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p-adic L-functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ III

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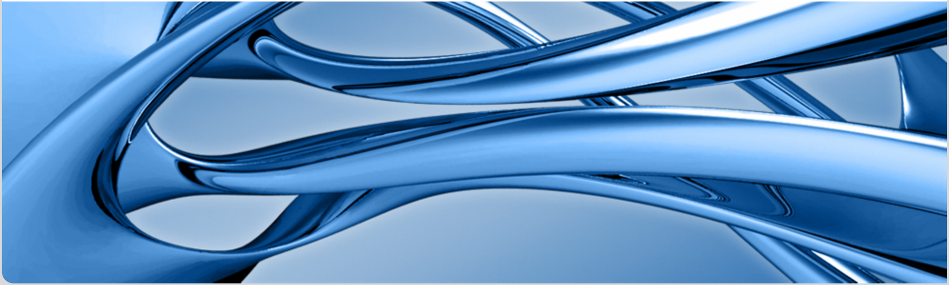

Overview

Previous lectures:

- $\mathrm{GL}(2)/F$ adelically (Kenichi)
- Rankin-Selberg L-functions following Jacquet, Piatetski-Shapiro and Shalika
- The relative modular symbol and algebraicity of special values
- Archmedean periods: Non-vanishing and period relations

Today's lectures:

- p-adic distributions attached to finite slope classes (Kazhdan-Mazur-Schmidt, Schmidt, J.)
- Boundedness in the nearly ordinary case (Schmidt, J.)
- Functional equation (J.)
- Manin congruences and independence of weight (J.)
- Interpolation formulae (Schmidt, J.)

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Cohomological setup

$F/\mathbf{Q}$ number field, $\mathbf{A}$ adèles over $\mathbf{Q}$, $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$

$$\begin{array}{rcl} G & = & G_{n+1} \times G_n \;=\; \mathrm{res}_{F/\mathbf{Q}} \mathrm{GL}(n+1) \times \mathrm{GL}(n) \\ H & = & G_n = \mathrm{res}_{F/\mathbf{Q}} \mathrm{GL}(n) \\ \Delta & : & H \to G,\; g \mapsto (\mathrm{diag}(g,1), g) \\ S & \subseteq & G \quad \text{maximal } \mathbf{Q}\text{-split torus in the center (rank 2)} \\ K_\infty & \subseteq & G(\mathbf{R}) \quad \text{max'l compact} \\ \widetilde{K}_\infty & = & S(\mathbf{R})^0 K_\infty \subseteq G(\mathbf{R}) \\ L_\infty & = & \widetilde{K}_\infty \cap H(\mathbf{R}) \;\subseteq\; H(\mathbf{R}) \quad \text{max'l compact} \\ K & \subseteq & G(\mathbf{A}^{(\infty)}) \quad \text{compact open} \\ L & \subseteq & H(\mathbf{A}^{(\infty)}) \quad \text{compact open} \\ \mathscr{X}(K) & = & G(\mathbf{Q}) \backslash G(\mathbf{A}) / \widetilde{K}_\infty K \\ \mathscr{Y}(L) & = & H(\mathbf{Q}) \backslash H(\mathbf{A}) / L_\infty L \\ i & : & \mathscr{Y}(L) \to \mathscr{X}(K) \quad \text{proper, whenever } L \subseteq K \end{array}$$

Cohomological setup

p a rational prime, $E/\mathbf{Q}_p$ finite, $\mathcal{O} \subseteq E$ its valuation ring

$$
\begin{array}{rcl}
B & = & TU \quad \text{upper Borel in } G \\
\mathfrak{u} & : & \mathcal{O}\text{-Lie algebra of } U \\
\lambda & : & B\text{-dominant } E\text{-rational weight of } G \\
L_{\lambda,E} & : & \text{irred. rep. of } G \text{ of heighest wt } \lambda \text{ over } E \\
v_0 & : & B\text{-lowest weight vector in } L_{\lambda,E} \\
L_{\lambda,\mathcal{O}} & = & U(\mathfrak{u}) \cdot v_0 \subseteq L_{\lambda,E} \quad \mathcal{O}\text{-lattice} \\
L_{\lambda,A} & & \text{for } A \in \{E, \mathcal{O}, E/\mathcal{O}, \mathcal{O}/p^{\beta}\mathcal{O}, p^{-\beta}\mathcal{O}/\mathcal{O}\} \\
g & \in & G(\mathbf{A}^{(\infty)}) \\
gL_{\mathcal{O}} & = & L_{\lambda,E} \cap g \cdot (L_{\mathcal{O}} \otimes_{\mathbf{Z}_p} \widehat{\mathbf{Z}}) \quad \text{for any } \mathcal{O}\text{-lattice } L_{\mathcal{O}} \subseteq L_{\lambda,E} \\
t_g & : & \mathscr{X}(gKg^{-1}) \to \mathscr{X}(K) \quad \text{right translation by } g \\
T_g & : & t_g^* \underline{L}_{\mathcal{O}} \to \underline{gL}_{\mathcal{O}} \quad \text{canonical map of associated sheaves} \\
t_g^{\lambda} & : & \Gamma(U, \underline{L}_{\mathcal{O}}) \to \Gamma(Ug^{-1}, \underline{gL}_{\mathcal{O}}) \quad \text{'normalized' pullback}
\end{array}
$$

Cohomological setup

For $x \in F_p^\times = (F \otimes_{\mathbf{Q}} \mathbf{Q}_p)^\times$ and $\alpha \geq \alpha' \geq 0$ put:

$$\begin{aligned} t_x &= \operatorname{diag}(x^n, x^{n-1}, \dots, x) \in \mathrm{GL}_n(F_p) = H(\mathbf{Q}_p) \\ I_{\alpha',\alpha} &= \{k \in G(\mathbf{Z}_p) \mid k \in B(\mathbf{Z}_p/p^\alpha) \text{ and } k \in U(\mathbf{Z}_p/p^{\alpha'})\} \\ U_p &= I_{\alpha',\alpha}\Delta(t_p)I_{\alpha',\alpha} = \bigsqcup_{u \in U(\mathbf{Z}_p)/t_p U(\mathbf{Z}_p) t_p^{-1}} u t_p I_{\alpha',\alpha} \\ &= \prod_{\mathfrak{p}|p} (U_{\mathfrak{p}} \otimes U'_{\mathfrak{p}})^{v_{\mathfrak{p}}(p)} \\ K_{\alpha',\alpha} &= K^{(p)} \times I_{\alpha',\alpha} \end{aligned}$$

p-optimal Hecke action: Put for $v \in L_{\lambda,E}$ and $t \in T(\mathbf{Q}_p)$:

$$t \bullet v := \lambda^\vee(t) \cdot (t \cdot v) = (-\lambda^{w_0})(t) \cdot (t \cdot v)$$

Then for any $\phi \in H_?^\bullet(\mathscr{X}(K_{\alpha',\alpha}); \underline{L}_{\lambda,A})$:

$$U_p \bullet \phi \in H_?^\bullet(\mathscr{X}(K_{\alpha',\alpha}); \underline{L}_{\lambda,A})$$

The modular symbol

Let $w_n \in \mathrm{GL}_n(\mathbf{Z})$ denote the antidiagonal matrix. Introduce

$$h_n = \begin{pmatrix} & & & 1 \\ & w_n & & \vdots \\ & & & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbf{Z})$$

$$h = (h_n \operatorname{diag}(t_{-1}, 1), \mathbf{1}_n) \in G(\mathbf{Z}_p)$$

$$g_\beta = h t_p^\beta \in G(\mathbf{Z}_p)$$

Proposition (Schmidt, J.)

For $\beta \geq \alpha \geq \alpha'$, $\alpha > 0$, $\mathfrak{I}_\beta := H(\mathbf{Q}_p) \cap g_\beta I_{\alpha',\alpha} g_\beta^{-1}$ *is independent of* α *and* α' *and*

$$(H(\mathbf{Z}_p) : \mathfrak{I}_\beta) = \prod_{v|p} \prod_{\mu=1}^{n} \left(1 - q_v^{-\mu}\right)^{-1} \cdot p^{\beta \frac{(n+2)(n+1)n + (n+1)n(n-1)}{6}}$$

$$\det \mathfrak{I}_\beta = 1 + p^\beta \mathcal{O}_p$$

The modular symbol

This implies:

$$C(p^\beta) \;=\; F^\times\backslash\mathbf{A}_F^\times/F_\infty^+\cdot\det\left(g_\beta K_{\alpha',\alpha}g_\beta^{-1}\cap H(\mathbf{A}^{(\infty)})\right)$$
$$\;=\; F^\times\backslash\mathbf{A}_F^\times/F_\infty^+\cdot\det\left(K^{(p)}\cap H(\mathbf{A}^{(p\infty)})\right)\cdot(1+p^\beta\mathcal{O}_p)$$

This class group parametrizes the connected components of $\mathscr{Y}(L_\beta)$ where

$$L_\beta \;:=\; g_\beta K_{\alpha',\alpha}g_\beta^{-1}\cap H(\mathbf{A}^{(\infty)})$$

only depends on β.
Assume there is $j\in\mathbf{Z}$ and a non-zero H-intertwining

$$\eta_j:\; L_{\lambda,E}\to(N_{F/\mathbf{Q}}\otimes\det)^{\otimes j}=:E_{(j)}$$

Fact: $\mathfrak{h}$ and $g_0\mathfrak{b}^-g_0^{-1}$ are *transversal*. Therefore, $\eta_j(g_0v_0)\neq 0$ and may define

$$\eta_{j,A}:\; L_{\lambda,A}\to A_{(j)},\quad v\mapsto\frac{\eta_j(v)}{\eta_j(g_0v_0)}$$

The modular symbol

For $x \in C(p^\beta)$, define the modular symbol

$$\mathscr{P}^{\lambda,j}_{A,x,\beta}: \quad H^{\dim \mathscr{Y}}_{\mathrm{c,ord}}(\mathscr{X}(K_{\alpha',\alpha}); \underline{L_{\lambda,A}}) \to A_{(j)},$$

$$\phi \mapsto \int_{\mathscr{Y}(L_\beta)[x]} \eta_{j,A} \circ i^* \left[(-\lambda^{w_0})(t_p^\beta) \cdot t^\lambda_{g_\beta}\right] (U_p^{-\beta} \bullet \phi).$$

Put

$$L^{x,\beta}_{\lambda,\mathcal{O}} = (-\lambda^{w_0})(t_p^\beta) \cdot d_x \cdot U(\mathfrak{u}^\beta_{\mathcal{O}}) \cdot g_\beta v_0, \quad L^{x,\beta}_{\lambda,A} := L^{x,\beta}_{\lambda,\mathcal{O}} \otimes A.$$

The elementary relation $L^{x,\beta}_{\lambda,A} \subseteq L^{1,0}_{\lambda,A}$ shows

$$(-\lambda^{w_0})(t_p^\beta) \cdot t^\lambda_{g_\beta}(U_p^{-\beta} \bullet \phi) \in L^{x,\beta}_{\lambda,A} \subseteq L^{1,0}_{\lambda,A}$$

Hence $\mathscr{P}^{\lambda,j}_{A,x,\beta}$ is well defined. Define:

$$\mu^{\lambda,j}_{A,\beta}(\phi) := \sum_{x \in C(p^\beta)} \mathscr{P}^{\lambda,j}_{A,x,\beta}(\phi) \cdot x \in A_{(j)} \otimes_{\mathcal{O}} \mathcal{O}[C(p^\beta)] =: A_{(j)}[C(p^\beta)]$$

The distribution relation

For any $\beta \geq \beta' > 0$ the canonical projection $C(p^\beta) \to C(p^{\beta'})$ induces an $\mathcal{O}$-linear epimorphism

$$\mathrm{res}^{\beta}_{\beta'} : \quad A_{(j)}[C(p^\beta)] \to A_{(j)}[C(p^{\beta'})]$$

Proposition (Schmidt, J.)

For any cohomology class ϕ and any $\beta \geq \beta' > 0$ we have the distribution relation

$$\mathrm{res}^{\beta}_{\beta'}\left(\mu^{\lambda,j}_{A,\beta}(\phi)\right) \;=\; \mu^{\lambda,j}_{A,\beta'}(\phi)$$

Lemma

Let $u \in U(\mathbf{Z}_p)$, $\beta > 0$. Then:

(i)
$$\exists k_u \in I_{\alpha,\alpha} : \quad ht_p^\beta \cdot ut_p \;=\; ht_p^{\beta+1} \cdot k_u \tag{1}$$

(ii) *For any $k_u = (k'_u, k''_u)$ in (1) the residue class $\det k'_u \pmod{p^{\beta+1}}$ is uniquely determined by $u \in U(\mathbf{Z}_p)/t_pU(\mathbf{Z}_p)t_p^{-1}$ and lies in $1 + p^\beta\mathcal{O}_p$.*

(iii)
$$U(\mathbf{Z}_p)/t_pU(\mathbf{Z}_p)t_p^{-1} \to (1+p^\beta\mathcal{O}_p)/(1+p^{\beta+1}\mathcal{O}_p), \quad u \;\mapsto\; \det k'_u,$$

is a surjective group homomorphism.

The distribution relation

Proof (of the Proposition): Using the Lemma, unfold

$$\begin{aligned}
\mathscr{P}^{\lambda,j}_{A,x,\beta}(\phi) &= \mathscr{P}^{\lambda,j}_{A,x,\beta}(U_p U_p^{-1} \bullet \phi) \\
&= \int\limits_{\mathscr{Y}(L_\beta)[x]} \eta_{j,A} \circ i^* \left[(-\lambda^{w_0})(t_p^\beta) \cdot t^\lambda_{g_\beta} \right] (U_p U_p^{-(\beta+1)} \bullet \phi) \\
&= \int\limits_{\mathscr{Y}(L_\beta)[x]} \sum_{u \in U(\mathbf{Z}_p)/t_p U(\mathbf{Z}_p) t_p^{-1}} \eta_{j,A} \circ i^* \left[(-\lambda^{w_0})(t_p^{\beta+1}) \cdot t^\lambda_{g_\beta u t_p} \right] (U_p^{-(\beta+1)} \phi) \\
&= \sum_u [\text{some index}]^{-1} \int\limits_{\mathscr{Y}(L_{\beta+1})[x \det(k_u')]} \eta_{j,A} \circ i^* \left[(-\lambda^{w_0})(t_p^{\beta+1}) \cdot t^\lambda_{g_{\beta+1}} \right] (U_p^{-(\beta+1)} \phi) \\
&= \sum_{y \pmod p} \mathscr{P}^{\lambda,j}_{A,x+yp^\beta,\beta+1}(\phi)
\end{aligned}$$

The p-adic measure

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By the previous Proposition we have a projective system $\left(\mu_{A,\beta}^{\lambda,j}(\phi)\right)_{\beta}$. Put

$$\begin{aligned} C_F(p^\infty) &= \varprojlim_{\beta} C_F(p^\beta) \\ \mu_A^{\lambda,j}(\phi) &= \varprojlim_{\beta} \mu_{A,\beta}^{\lambda,j}(\phi) \end{aligned}$$

Summing up, we obtain an $\mathcal{O}$-linear map

$$\mu_A^{\lambda,j}: \quad H_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(\mathscr{X}(K_{\alpha',\alpha}); \underline{L}_{\lambda,A}) \to A_{(j)}[[C_F(p^\infty)]]$$

For $A = \mathcal{O}$ we obtain for every nearly ordinary ϕ a p-adic **measure**

$$\mu_{\mathcal{O}}^{\lambda,j}(\phi) \in \mathcal{O}[[C_F(p^\infty)]]$$

For $A = E$ we obtain for every finite slope ϕ a p-adic distribution $\mu_E^{\lambda,j}(\phi)$ whose growth is bounded in terms of the slope.

Functional equation

Consider the involution of $\mathrm{GL}(n)$:

$$\iota: \quad g \mapsto w_n {}^t g^{-1} w_n$$

stabilizes B_n, T_n, U_n, hence induces an involution on the Hecke algebra at p sends λ to $\lambda^\vee$, hence induces identifications $L_{\lambda^\vee,A} \cong \iota^*(L_{\lambda,A})$, get

$$(-)^\vee: \quad L_{\lambda,A} \to L_{\lambda^\vee,A}, \; v \mapsto v^\vee$$

likewise for sheaves, since ι stabilizes $\mathscr{X}(K_{\alpha',\alpha})$ and $\mathscr{Y}_n(L_\beta)$. Therefore, ι induces an involution

$$H^{\dim \mathscr{Y}}_{\mathrm{c,ord}}(\mathscr{X}(K_{\alpha',\alpha}); \underline{L}_{\lambda,A}) \to H^{\dim \mathscr{Y}}_{\mathrm{c,ord}}(\mathscr{X}(K_{\alpha',\alpha}); \underline{L}_{\lambda^\vee,A}), \quad \phi \mapsto \phi^\vee$$

Proposition (Functional equation, J.)

$$\mu_A^{\lambda,j}(\phi)(x) \;=\; \mu_A^{\lambda^\vee,\mathbf{w}-j}(\phi^\vee)((-1)^n x^{-1})$$

Proof: Relies on $\iota(g_\beta) = w_0 \cdot {}^t g_\beta^{-1} \cdot w_0 \in ((\mathbf{1}_{n+1}, t_p^{\beta}) I_{\alpha',\alpha} (\mathbf{1}_{n+1}, t_p^{-\beta}) \cap H) g_\beta I_{\alpha',\alpha}$

Manin congruences

Goal: Relate $\mu_A^{\lambda,j_1}(\phi)$ and $\mu_A^{\lambda,j_2}(\phi)$ for $j_1 \neq j_2$.

Observe that by construction,

$$i^*\left[(-\lambda^{w_0})(t_p^\beta)\cdot t_{g_\beta}^\lambda\right](U_p^{-\beta}\bullet\phi)\ \in\ H_c^{\dim\mathcal{Y}}(\mathscr{Y}(L_\beta);\underline{L}_{\lambda,\mathcal{O}}^{x,\beta})$$

where

$$\begin{aligned} L_{\lambda,\mathcal{O}}^{x,\beta} &= d_x h\cdot\left(t_p^\beta\bullet L_{\lambda,\mathcal{O}}\right)\\ &= (-\lambda^{w_0})(t_p^\beta)\cdot d_x g_\beta\cdot L_{\lambda,\mathcal{O}} \end{aligned}$$

Therefore, we have to answer the question

$$\eta_{j,\mathcal{O}}(L_{\lambda,\mathcal{O}}^{x,\beta})\ =\ ?\pmod{p^\beta L_{\lambda,\mathcal{O}}}$$

In the case of $\mathrm{GL}(2)$, Manin solved this by explicit computation in $\mathrm{Sym}^{k-2}\,\mathcal{O}^2$
This approach also works in the presence of complex places (Namikawa, 2016)
This was also successful for $n=2$, $\lambda=(1,0,-1)\otimes(1,0)$ (Schwab, 2015)

Manin congruences

Let $\beta \geq 0$. Recall $g_\beta = ht_p^\beta$. Write ${}^{g_\beta}a = g_\beta a g_\beta^{-1}$.

Lemma

(i) *We have* ${}^{g_\beta}\mathfrak{b}_E^- = {}^{g_0}\mathfrak{b}_E^-$

(ii) *The subgroups H and* ${}^{g_\beta}B^-$ *are transversal, i.e.*

$$\mathfrak{g}_E = \mathfrak{h}_E \oplus {}^{g_\beta}\mathfrak{b}_E^-$$

(iii)

$$U(\mathfrak{g}_E) = U(\mathfrak{h}_E) \otimes_E U({}^{g_\beta}\mathfrak{b}_E^-)$$

(iv)

$$U({}^{g_\beta}\mathfrak{u}_{\mathcal{O}}) \subseteq \left(\mathcal{O} + p^\beta U(\mathfrak{h}_{\mathcal{O}})\right) \otimes_{\mathcal{O}} \left(\mathcal{O} + p^\beta U({}^{g_0}\mathfrak{b}_{\mathcal{O}}^-)\right)$$

This is of relevance because

$$L_{\lambda,\mathcal{O}}^{x,\beta} = (-\lambda^{w_0})(t_p^\beta) \cdot d_x \cdot U({}^{g_\beta}\mathfrak{u}_{\mathcal{O}}) \cdot g_\beta v_0.$$

Manin congruences

A direct computation using the Lemma shows

Proposition

(i) *For every non-zero H-invariant* $\eta_j : L_{\lambda,E} \to E_{(j)}$ *we have* $\eta_j(g_0 v_0) \neq 0$.

(ii) *For all* $x \in \mathcal{O}^\times$, $\beta \geq 0$, $v \in L_{\lambda,\mathcal{O}}^{x,\beta}$, *there is a constant* $\Omega_p^{\beta,v} \in \mathcal{O}$ *with:*

$$\eta_j(v) \equiv N_{F/\mathbf{Q}}(x)^j \cdot \Omega_p^{\beta,v} \cdot \eta_j(g_0 v_0) \pmod{\mathcal{O} \cdot p^\beta \eta_j(g_0 v_0)}$$

(iii) *For all* j_1 *and* j_2 *admitting H-invariant functionals:*

$$\eta_{j_1,\mathcal{O}}(v) \cdot N_{F/\mathbf{Q}}^{j_2}(x) \equiv \eta_{j_2,\mathcal{O}}(v) \cdot N_{F/\mathbf{Q}}^{j_1}(x) \pmod{p^\beta}$$

Theorem (J., 2017)

Assume that two non-zero H-linear $\eta_{j_1}, \eta_{j_2} : L_{\lambda,E} \to E_{(j_i)}$ *are given. Then*

$$\omega_F^{j_2}(x)\langle x\rangle_F^{j_2} \cdot \mu_{\mathcal{O}}^{\lambda,j_1}(\phi)(x) = \omega_F^{j_1}(x)\langle x\rangle_F^{j_1} \cdot \mu_{\mathcal{O}}^{\lambda,j_2}(\phi)(x)$$

Independence of weight

For $A = p^{-\alpha}\mathcal{O}/\mathcal{O}$, our construction also yields a map

$$\mu_\alpha^{\lambda,j}: \quad H_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(\mathscr{X}(K_{\alpha,\alpha}); \underline{L}_{\lambda, p^{-\alpha}\mathcal{O}/\mathcal{O}}) \to (p^{-\alpha}\mathcal{O}/\mathcal{O})_{(j)}[[C(p^\infty)]]$$

Passing to the direct limit gives us

$$\mu^{\lambda,j}: \quad \mathcal{H}_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(K_{\infty,\infty}; \underline{L}_{\lambda,E/\mathcal{O}}) \to (E/\mathcal{O})_{(j)}[[C(p^\infty)]]$$

Theorem (Independence of weight, J., 2017)

For any λ with $(L_{\lambda,E})^H \neq 0$ we have a commuting square

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(K_{\infty,\infty}; \underline{L}_{\lambda,K/\mathcal{O}}) & \xrightarrow{\mu^{\lambda,0}} & (K/\mathcal{O})_{(0)}[[C(p^\infty)]] \\ \pi_\lambda \downarrow & & \| \\ \mathcal{H}_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(K_{\infty,\infty}; \underline{K/\mathcal{O}}) & \xrightarrow{\mu^{0,0}} & (K/\mathcal{O})_{(0)}[[C(p^\infty)]] \end{array}$$

where π_λ is Hida's weight comparison map.

Non-abelian measures

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By specialization, we may consider

$$L_p^{\mathrm{univ}} := \int_{C(p^\infty)} d\mu^{\lambda,0} \quad \in \quad \mathrm{Hom}_{\mathcal{O}}(H_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(K_{\infty,\infty}; \underline{L}_{\lambda, E/\mathcal{O}}), E/\mathcal{O})$$
$$= \quad H_{\mathrm{ord}}^{\dim \mathscr{X} - \dim \mathscr{Y}}(K_{\infty,\infty}; \mathcal{O})$$

This is independent of λ. Now $\dim \mathscr{X} - \dim \mathscr{Y}$ is the *top degree*.
Specialization à la Hida recovers the previously constructed measures $\mu_{\mathcal{O}}^{\lambda,j}(\phi)$.
$F/\mathbf{Q}$: CM or totally real or assume existence of Galois representations for torsion classes for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$
$\mathfrak{m}$: non-Eisenstein maximal ideal in Hida's universal nearly ordinary Hecke algebra $\mathbf{h}_{\mathrm{ord}}(K_{\infty,\infty}; \mathcal{O})$, i.e. the residual Galois representation is of the form

$$\overline{\rho}_{\mathfrak{m}} = \overline{\rho}_{n+1} \otimes \overline{\rho}_n$$

with absolutely irreducible $\overline{\rho}_{n+1}$ and $\overline{\rho}_n$ of dimensions $n+1$ and n.
Conjecturally,

$$H_{\mathrm{ord}}^{\dim \mathscr{X} - \dim \mathscr{Y}}(K_{\infty,\infty}; \underline{L}_{\lambda^\vee, \mathcal{O}})_{\mathfrak{m}} \cong \mathbf{h}_{\mathrm{ord}}(K_{\infty,\infty}; \mathcal{O})_{\mathfrak{m}}$$

and $L_{p,\mathfrak{m}}^{\mathrm{univ}} \in \mathbf{h}_{\mathrm{ord}}(K_{\infty,\infty}; \mathcal{O})_{\mathfrak{m}}$.

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Setup

$F/\mathbf{Q}$ a *totally real* number field

$$
\begin{aligned}
G &= G_{2n} = \operatorname{res}_{F/\mathbf{Q}} \mathrm{GL}(2n) \\
H &= G_n \times G_n = \operatorname{res}_{F/\mathbf{Q}} \mathrm{GL}(n) \times \mathrm{GL}(n) \\
\Delta &: H \to G,\ g \mapsto \operatorname{diag}(g, g) \\
S &\subseteq G \quad \text{maximal } \mathbf{Q}\text{-split torus in the center (rank 1)} \\
K_\infty &\subseteq G(\mathbf{R}) \quad \text{max'l compact} \\
\widetilde{K}_\infty &= S(\mathbf{R})^0 K_\infty \subseteq G(\mathbf{R}) \\
L_\infty &= \widetilde{K}_\infty \cap H(\mathbf{R}) = S(\mathbf{R})^0 \cdot [\text{max'l compact}] \\
K &\subseteq G(\mathbf{A}^{(\infty)}) \quad \text{compact open} \\
L &\subseteq H(\mathbf{A}^{(\infty)}) \quad \text{compact open} \\
\mathscr{X}(K) &= G(\mathbf{Q}) \backslash G(\mathbf{A}) / \widetilde{K}_\infty K \\
\mathscr{Y}(L) &= H(\mathbf{Q}) \backslash H(\mathbf{A}) / L_\infty L \\
i &: \mathscr{Y}(L) \to \mathscr{X}(K) \quad \text{proper, whenever } L \subseteq K
\end{aligned}
$$

Setup

p a rational prime, $E/\mathbf{Q}_p$ finite, $\mathcal{O} \subseteq E$ its valuation ring

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\begin{aligned}
B &= TU \quad \text{upper Borel in } G \\
\mathfrak{u} &: \mathcal{O}\text{-Lie algebra of } U \\
\lambda &: B\text{-dominant } E\text{-rational weight of } G \\
P &= HN \quad \text{upper maximal } (n,n)\text{-parabolic in } G \\
t_p &= \mathrm{diag}(p \cdot \mathbf{1}_n, \mathbf{1}_n) \in \mathrm{GL}_{2n}(F_p) = G(\mathbf{Q}_p) \\
I_{\alpha',\alpha} &= \{k \in G(\mathbf{Z}_p) \mid k \in B(\mathbf{Z}_p/p^{\alpha}) \text{ and } k \in U(\mathbf{Z}_p/p^{\alpha'})\} \\
U_p &= I_{\alpha',\alpha}\Delta(t_p)I_{\alpha',\alpha} = \bigsqcup_{u \in N(\mathbf{Z}_p)/t_pN(\mathbf{Z}_p)t_p^{-1}} ut_pI_{\alpha',\alpha} \\
&= \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}^{v_{\mathfrak{p}}(p)} \\
K_{\alpha',\alpha} &= K^{(p)} \times I_{\alpha',\alpha} \\
h &= \begin{pmatrix} \mathbf{1}_n & w_n \\ \mathbf{0}_n & w_n \end{pmatrix}
\end{aligned}
$$

The modular symbol

Put again $g_\beta = ht_p^\beta$ and observe

$$H(\mathbf{Q}_p) \cap g_\beta I_{0,\alpha} g_\beta^{-1} \;=\; \{\mathrm{diag}(h_1, h_2) \mid h_1 h_2^{-1} \in \mathbf{1}_n + M_n(\mathcal{O}_p)\}$$

Define $C(p^\beta)$ appropriately and define the modular symbol as before:

$$\mathscr{P}^{\lambda,j}_{A,x,\beta} : \quad H_{\mathrm{c,ord}}^{\dim \mathscr{Y}}(\mathscr{X}(K_{\alpha',\alpha}); \underline{L_{\lambda,A}}) \to A_{(j)},$$

$$\phi \;\mapsto\; \int_{\mathscr{Y}(L_\beta)[x]} \eta_{j,A} \circ i^* \left[(-\lambda^{w_0})(t_p^\beta) \cdot t_{g_\beta}^\lambda\right] (U_p^{-\beta} \bullet \phi).$$

Likewise, define finite measures:

$$\mu^{\lambda,j}_{A,\beta}(\phi) \;:=\; \sum_{x \in C(p^\beta)} \mathscr{P}^{\lambda,j}_{A,x,\beta}(\phi) \cdot x \;\in\; A_{(j)}[C(p^\beta)]$$

Properties

(jt. with M. Dimitrov and A. Raghuram)

Put

$$\mu_{\mathcal{O}}^{\lambda,j}(\phi) := \varprojlim_{\beta} \mu_{\mathcal{O},\beta}^{\lambda,j}(\phi) \in \mathcal{O}[[C(p^\infty)]]$$

- So far have to assume $\alpha' = 0$.
- This is a bounded measure in the ordinary case.
- Satisfies Manin's congruences.
- Independent of weight as well.
- Have an interpolation formula for cuspidal automorphic regular algebraic Π of $\mathrm{GL}_{2n}(\mathbf{A}_F)$ admitting a Shalika model, i.e. transfers from globally generic cuspidal representations of $\mathrm{GSpin}_{2n+1}(\mathbf{A}_F)$, which are U_p-ordinary, P-regular and spherical at p.

To be continued.

Thank you for your attention.