FSI: from the Immersed Boundary Method to a Fictitious Domain approach with Lagrange multiplier

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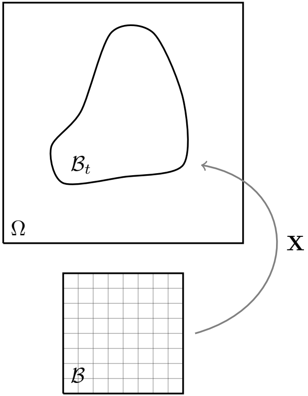
Fluid-structure interaction

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$
$\mathbf{x}$ Euler. var. in Ω

$\mathcal{B}_t$ deformable structure domain
$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$
$\mathbf{s}$ Lagrangian var. in $\mathcal{B}$
$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$\mathbf{X}(\mathbf{s}, t) = \mathbf{X}_0(\mathbf{s}) + \boldsymbol{\eta}(\mathbf{s}, t)$
with $\boldsymbol{\eta}$ displacement
$\mathbb{F}$ deformation gradient



the material velocity $\mathbf{u}(\mathbf{x}, t)$ is:

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{s}, t) \quad \text{where } \mathbf{x} = \mathbf{X}(\mathbf{s}, t)$$

FSI problem

$$
\begin{aligned}
&\rho_f \dot{\mathbf{u}}_f = \rho_f \left(\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{u}_f \cdot \nabla \mathbf{u}_f \right) = \operatorname{div} \boldsymbol{\sigma}_f && \text{in } \Omega \setminus \mathcal{B}_t \\
&\operatorname{div} \mathbf{u}_f = 0 && \text{in } \Omega \setminus \mathcal{B}_t \\
&\rho_s \dot{\mathbf{u}}_s = \operatorname{div} \boldsymbol{\sigma}_s && \text{in } \mathcal{B}_t \\
&\mathbf{u}_f = \mathbf{u}_s && \text{on } \partial\mathcal{B}_t \\
&\boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_s \mathbf{n}_s && \text{on } \partial\mathcal{B}_t \\
&+ \text{ initial and boundary conditions}
\end{aligned}
$$

Recall:

$$\mathbf{X}(t) : \mathcal{B} \to \mathcal{B}_t$$

$$\mathbf{u}_s(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{s}, t) \quad \text{where } \mathbf{x} = \mathbf{X}(\mathbf{s}, t)$$

Numerical approaches to FSI

Boundary fitted approaches The fluid problem is solved on a mesh that deforms around a Lagrangian structure mesh, using *arbitrary Lagrangian–Eulerian* (ALE) coordinate system
In case of large deformation the boundary fitted fluid mesh can become severely distorted

Non boundary fitted approaches

- fictitious domain ⟨Glowinski–Pan–Périaux '94, Yu '05⟩
- level set method ⟨Chang–How–Merriman–Osher '96⟩
- immersed boundary method ⟨Peskin '02⟩
- immersogeometric FSI (thin structures) ⟨Kamenski–Hsu–Schillinger–Evans–Aggarwal–Bazilevs–Sacks–Hughes '15⟩
- Nitsche X-FEM ⟨Burman–Fernández '14, Alauzet–Fabrèges–Fernández–Landajuela '16⟩

Our research originates from the *immersed boundary method* IBM and moved towards a fictitious domain approach

Outline

Finite element Immersed Boundary Method (FE-IBM)

Initial analysis of the FE-IBM

Approximation of FE-IBM

Mass conservation

An interface problem (towards a fully variational approach)

IBM - Immersed Boundary Method

⟨Peskin ’72-’77⟩
⟨McQueen–Peskin ’83-⟩
⟨Peskin ’02⟩

- Introduced by Peskin for the simulation of the blood flow in the heart.
- Applied to biological problems, where a fluid interacts with a flexible structure.
- The structure is a part of the fluid with additional forces and mass.
- The Navier–Stokes equations are solved in the whole domain (fluid + solid) by *finite differences*.
- The Dirac delta function is used to localize forces and masses in the solid domain.
- The immersed body has a fiber like structure.

Model assumptions

- Incompressible fluid:

$$\boldsymbol{\sigma}_f = -p_f \mathbb{I} + \nu_f \nabla_{sym} \mathbf{u}_f$$

$(\nabla_{sym} = \nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$

- Visco-elastic incompressible material:

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^f + \boldsymbol{\sigma}_s^s$$

with

$$\boldsymbol{\sigma}_s^f = -p_s \mathbb{I} + \nu_s \nabla_{sym} \mathbf{u}_s$$

and $\boldsymbol{\sigma}_s^s$ elastic part of the stress.

The Piola–Kirchhoff stress tensor takes into account the change of variable

$$\mathbb{P} = |\mathbb{F}| \boldsymbol{\sigma}_s^s \mathbb{F}^{-\top}$$

and is related with the potential energy density W by

$$\mathbb{P}(\mathbb{F}) = \frac{\partial W}{\partial \mathbb{F}}$$

IBM - Immersed Boundary Method

Problem formulation

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nu_f \Delta \mathbf{u} + \nabla p = \mathbf{d} + \mathbf{F}^{FSI} + \mathbf{t} \qquad \text{in } \Omega \times]0, T[$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega \times]0, T[$$

$$\mathbf{d}(\mathbf{x}, t) = (\rho_s - \rho_f) \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2} \delta(\mathbf{x} - \mathbf{X}(s, t)) ds$$ excess mass density

$$\mathbf{F}^{FSI}(\mathbf{x}, t) = \int_{\mathcal{B}} \nabla_s \cdot \mathbb{P}(s, t) \delta(\mathbf{x} - \mathbf{X}(s, t)) ds$$ inner force density

$$\mathbf{t}(\mathbf{x}, t) = - \int_{\partial\mathcal{B}} \mathbb{P}(s, t) \mathbf{N}(s, t) \delta(\mathbf{x} - \mathbf{X}(s, t)) ds$$ transm. force dens.

$$\frac{\partial \mathbf{X}}{\partial t}(s, t) = \mathbf{u}(\mathbf{X}(s, t), t) \quad \text{in } \mathcal{B} \times]0, T[$$ motion of the immersed body

FE-IBM Finite elements for IBM

⟨B.–Gastaldi ’03⟩
⟨B.–Gastaldi–Heltai ’04-’07⟩
⟨Heltai ’08⟩
⟨B.–Gastaldi–Heltai–Peskin ’08⟩
⟨B.–Cavallini–Gastaldi ’12⟩

- Variational formulation of the FSI force
- No need to approximating the Dirac delta functions
- Better interface approximation (less diffusion, sharp pressure jump)
- The fluid equations can be approximated with standard mixed schemes ($Q_2 - P_1$, Hood–Taylor, Bercovier–Pironneau, ...)

Variational definition of the source term

Everything started from this simple remark (two space dimensions, co-dimension one)

$\mathbf{X}(s,t)$ position of the immersed boundary $\mathcal{B}_t$

$$\mathbf{F}(\mathbf{x},t) = \int_0^L \kappa \frac{\partial^2 \mathbf{X}(s,t)}{\partial s^2} \delta(\mathbf{x} - \mathbf{X}(s,t))\, ds$$

Lemma

Assume that, for all $t \in [0,T]$, the curve $\mathcal{B}_t$ is Lipschitz continuous. Then for all $t \in]0,T[$, the force density $\mathbf{F}(t)$ is a distribution function belonging to $H^{-1}(\Omega)^2$ defined as follows: for all $\mathbf{v} \in H_0^1(\Omega)$

$$_{H^{-1}}\langle \mathbf{F}(t), \mathbf{v} \rangle_{H_0^1} = \int_0^L \kappa \frac{\partial^2 \mathbf{X}(s,t)}{\partial s^2} \mathbf{v}(\mathbf{X}(s,t))\, ds \quad \forall t \in]0,T[$$

Existence of the solution (1D)

⟨B.–Gastaldi ’03⟩

Existence of the solution for a simplified 1D problem:

Find $u : [a,b] \times [0,T] \to \mathbb{R}$ and $\mathbf{X} : [0,T] \to [a,b]$ such that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \mu u_{xx} = F \quad \text{in }]a,b[\times]0,T[\\
&F(x,t) = f(t)\delta(x - X(t)) \quad \forall x \in]a,b[,\ t \in]0,T[\\
&\mathbf{X}'(t) = u(\mathbf{X}(t),t) \quad \forall t \in]0,T[\\
&u(a,t) = u(b,t) = 0\ \forall t \in]0,T[\\
&u = u_0 \text{ in }]a,b[\qquad \mathbf{X}(0) = \mathbf{X}_0
\end{aligned}
$$

Schauder theorem

We set $\mathbb{X} = \{\mathbf{X} \in C^0([0,T]) : \mathbf{X}(0) = \mathbf{X}_0\}$

Given $\mathbf{X} \in \mathbb{X}$, $u(t) \in H_0^1(a,b)$ is the solution to:

$$\frac{d}{dt}(u(t),v) + \mu(u_x(t),v_x) = \langle F,v\rangle \quad [= f(t)v(\mathbf{X}(t))] \quad \forall v \in H_0^1(a,b) \qquad \textbf{(P1)}$$

$$u(0) = u_0 \quad \text{in }]a,b[$$

Then $\mathbf{X} = \mathbb{T}(\mathbf{X})$ solves

$$\mathbf{X}'(t) = u(\mathbf{X}(t),t) \quad \forall t \in [0,T] \quad \mathbf{X}(0) = X_0 \qquad \textbf{(P2)}$$

Theorem

There exists a fixed point of $\mathbb{T}$ *in the convex and compact subset* $B = \{\mathbf{Y} \in \mathbb{X} : \mathbf{Y}(t) \in [a,b],\ \|\mathbf{Y}'\|_{L^2(0,T)} \leq K\}$ *of* $\mathbb{X}$

Schauder theorem (cont'ed)

Step 1.
There exists a unique solution u to problem (**P**1)

If $\mathbf{X}' \in L^2(0,T)$, then there exists a summable function $\ell(t)$ so that

$$|u(x,t) - u(y,t)| \leq \ell(t)|x-y| \quad \forall (x,t),(y,t) \in [a,b] \times [0,T]$$

Step 2.
Let $\mathbf{X}_0 \in]a,b[$. There exists a unique solution $\mathbf{X}$ to equation (**P**2) defined in $[0,T]$, with $\mathbf{X}' \in L^2(0,T)$ and

$$\mathbf{X}(t) \in]a,b[\quad \forall t \in [0,T] \qquad \|\mathbf{X}'\|_{L^2(0,T)} \leq K$$

Stability

⟨B.–Cavallini–Gastaldi ’11⟩

Recalling that

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

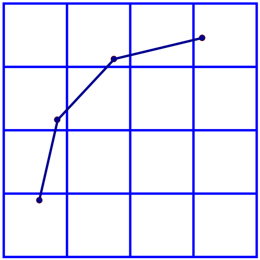
it holds

$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\nabla \mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}(\rho_s - \rho_f)\frac{d}{dt}\left\|\frac{\partial \mathbf{X}}{\partial t}\right\|_B^2 = 0$$

where E is the total elastic potential energy

$$E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbf{F}(s,t))\,ds$$

Finite element approximation



- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair

$$V_h \subset H^1_0(\Omega)^d$$
$$Q_h \subset L^2_0(\Omega)$$

- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- Piecewise linear finite element space for $\mathbf{X}$
$S_h = \{\mathbf{Y} \in C^0(\mathcal{B};\Omega) : \mathbf{Y} \in P1\}$

Notation

- T_k, $k = 1, \dots, M_e$ elements of $\mathcal{S}_h$
- $\mathbf{s}_j$, $j = 1, \dots, M$ vertices of $\mathcal{S}_h$
- $\mathcal{E}_h$ set of the edges e of $\mathcal{S}_h$

Discrete source term

Source term:

$$\langle \mathbb{F}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbf{F}_h(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}_h(s,t))\, ds \quad \forall \mathbf{v} \in V_h$$

$\mathbf{X}_h$ p.w. linear $\Rightarrow$ $\mathbf{F}_h$, $\mathbb{P}_h$ p.w. constant
By integration by parts

$$\begin{aligned}\langle \mathbb{F}_h(t), \mathbf{v} \rangle_h &= -\sum_{k=1}^{M_e} \int_{T_k} \mathbb{P}_h : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \\ &= -\sum_{k=1}^{M_e} \int_{\partial T_k} \mathbb{P}_h \mathbf{N} \mathbf{v}(\mathbf{X}(s,t))\, dA\end{aligned}$$

that is

$$\langle \mathbb{F}_h(t), \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!] \cdot \mathbf{v}(\mathbf{X}(s,t))\, dA$$

$[\![\mathbb{P}]\!] = \mathbb{P}^+\mathbf{N}^+ + \mathbb{P}^-\mathbf{N}^-$ jump of $\mathbb{P}$ across e for internal edges
$[\![\mathbb{P}]\!] = \mathbb{P}\mathbf{N}$ *jump* when $e \subset \partial\mathcal{B}$

The *semidiscrete* problem reads:

find $(\mathbf{u}_h, p_h) :]0,T[\to V_h \times Q_h$ and $\mathbf{X}_h : [0,T] \to S_h$ such that

$$\left\{\begin{aligned} &\rho_f \frac{d}{dt}(\mathbf{u}_h(t),\mathbf{v}) + a(\mathbf{u}_h(t),\mathbf{v}) + b(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}) \\ &\quad -(\operatorname{div}\mathbf{v}, p_h(t)) = -\int_{\mathcal{B}} (\rho_s - \rho_f)\frac{\partial^2 \mathbf{X}_h}{\partial t^2}\mathbf{v}(\mathbf{X}_h(s,t))ds \\ &\qquad -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]\cdot\mathbf{v}(\mathbf{X}_h(s,t))dA \qquad \forall \mathbf{v}\in V_h \\ &(\operatorname{div}\mathbf{u}_h(t), q) = 0 \qquad \forall q \in Q_h \end{aligned}\right.$$

$$\frac{d\mathbf{X}_{hi}}{dt}(t) = \mathbf{u}_h(\mathbf{X}_{hi}(t),t) \quad \forall i = 1,\dots,M$$

$$\mathbf{u}_h(0) = \mathbf{u}_{0h} \text{ in } \Omega$$

$$\mathbf{X}_{hi}(0) = \mathbf{X}_0(s_i) \quad \forall i = 1,\dots,M$$

Fully discrete problem (Backward Euler)

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ e $\mathbf{X}_h^{n+1} \in S_h$ such that

$$\langle \mathbb{F}_h^{n+1}, \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^{n+1} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s))\, dA \qquad \forall \mathbf{v} \in V_h$$

$$\textbf{NS} \begin{cases} \rho_f \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\ -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\mathbf{X}_h^{n+1} - 2\mathbf{X}_h^n + \mathbf{X}_h^{n-1}}{\Delta t^2} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s))\, ds \\ \quad + \langle \mathbb{F}_h^{n+1}, \mathbf{v} \rangle_h \qquad \forall \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h \end{cases}$$

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n+1}) \quad \forall i = 1, \ldots, M$$

Fully discrete problem (Modified Backward Euler)

Step 1. $\langle \mathbb{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\, dA \qquad \forall \mathbf{v} \in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$\text{NS} \begin{cases} \rho_f \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ \quad -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\ \quad -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\mathbf{X}_h^{n+1} - 2\mathbf{X}_h^n + \mathbf{X}_h^{n-1}}{\Delta t^2} \cdot \mathbf{v}(\mathbf{X}_h^n(s))ds \\ \qquad + \langle \mathbb{F}_h^n, \mathbf{v}\rangle_h \qquad\qquad \forall \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 \qquad\qquad \forall q \in Q_h \end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1, \dots, M$

Using **Step 3** in **Step 2** we get:

Step 1. $\langle \mathbb{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\, dA \qquad \forall \mathbf{v} \in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$\text{NS}\begin{cases} \rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}, \mathbf{v}\right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ \quad -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\ \quad -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\mathbf{u}_h^{n+1}(\mathbf{X}_h^n(s)) - \mathbf{u}_h^n(\mathbf{X}_h^{n-1}(s))}{\Delta t} \cdot \mathbf{v}(\mathbf{X}_h^n(s))ds \\ \qquad + \langle \mathbb{F}_h^n, \mathbf{v}\rangle_h \qquad\qquad \forall \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 \qquad\qquad \forall q \in Q_h \end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1, \ldots, M$

Discrete Energy Estimate

⟨B.–Cavallini–Gastaldi '11⟩

Artificial Viscosity Theorem

Let $\mathbf{u}_h^n$, p_h^n and $\mathbf{X}_h^n$ be a solution to the FE-IBM, then

$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + (\mu + \mu_a)\|\nabla \mathbf{u}_h^{n+1}\|_0^2 + \frac{1}{\Delta t}\left(E\left[\mathbf{X}_h^{n+1}\right] - E\left[\mathbf{X}_h^n\right]\right) + \frac{1}{2\Delta t}(\rho_s - \rho_f)\left(\|\mathbf{u}_h^{n+1}(\mathbf{X}_h^n)\|_{0,\mathcal{B}}^2 - \|\mathbf{u}_h^n(\mathbf{X}_h^{n-1}\|_{0,\mathcal{B}}^2\right) \le 0$$

CFL Conditions: $\mu + \mu_a \ge 0$, $\rho_s \ge \rho_f$ (might be relaxed)

CFL condition

BE is unconditionally stable, while MBE requires the term μ_a to be not too large

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^n$$

$$L^n := \max_{T_k \in S_h} \left\{ \max_{\mathbf{s}_j, \mathbf{s}_i \in V(T_k)} |\mathbf{X}_{hj}^n - \mathbf{X}_{hi}^n| \right\}$$

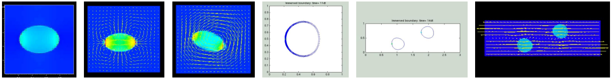
space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \le C h_x h_s$
2	2	$L^n \Delta t \le C h_x$
3	2	$L^n \Delta t \le C h_x^2$
3	3	$L^n \Delta t \le C h_x^2 / h_s$

Some numerical results

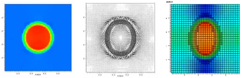
Original 2D code in Fortran 77, ported to DEAL.II (c++) (`www.dealii.org`) by L. Heltai ($Q_2 - P_1$)

2D

Codimension 1

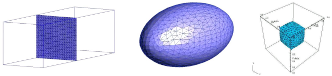


Codimension 0



3D

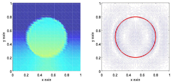
Codimension 1



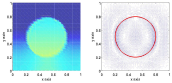
More numerical results

Fortran 90 code written by N. Cavallini ($P_1 \mathrm{iso} P_2 - P_1^c$)

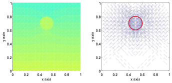
Densities: $\rho_s = 21$ and $\rho_f = 1$



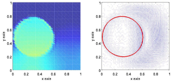
$\kappa = 1$



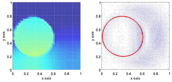
$\kappa = 0.1$



$\kappa = 0.1$

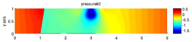


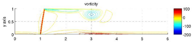
$\kappa = 1$



$\kappa = 0.1$

Heart valve (Auricchio–B.–Cavallini–Gastaldi–Lefieux)





Mass conservation of the IBM

⟨B.–Cavallini–Gardini–Gastaldi ’12⟩

Well-known and studied problem

The discrete divergence free condition is imposed in a weak sense

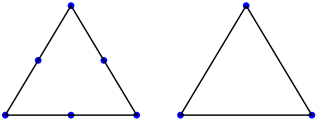
$$\int_\Omega \operatorname{div} \mathbf{u}_h q_h \, d\mathbf{x} = 0 \quad \forall q_h \in Q_h$$

which is not exact unless $\operatorname{div}(V_h) \subset Q_h$

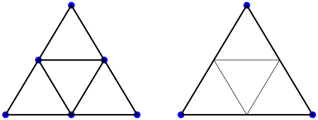
Basic remark

Discontinuous pressure schemes enjoy *local* mass conservation properties (average of divergence is zero element by element)

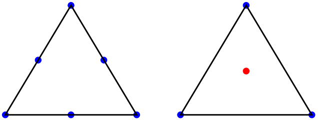
Our elements



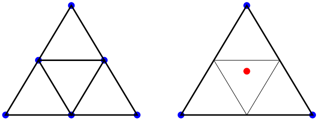
Hood–Taylor



$P_1 \mathrm{iso} P_2 - P_1^c$



Enhanced Hood–Taylor



Enhanced $P_1 \mathrm{iso} P_2 - P_1^c$

We actually considered generalized Hood–Taylor in two and three dimensions $P_{k+1} - P_k^c$ $(k \geq 1)$

Not a new idea

Local mass conservation is guaranteed by extra degree of freedom: add piecewise constant pressures

Analysis of our elements

Known facts

Hood–Taylor

- Introduced in 1973 ⟨Hood–Taylor ’73⟩
- First analysis ⟨Bercovier–Pironneau ’79, Verfürth ’84⟩
- Full analysis with some restrictions on boundary elements ⟨Scott–Vogelius ’85, Brezzi–Falk ’91⟩
- General analysis for the $P_{k+1} - P_k^c$ element with no restrictions (mesh contains at least 3 elements) ⟨B. ’94⟩

P_1 iso $P_2 - P_1^c$

- Same analysis as for the Hood-Taylor element can be carried on ⟨Bercovier–Pironneau ’79, Brezzi–Fortin ’91⟩
- Error estimates are suboptimal (unbalanced spaces); ease of implementation makes it appealing, in particular in 3D

Analysis of our elements (cont'ed)

Pressure enhancement

- Numerical evidence for lowest order Hood-Taylor (triangles and squares)
  ⟨Gresho–Lee–Chan–Leone '80⟩
  ⟨Griffiths '82⟩
  ⟨Tidd–Thatcher–Kaye '88⟩
- Proof of inf-sup for lowest order Hood-Taylor (triangles and squares)
  ⟨Thatcher '90, Pierre '94, Quin–Zhang '05⟩

Analysis of our elements (cont'ed)

Theorem (B.–Cavallini–Gardini–Gastaldi '12)

The generalized enhanced Hood-Taylor scheme

$$P_{k+1} - (P^c_k + P_0)$$

in two ($k \geq 1$) and three ($k \geq 2$) dimensions and the enhanced

$$P_1 iso P_2 - (P^c_1 + P_0)$$

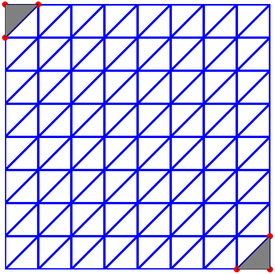
in two dimensions satisfy the inf-sup condition

Minimal restriction on the mesh: each element has at least one internal vertex.

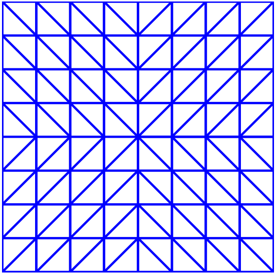
Mesh restrictions

2D: let us understand the restrictions

- Standard schemes: the mesh needs at least three elements
- Enhanced schemes: each element needs at least an internal vertex



Uniform mesh



Symmetric mesh

Mass conservation and FE-IBM

Inflated balloon test case

$$\mathbf{X}_0(s) = \begin{pmatrix} R\cos(s/R) + 0.5 \\ R\sin(s/R) + 0.5 \end{pmatrix}, \quad s \in [0, 2\pi R]$$

$$\langle \mathbb{F}(t), \mathbf{v} \rangle = -\kappa \int_0^{2\pi R} \frac{\partial \mathbf{X}(s,t)}{\partial s} \frac{\partial \mathbf{v}(\mathbf{X}(s,t))}{\partial s}$$

$$p(\mathbf{x}, t) = \begin{cases} \kappa(1/R - \pi R), & |\mathbf{x}| \le R \\ -\kappa\pi R, & |\mathbf{x}| > R \end{cases} \quad \forall t \in]0, T[$$

$T = 10^{-1}$

$\rho_f = \rho_s = 1$

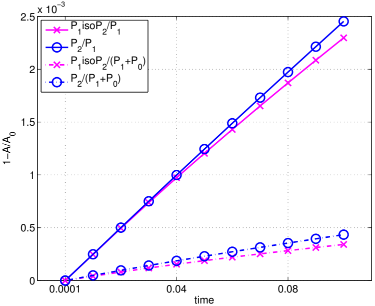
$\mu = 1$

$\kappa = 1$

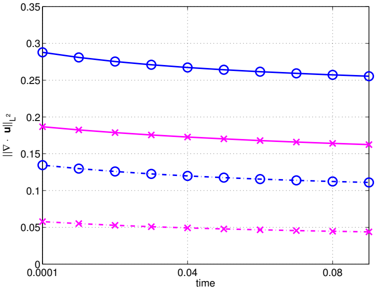
$h_x = 1/32$

$h_s = 2\pi R/1024$

Area loss w.r.t. time



Area loss

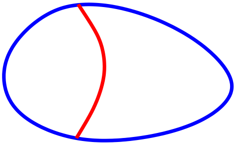


Divergence norm

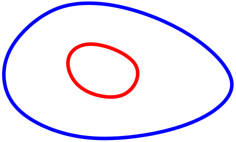
An interface problem

⟨Auricchio–B.–Gastaldi–Lefieux–Reali ’13⟩
⟨B.–Gastaldi–Ruggeri ’13⟩

Let us consider a standard interface problem



$$
\begin{aligned}
&-\operatorname{div}(\beta_1 \nabla u_1) = f_1 && \text{in } \Omega_1\\
&-\operatorname{div}(\beta_2 \nabla u_2) = f_2 && \text{in } \Omega_2\\
&u_1 = u_2 && \text{on } \Gamma\\
&\beta_1 \nabla u_1 \cdot \mathbf{n}_1 + \beta_2 \nabla u_2 \cdot \mathbf{n}_2 = 0 && \text{on } \Gamma\\
&u_1 = 0 && \text{on } \partial\Omega_1 \setminus \Gamma\\
&u_2 = 0 && \text{on } \partial\Omega_2 \setminus \Gamma
\end{aligned}
$$



Mixed formulation

Notation: $\Omega = \Omega_1 \cup \Omega_2$

Find $u \in H_0^1(\Omega)$, $u_2 \in H^1(\Omega_2)$, and $\lambda \in \Lambda = [H^1(\Omega_2)]^*$ such that

$$\begin{aligned}
&\int_\Omega \beta \, \nabla u \cdot \nabla v \, d\mathbf{x} + \langle \lambda, v|_{\Omega_2} \rangle = \int_\Omega f v \, d\mathbf{x} && \forall v \in H_0^1(\Omega) \\
&\int_{\Omega_2} (\beta_2 - \beta) \, \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} - \langle \lambda, v_2 \rangle = \int_{\Omega_2} (f_2 - f) v_2 \, d\mathbf{x} && \forall v_2 \in H^1(\Omega_2) \\
&\langle \mu, u|_{\Omega_2} - u_2 \rangle = 0 && \forall \mu \in \Lambda
\end{aligned}$$

Equivalent to interface problem if $\beta|_{\Omega_1} = \beta_1$ and $f|_{\Omega_1} = f_1$

We get $u|_{\Omega_1} = u_1$

Alternative mixed formulation

Find $u \in H^1_0(\Omega)$, $u_2 \in H^1(\Omega_2)$, and $\psi \in H^1(\Omega_2)$ such that

$$\int_\Omega \beta \nabla u \cdot \nabla v \, d\mathbf{x} + ((\psi, v|_{\Omega_2})) = \int_\Omega fv \, d\mathbf{x} \qquad \forall v \in H^1_0(\Omega)$$

$$\int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} - ((\psi, v_2)) = \int_{\Omega_2} (f_2 - f) v_2 \, d\mathbf{x} \qquad \forall v_2 \in H^1(\Omega_2)$$

$$((\varphi, u|_{\Omega_2} - u_2)) = 0 \qquad \forall \varphi \in H^1(\Omega_2)$$

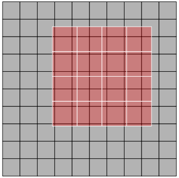
where $((\cdot, \cdot))$ denotes the scalar product in $H^1(\Omega_2)$

Remark

The two mixed formulations are equivalent but give rise to different discrete schemes

Approximation of mixed formulations

Two meshes: $\mathcal{T}_h$ for Ω and $\mathcal{T}_{2,h}$ for Ω_2



Three finite element spaces:
V_h continuous p/w linears on $\mathcal{T}_h$
$V_{2,h}$ continuous p/w linears on $\mathcal{T}_{2,h}$
$\Lambda_h = V_{2,h}$

Several other choices are possible

Remark

First mixed formulation makes use of V_h*,* $V_{2,h}$*, and* Λ_h
(duality represented by scalar product in $L^2(\Omega_2)$*)*
Second mixed formulation makes use of V_h*,* $V_{2,h}$*, and* $V_{2,h}$

Matrix form of the problem

$$\begin{pmatrix} A & B^\top \\ B & 0 \end{pmatrix}$$

Stability of the approximation

We need to show the ellipticity in the kernel and the inf-sup condition

ELKER

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \kappa_1 \left(\|v\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega_2)} \right)$$

$$\forall (v, v_2) \in \mathbb{K}_h$$

where the kernel $\mathbb{K}_h$ is defined as

$$\mathbb{K}_h = \{(v, v_2) \in V_h \times V_{2,h} : (\mu, v|_{\Omega_2} - v_2) = 0 \ \forall \mu \in \Lambda_h\}$$

or, for the second formulation,

$$\mathbb{K}_h = \{(v, v_2) \in V_h \times V_{2,h} : ((\varphi, v|_{\Omega_2} - v_2)) = 0 \ \forall \varphi \in V_{2,h}\}$$

Stability of the approximation (cont'ed)

INFSUP1

$$\sup_{(v,v_2)\in V_h\times\Lambda_h} \frac{(\mu, v|_{\Omega_2} - v_2)}{\left(\|v\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega_2)}\right)^{1/2}} \geq \kappa_2 \|\mu\|_\Lambda \quad \forall \mu \in \Lambda_h$$

INFSUP2

$$\sup_{(v,v_2)\in V_h\times V_{2,h}} \frac{((\varphi, v|_{\Omega_2} - v_2))}{\left(\|v\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega_2)}\right)^{1/2}} \geq \kappa_2 \|\varphi\|_{H^1(\Omega_2)} \quad \forall \varphi \in V_{2,h}$$

Stability of the approximation (cont'ed)

Theorem

If $\beta_2 - \beta|_{\Omega_2} \geq \eta_0 > 0$ *then ELKER holds true for both formulations, uniformly in* h *and* h_2

Remark

For the second mixed formulation, ELKER holds true without assumptions on β *if* $h_2/h^{d/2}$ *is small enough and* $\mathcal{T}_h$ *is quasi-uniform*

Stability of the approximation (cont'ed)

Theorem

If the mesh sequence $\mathcal{T}_{2,h}$ is quasi-uniform, then INFSUP1 holds true, uniformly in h and h_2

Theorem

INFSUP2 holds true, uniformly in h and h_2 without any additional assumptions on the mesh sequence

Conclusions (Part I)

- FE-IBM allows for natural treatment of Dirac delta function
- Superior CFL condition with respect to ALE formulations
- Superior mass conservation property
- Mixed approach for an interface problem
- Towards a DLM fictitious domain formulation