\mathcal{C}^0$ reconstruction of normal field $\frac{\nabla\phi}{|\nabla\phi|}$ via a mass lumped L^2 projection.

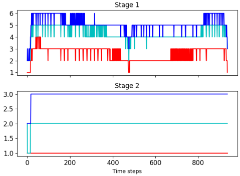
Numerical examples: level-set

2D periodic vortex

Newton tolerance of: 1×10^{-12}, 1×10^{-8} and 1×10^{-4}.

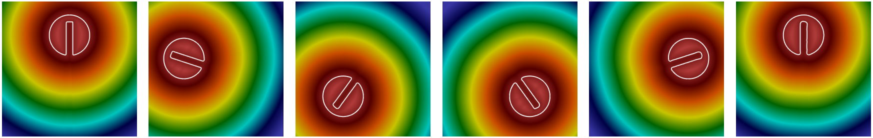


Solution at $t = 0, 4, 8$



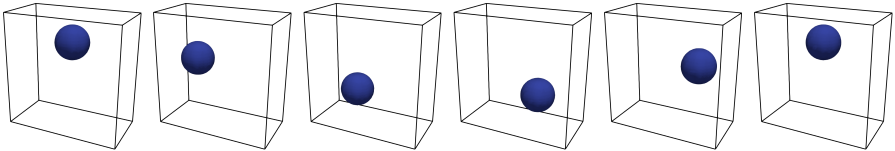
Newton iterations

Zalesak's disk

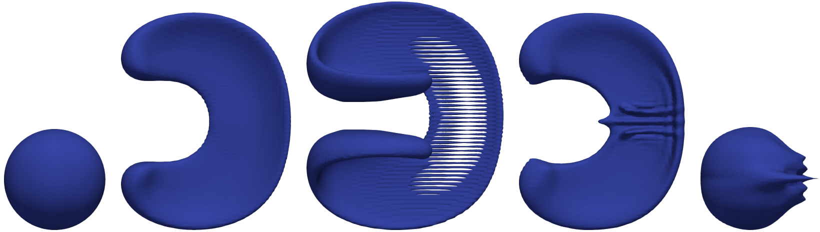


Numerical examples: level-set

3D solid rotation



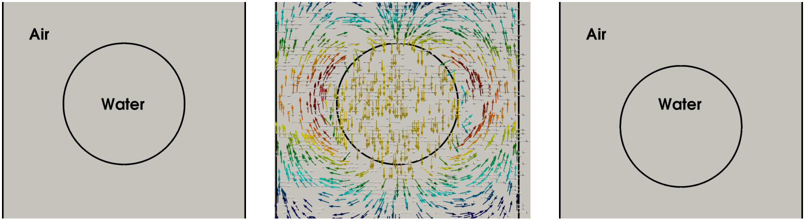
LeVeque test



Multiphase flow

Multiphase flow: general strategy.

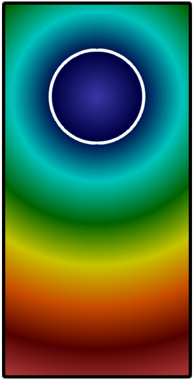
1. Representation of the air-water interface.
2. Reconstruct density and viscosity fields.
3. Solve the Navier-Stokes (NS) equations to get a velocity field.
4. Use the NS velocity field within **monolithic level-set**.
5. Repeat until the final time.



Multiphase flow: [1] representation of air-water interface.

We consider level-sets to represent the interface.

- The **level-set** is a distance function to the interface.
- The **zero countour value** represents the interface.
- Positive values represent air.
- Negative values represent water.



Domain: $\Omega = (0,1) \times (0,2)$.

Level-set function: $\phi(x,y)$.

Interface: $\Gamma := \{(x,y) \in \Omega \mid \phi(x,y) = 0\}$.

Multiphase flow: [2] reconstruct density and viscosity fields.

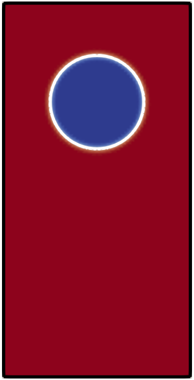
Given the level set $\phi(x, t)$ we reconstruct the density and viscosity fields via (smoothed) Heaviside functions.

Smoothed Heaviside function:

$$H_\epsilon(\phi) \approx \begin{cases} 0, & \text{if } \phi < 0, \\ 0.5, & \text{if } \phi = 0, \\ 1, & \text{if } \phi > 0, \end{cases}$$

Material parameters:

$$\rho(x, t) = \rho_A H_\epsilon(\phi) + \rho_W (1 - H_\epsilon(\phi)),$$
$$\mu(x, t) = \mu_A H_\epsilon(\phi) + \mu_W (1 - H_\epsilon(\phi)).$$



Multiphase flow: [3] solve the Navier-Stokes (NS) equations.

We solve the Navier-Stokes equations to obtain **velocity** and **pressure** fields.

$$\rho\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\mu\varepsilon(\mathbf{u}) + \nabla p = \mathbf{f},$$

‘plus’ boundary and initial conditions and other physics such as surface tension.

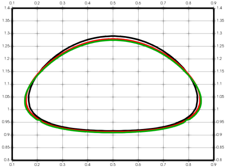
Numerical methods

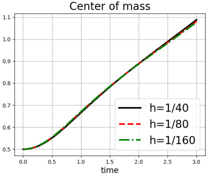
- **Time discretization:** projection scheme by [Guermond and Salgado(2009)].
- **Space discretization:** CG-FEM with P2-P1 spaces.
- **Stabilization:** entropy viscosity by [Cappanera et al.(2017)].
- **Surface tension:** semi implicit approximation by [Hysing(2006)].

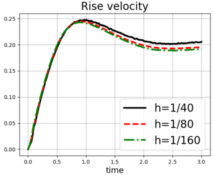
Numerical examples: multiphase flow
Rising bubble with surface tension

We follow [Hysing et al.(2009)] and reproduce two benchmarks.

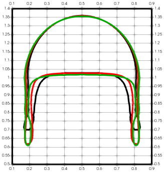
Test case 1: $\rho_W = 1000, \rho_A = 100, \mu_W = 10, \mu_A = 1, g = 0.98, \sigma = 24.5.$

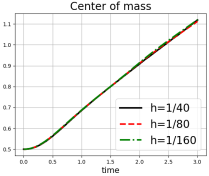


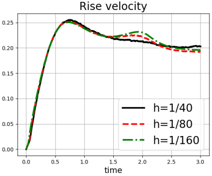




Test case 2: $\rho_W = 1000, \rho_A = 1, \mu_W = 10, \mu_A = 0.1, g = 0.98, \sigma = 1.96.$

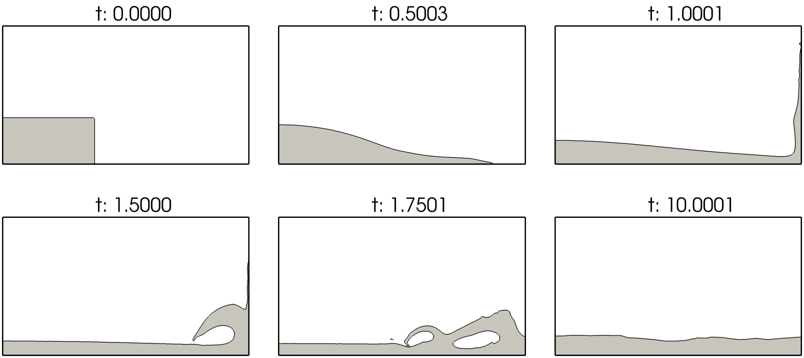






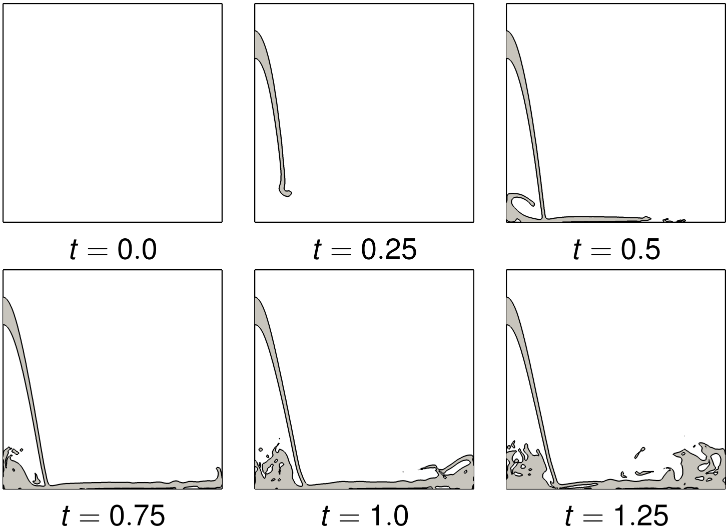
Numerical examples: multiphase flow
Dambreak with Colagrossi's setup

See [A. Colagrossi and M. Landrini (2003)].



Multiphase flow: numerical examples.

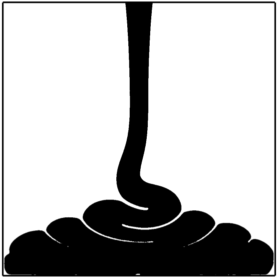
Filling a 2D tank

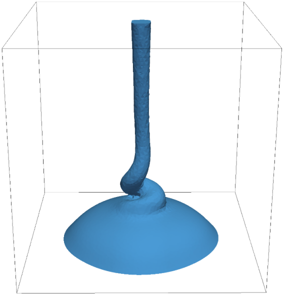


Numerical examples: multiphase flow
Buckling flow in 2D and 3D

See for instance [Ville et al.(2011), Tome and McKee(1999), Bonito et al.(2016)].
The material parameters are

$$\rho_W = 1800, \quad \mu_W = 500, \quad \rho_A = 1, \quad \mu_A = 2 \times 10^{-5}, \quad g = 9.81$$

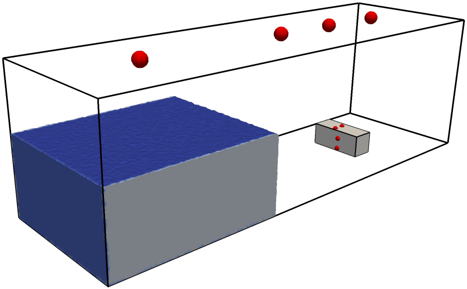


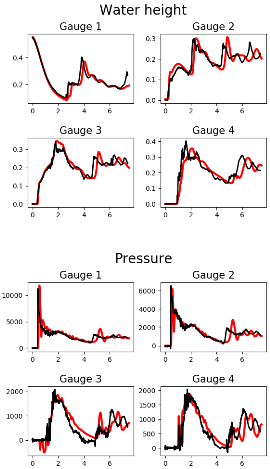


Numerical examples: multiphase flow

3D Marin problem

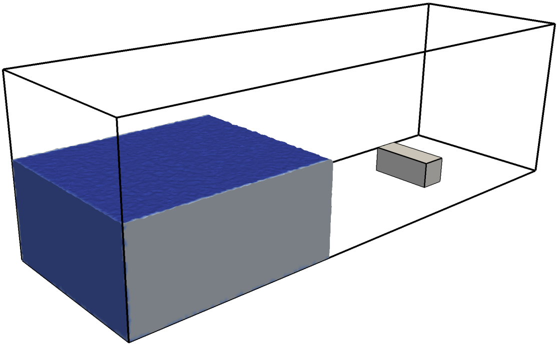
See for instance [Elias and Coutinho(2007), Kleefsman et al.(2005), Kees et al.(2011)].





Numerical examples: multiphase flow

3D Marin problem



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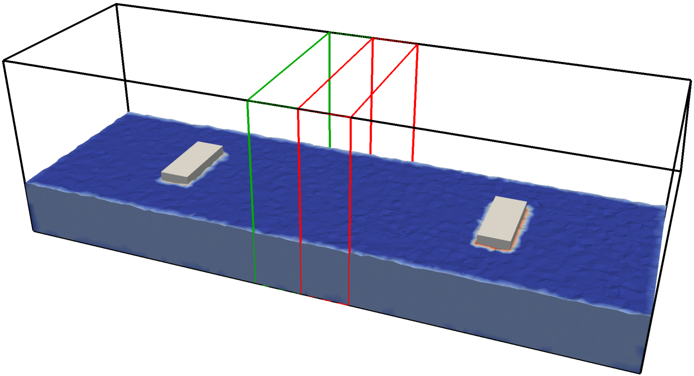

Conclusions

What do we have?

- Monolithic level-set/volume of fluid model.
- Phase conservative model.
- No need for interface reconstruction.
- Merge of: cons. level-set by [Kees et. al.] and elliptic redistancing by [Basting and Kuzmin]

What are we missing?

- We still have one parameter λ.

  Explore use of optimal control theory as in [Basting and Kuzmin(2014)].
- Use of smoothed (instead of sharp) Heavisides.

  Use advanced adaptive composite quadrature rules as in [Tornberg(2002)].

Thank You!