STRUCTURE AND GEOMETRY OF POLISH GROUPS

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1 Introduction

The principal aim of the workshop was to bring together people from different areas of mathematics for whom Polish groups intervene in an essential manner in their research or for whom these may even be the main object of study.

To fix ideas, a Polish group is a separable topological group whose topology may be induced by a complete metric. On the one hand, these appear naturally all over mathematics as topological transformation groups of infinite structures. For example,

$$\mathrm{Homeo}(\mathcal{M}),\ \ \mathrm{Diff}(\mathcal{M}),\ \ \mathrm{Isom}(X, \|\cdot\|),\ \ \mathrm{Aut}(\mathbf{A}),$$

where $\mathcal{M}$ is a compact (smooth) manifold, X is a separable Banach space and $\mathbf{A}$ is some countable discrete structure. Thus, while oftentimes the automorphism groups of mathematical structures become locally compact or even Lie groups, this requires some combination of finite-dimensionality of the phase space and preservation of a metric structure. Weakening this, immediately brings us to the realm of non-locally compact transformation groups such as the ones above. On the other hand, a large source of examples of Polish groups are the separable Banach spaces, which in themselves represent a deep object of study with an entirely different set of problems, tools and applications.

Given their ubiquity, it is not surprising that Polish groups have received substantial attention. Indeed, various subclasses such as Banach spaces, locally compact groups, infinite-dimensional Lie groups, homeomorphism and diffeomorphism groups have been intensely studied from various perspectives. This variety of perspectives have not always had very similar objectives, but recently, essentially in the last 20 years, Polish groups as such have come together as a unified class of objects to be studied and given birth to a deep, wide ranging and interesting theory.

Part of the impetus for this came from logic and, in particular, descriptive set theory. Here the exact condition of being Polish is essential in the work of classifying the complexities of orbital equivalence relations induced by continuous group actions, which is related to the determination of complexities of classification problems. Indeed a central theme in descriptive set theory over the last three decades has been the treatment of complexity of abstract mathematical structures through a notion of relative complexity of their associated classification problems. More specifically, given two classes of mathematical objects $\mathbf{A}$ and $\mathbf{B}$, one may ask whether the objects of $\mathbf{B}$ may be used as complete invariants for the objects of $\mathbf{A}$, in the sense that there is a reasonably explicit map associating to each $A \in \mathbf{A}$ some $B \in \mathbf{B}$ so that the objects of $\mathbf{A}$ are isomorphic if and only if the associated invariants from $\mathbf{B}$ are isomorphic. Formulated precisely within descriptive set theory, this gives rise to a notion of Borel reducibility of equivalence relations on Polish spaces, where the complexity of the objects themselves becomes encoded by the complexity of the associated isomorphism equivalence relation. In any case, one particularly important realm of mathematical objects are those whose associated isomorphism equivalence relation can be realised as the orbit equivalence relation of a continuous action

$$G \curvearrowright X$$

of a Polish group G on a Polish space X. This is for example the case of separable Banach spaces up to linear isometry, separable C^*-algebras up to isomorphism, compact metric spaces up to homeomorphism and all classes of countable models of any particular first-order theory of logic. On the other hand, the class of separable Banach spaces considered up to linear isomorphism is too complex to be realised in such a manner.

While initially unrelated to the study of Polish groups, this theory of complexity ultimately requires a structural study of the orbit equivalence relations induced by the continuous actions of Polish groups [2, 15]. And again, such a study naturally involves and depends on the dynamics and structure of the Polish groups themselves, for example leading to Greg Hjorth's influential theory of *turbulence* in the setting of Polish group actions [8].

Though this motivation is still relevant, much of recent work on Polish groups within logic has focused on other aspects and the subject has similarly flourished elsewhere. In the following, we will describe a few of the many topics that are currently under investigation.

2 Overview of the Field

2.1 Topological dynamics, ergodic theory and harmonic analysis

A program in ergodic theory long advocated by G. Mackey and B. Weiss has been to investigate ergodic theoretical concepts under the more precise language of descriptive set theory, meaning without discarding sets of measure zero. As an example, consider an action $G \curvearrowright ([0,1], \lambda)$ of a group G by measure-preserving transformations, i.e., so that each $g \in G$ defines, up to measure zero, a measure-preserving transformation. One may ask when such an action may be exactly implemented, i.e., implemented by a Borel measurable action $G \curvearrowright [0,1]$. Though this may fail for a general Polish group G, Mackey showed that this can always be done provided G is locally compact and recent work by a number of people, including H. Becker, E. Glasner, A. Kwiatkowska, J. Moore, S. Solecki, A. Törnquist, B. Tsirelson and B. Weiss has focused on detailing the maximal extent of Mackey's result.

A particularly fertile line of research in this area has revolved around the concept of *extreme amenability*. Here a topological group G is extremely amenable if every continuous G-action on a compact Hausdorff space has a fixed point. As shown by W. Veech, no non-trivial locally compact group is extremely amenable, but examples of Polish extremely amenable groups were constructed by J.P.R. Christensen and W. Herer. Connections between the concentration of measure phenomenon and the extreme amenability of the unitary group were established by M. Gromov and V. Milman, but the theory really started developing with the work of V. Pestov in the 1990s and the subsequent connection between extreme amenability of automorphism groups of first-order structures and structural Ramsey theory of finite structures developed in an influential paper by A. Kechris, Pestov and S. Todorcevic. The determination of model theoretical classes with structural Ramsey properties and further development of the dynamical aspects of the theory continues to be the focus of research with a number of younger people working in the area, e.g., J. Melleray, L. Nguyen van The, T. Tsankov and A. Zucker.

Other studies with a harmonic analytical flavour have focused on compactifications of Polish groups, in particular, V. Uspenskii has developed a substantial theory of *Roelcke precompact* groups, i.e., precompact in the Roelcke uniformity. Moreover, as the Roelcke precompact automorphism groups turn out to be essentially the automorphism groups of countably categorical structures, this perspective is particularly fruitful in connection with model theory, where Tsankov has been able to classify unitary representations of such groups and, in collaboration with I. Ben Yaacov, studied weakly almost periodic functions on these. Again this ties up with research by Glasner and M. Megrelishvili on compactifications associated to dynamical systems.

2.2 Topological rigidity

A phenomenon surprisingly prevalent in Polish groups is *topological rigidity*, that is, that the topological structure on the groups is completely determined by the underlying algebraical features of these. In many ways, such matters have been known for many years in the measure-theoretical context, for example, in the form of the automatic continuity of measurable homomorphisms between locally compact groups. Similarly, motivated by results of G. Ahlbrandt and M. Ziegler, many automorphism groups of countable first-order structures have been shown to have an essentially unique permutation group structure.

However, more recently, elaborating on earlier work of D. Lascar, W. Hodges, I. Hodkinson and S. Shelah, a much stronger order of rigidity was shown to hold in groups with large conjugacy classes, specifically with *ample generics*, by A. Kechris and C. Rosendal. Here a Polish group G is said to have ample generics if, for every n, the action $G \curvearrowright G^n$ by diagonal conjugacy,

$$g \cdot (f_1, \ldots, f_n) = (gf_1g^{-1}, \ldots, gf_ng^{-1}),$$

has a comeagre orbit. Now, as shown by Kechris and Rosendal, every homomorphism from such a group into a separable topological group is automatically continuous. The extent of this phenomenon has subsequently been greatly expanded, e.g., to all homeomorphism groups of compact manifolds by K. Mann, Rosendal and Solecki and a host of other naturally occurring transformation groups by Ben Yaacov, A. Berenstein, Melleray, Sabok and others.

The underlying technique of ample generics or simply large conjugacy classes in Polish groups remains of vital importance, both in terms of the structural rigidity mentioned above, but also for the understanding of generic properties of single automorphisms of the underlying phase space of the transformation groups. Again, the limits and extent of this have been studied by, e.g., A. Kaichouh, A. Kwiatkowska, F. LeMaitre, M. Malicki, K. Slutsky and P. Wesolek.

Also, in geometric topology several studies revolving around the question of which homeomorphism or diffeomorphism groups may act on what manifolds have been conducted. Questions of this type appear, for example, in the work of E. Ghys, but are perhaps best motivated by problems of extension. E.g., if M and N are compact manifolds with $M \subseteq N$, when and how does the action of $\mathrm{Homeo}(M)$ on M extend to an action on N and, if so, is the extension always continuous? Such questions have been studied, e.g., by D. Epstein and V. Markovic and more recently by S. Hurtado, Mann and E. Militon and automatic continuity proves to be a fundamental tool in this context.

2.3 Geometric group theory

Geometric group theory or large scale geometry of finitely generated groups stimulated primarily by the work of Mikhail Gromov [7] is currently an area of high activity and interest, which permeates many areas of mathematics. Over the last 15 years, the large scale geometry of locally compact groups has also come into focus, both in the setting of Lie groups and for totally disconnected groups (a book surveying this is in preparation by Y. de Cornulier and P. de la Harpe). Research in this area connects naturally with the study of harmonic analytic aspects of locally compact groups and C^*-algebra techniques, e.g., rigidity and antirigidity properties such as Kazhdan's property (T) and the Haagerup property as evidenced in work by B. Bekka, de la Harpe, Y. Shalom, R. Tessera and A. Valette among many others. Lately, under the impetus of papers by D. Fisher and G. Margulis and by U. Bader, A. Furman, T. Gelander and N. Monod, the study of linear and affine actions on Banach spaces more general than Hilbert space has received a substantial amount of attention leading to an expanded toolset of relevance to the broader class of Polish groups.

Whereas non-locally compact Polish groups do not immediately lend themselves to the techniques of geometric group theory, recent discoveries by Rosendal allow one to transfer a large part of the machinery from the discrete or locally compact setting to all Polish groups without exception. This provides a unified conceptual framework in which geometric group theory and geometric non-linear functional analysis, i.e., the non-linear analysis of Banach spaces, are different instances of a single category and reveals deeper coarse geometric features of general Polish groups.

2.4 Other areas

Many other areas have a vested interest in the structure of Polish groups or provide ideas for its further development. Of specific examples, one can mention the general study of infinite closed permutation groups, i.e., closed subgroups of the group all permutations of an infinite set, and its connections with model theory represented by researchers such as P. Cameron, D. Macpherson, P. Neumann, K. Tent and S. Thomas. Similarly, the permutation groups among locally compact groups are exactly the totally disconnected locally compact groups, a topic that has received substantial attention over the last 20 years through seminal work of G. Willis and was recently the subject of an Arbeitsgemeinschaft in Oberwolfach organised by P.-E. Caprace and Monod. Polish groups also appear naturally in the context of ergodic theory and topological dynamics, notably as full groups of group actions; these groups have been under intense scrutiny in the past ten years or so. And, finally, infinite-dimensional Lie groups, which from the pure side has received considerable attention, e.g., by H. Glockner and K.-H. Neeb.

3 Open Problems

3.1 Actions of the unitary group

In the course of the last few years, a number of the central open problems in descriptive classification theory, i.e., the study of Borel reducibility of equivalence relations, have been solved. Namely, the determination of the exact complexities of the classification of separable, simple nuclear C^*-algebras by Marcin Sabok [13] and the classification of compact metric spaces up to homeomorphism by Joseph Zielinski [17]. In both cases, the results can be deemed positive in the sense that the outcome conformed to expectations. Namely, both relations are bireducible with the Polish orbit equivalence relation of maximal complexity, which is that induced by an action of an injectively universal Polish group. These two results largely settle the central issues regarding non-commutative topology, i.e., C^*-algebra (recall that via the Banach–Stone theorem and Gelfand duality, homeomorphism of compact spaces is essentially equivalent to isomorphism of commutative C^*-algebras). On the other hand, several important questions regarding complexity in measure theory remain open that can ultimately be reformulated in terms of actions of the unitary group of separable Hilbert space, $U(\mathcal{H})$. For this reason, the following problem is of central importance.

Problem: *Let E be the orbit equivalence relation of maximal complexity induced by a continuous action of the unitary group $U(\mathcal{H})$. Is E bireducible with the orbit equivalence relation of maximal complexity among all Polish group actions?*

3.2 Automatic continuity

The study of topological rigidity and, in particular, automatic continuity in the context of Polish groups remains an active topic with an almost inexhaustible range of problems. While certain general techniques for proving automatic continuity are available, for example, ample generics [9, 11] and topometric ample generics [3], for many other groups no such tools are available and one must instead resort to a detailed combinatorial analysis of the group in question. However, since obviously a general approach is preferable, rather than the piecemeal accumulation of positive results for individual groups, it is important to delineate just where the existing general techniques are applicable. In this connection, Phillip Wesolek [16] showed that ample generics are never present in locally compact groups, in fact, that no non-trivial locally compact group can have a comeagre conjugacy class. Nevertheless, though excluding the locally compact groups from those where general methods apply, it is still unknown if there are any examples of non-trivial locally compact Polish groups satisfying automatic continuity.

Problem: *Is there a locally compact Polish group $G \neq 1$ so that every homomorphism $\pi\colon G \to H$ into a separable topological group H is continuous?*

The literature on Lie and, in particular, algebraic groups contains a multitude of examples of topological rigidity similar to automatic continuity, perhaps most prominently the work of Armand Borel and Jaques Tits [5]. However, these results always involve some further restriction on the target group beyond separability. It thus remains an interesting open problem whether this can be obtained in all generality.

Another general instance of automatic continuity has evolved from seminal work of Richard M. Dudley [6] showing that every group homomorphism from a Polish or locally compact group to $\mathbb{Z}$ must have open kernel. In fact, Dudley shows a bit more, by generalising the target to encompass other discretely normed groups. This result has been rediscovered in various disguises and extended in many directions over the years, but it seems possible that these many a priori disparate results can in fact be brought under a single heading. Indeed, a positive answer to the following problem would do exactly that.

Problem: *Suppose G is a Polish group written as a non-trivial free product with amalgamation,*

$$G = A *_C B,$$

i.e., so that A and B are proper subgroups of G. Must C be open in G?

Alternatively, by virtue of Bass–Serre theory, this problem can be reformulated to be an equivalent problem about the topological size of stabilisers for actions on combinatorial trees.

Problem: *Suppose G is a Polish group acting without inversion on a combinatorial tree X. Must G either fix an end of X or have an open subgroup fixing a vertex?*

Regarding this, one can mention the result of Konstantin Slutsky [14] that offers a positive answer in the special case when G is a free product without amalgamation, that is, any Polish group G which can be written as a non-trivial free product $G = A * B$ is necessarily discrete.

3.3 Topological similarity, induced conjugacy and HNN constructions

The preceding two problems on automatic continuity are not surprisingly related to HNN constructions and thus ultimately to the problems for which these were initially invented, namely conjugacy of subgroups. In the case of Polish groups, there is no good theory of HNN extensions and indeed it it far from clear to which extent such constructions are even possible. One motivating problem for developing such tools is that of induced conjugacy of topologically similar elements.

Here two elements g, f of a Polish group G are said to be *topologically similar* if

$$g^n \mapsto f^n \quad \text{for } m \in \mathbb{Z}$$

defines an isomorphism between the cyclic topological subgroups of G generated by g and f respectively. Alternatively, for every sequence $s_n \in \mathbb{Z}$, one has

$$g^{s_n} \to 1 \quad \Leftrightarrow \quad f^{s_n} \to 1.$$

Now clearly if g and f are conjugate, then they are topologically similar, while for example any two non-identity elements of $\mathbb{R}$ are topologically similar (as they generate infinite cyclic discrete subgroups), but only conjugate if actually equal. But, in fact, topological similarity is an invariant even for *induced conjugacy* in the sense that if G is embedded as a closed subgroup of a larger topological group H and g and f are conjugate by an element of H, then g and f must be topologically similar. This is of course also a reflection of the fact that topological similarity is only weakly dependent on the ambient group in which the elements are considered. However, whether the reverse implication holds seems ultimately to depend on the possibility of creating HNN extensions for Polish groups.

Problem: *Suppose g and F are topologically similar elements of a Polish group G. Must they be conjugate inside a larger Polish group H in which G is contained as a closed subgroup?*

4 Presentation Snapshots

Aleksandra Kwiatkowska: In her talk, Aleksandra Kwiatkowska (Bonn) reported on her recent joint work with Maciej Malicki on the size of conjugacy classes in the automorphism groups

$$\mathrm{Aut}(\mathbf{M})$$

of homogeneous countable model-theoretical structures $\mathbf{M}$. As discussed above, the existence of comeagre conjugacy classes or even ample generics in a Polish group G has a tremendous impact on the structure of the group and, in particular, imposes strong topological rigidity in form of automatic continuity of group homomorphisms defined on G. Simultaneously, comeagre conjugacy classes in an automorphism group $G = \mathrm{Aut}(M)$ also expresses homogeneity of the underlying structure $\mathbf{M}$, since not only are the elements of $\mathbf{M}$ similar, but even the automorphisms of $\mathbf{M}$ are generically conjugate. Now, even in very homogeneous structures, one may have a comeagre conjugacy class without having ample generics. This was originally established by I. Hodkinson for $\mathrm{Aut}(\mathbb{Q}, <)$ (with a proof published by J. Truss). Indeed, Hodkinson showed that every diagonal conjugacy class of pairs (g_1, g_2) is meagre in $\mathrm{Aut}(\mathbb{Q}, <) \times \mathrm{Aut}(\mathbb{Q}, <)$. However, the proof of this relied on distinguishing the exact orbit structure of pairs (g_1, g_2) and thus really is a result about the *permutation group* structure

$$\mathrm{Aut}(\mathbb{Q}, <) \curvearrowright \mathbb{Q}.$$

Subsequently, K. Slutsky was able to provide a stronger result using coarser invariants defined directly on the *topological group* $\mathrm{Aut}(\mathbb{Q}, <)$ without reference to its permutation group structure. Namely, even each topological similarity class (with the appropriate generalisation from dimension 1) of pairs (g_1, g_2) is meagre in $\mathrm{Aut}(\mathbb{Q}, <) \times \mathrm{Aut}(\mathbb{Q}, <)$.

Via an analysis of Slutsky's proof, Kwiatkowska and Malicki have arrived at a very interesting and unexpected phenomenon for common automorphism groups. Under an additional assumption of strong amalgamation for $\mathbf{M}$, they show that every n-tuple $(g_1, \ldots, g_n)$ in $\mathrm{Aut}(\mathbf{M})$ satisfies a trichotomy (with non-exclusive conditions),

1. the closed subgroup $\overline{\langle g_1, \ldots, g_n \rangle}$ is discrete,
2. the closed subgroup $\overline{\langle g_1, \ldots, g_n \rangle}$ is compact,
3. $(g_1, \ldots, g_n)$ has a meagre topological similarity class.

Thus, with the exception of generating a discrete or precompact subgroup of G, every tuple $(g_1, \ldots, g_n)$ with a meagre conjugacy class must actually be distinguished from most other tuples in G by its similarity type.

One surprising aspect of this result is that despite using techniques dealing exclusively with genericity properties of elements, at the end, it establishes a dynamical trichotomy for all elements of the group. This is a very promising result and it certainly feels like more should come of this, perhaps in more general Polish groups or via analysing similarity types of tuples generating precompact subgroups.

For instance, combining this trichotomy with other known results of the literature, one can show that, for a class of homogenous structures M whose age satisfies a free form of amalgamation and so that the generic n-tuple generates a non-discrete subgroup, the following three properties are equivalent:

1. M has the Hrushovski property,
2. $\mathrm{Aut}(M)$ has ample generics,
3. $\mathrm{Aut}(M)$ has ample similarity classes.

Andrew Zucker: In a clear and engaging talk, Andrew Zucker (graduate student at Carnegie Mellon University) presented a very elegant and natural proof of a recent result of Ben-Yaacov, Melleray, and Tsankov [4] on the topological dynamics of Polish groups. Following the work of Kechris, Pestov and Todorcevic [10] on extremely amenable Polish groups, there has been a large body of work by many authors on providing a structure theory for universal minimal flows of Polish groups. Whereas [10] focused on the class of extremely amenable groups, subsequent papers have included the broader class of Polish groups whose universal minimal flow is metrisable, but possibly non-trivial (see, for example [1]). While Melleray, Nguyen Van Thé and Tsankov [12] were able to provide an explicit description and construction of the universal minimal flow of a Polish group in case this latter is metrisable, they left open a problem which was originally stated in [1]. Namely, if the universal minimal flow $M(G)$ of a Polish group G is metrisable, must the tautological action

$$G \curvearrowright M(G)$$

have a comeagre orbit? A positive answer to this problem was eventually established by Ben Yaacov, Melleray, and Tsankov in [4]. Nevertheless, the proof of this was not simple and required the understanding of a topometric structure associated to the minimal flow. The new proof by Zucker, which he presented in entirety during his talk, feels completely natural and follows a clear logic throughout. Indeed, as in [4], Zucker begins from an explicit criterion for when a continuous G-flow has a comeagre orbit in terms of a condition of *local topological transitivity* of the flow. Assuming that this fails, one obtains an identity neighbourhood V in the group G and an open subset U of the phase space $M(G)$ so that the partial action $V \curvearrowright U$ is never topologically transitive on any open subset. With this, he is able to show that there are too many sufficiently separated open subsets of $M(G)$ for this latter to be metrisable. As always, new and easier proofs lead to a better understanding of the material at hand and this should be no exception.

Phillip Wesolek: As a conference participant representing the exciting research currently being done on totally disconnected locally compact groups, Phillip Wesolek gave a talk about his joint work with Adrien Le Boudec on *tree almost automorphism groups*. The rooted tree $\mathcal{T}_{d,k}$ is such that the root has k children and all other vertices have d children. The automorphism group of a rooted tree is a well-behaved profinite group. However, one can consider maps on $\mathcal{T}_{d,k}$ which respect the tree structure outside a finite set. These 'almost automorphisms' form a large and exotic locally compact group.

Specifically, let $\partial\mathcal{T}_{d,k}$ be the boundary of the tree $\mathcal{T}_{d,k}$ equipped with the visual metric; this metric space is nothing but the Cantor space equipped with a modified Hamming distance. An *almost automorphism* of $\mathcal{T}_{d,k}$ is an element $g \in \mathrm{Homeo}(\partial\mathcal{T}_{d,k})$ such that there is a clopen partition $B_1, \ldots, B_n$ of $\partial\mathcal{T}_{d,k}$ into metric balls for which $g\restriction_{B_i} \colon B_i \to g(B_i)$ is a homothety. Here a homothety is a map on a metric space that uniformly contracts or dilates the distance. The collection of almost automorphisms forms a group and is denoted by $\mathrm{AAut}(\mathcal{T}_{d,k})$. The group $\mathrm{AAut}(\mathcal{T}_{d,k})$ contains $\mathrm{Aut}(\mathcal{T}_{d,k})$ and admits a totally disconnected locally compact group topology such that the inclusion $\mathrm{Aut}(\mathcal{T}_{d,k}) \hookrightarrow \mathrm{AAut}(\mathcal{T}_{d,k})$ is continuous and open. By work of, on the one hand, Pierre-Emmanuel Caprace and Tom de Medts and, on the other, Christophe Kapoudjian, when equipped with this topology $\mathrm{AAut}(\mathcal{T}_{d,k})$ is compactly generated and simple.

An almost automorphism is then said to be *elliptic* if it setwise stabilises a clopen partition of $\partial\mathcal{T}_{d,k}$ into metric balls and acts as a homothety on each ball. Obversely, an almost automorphism g is a *translation* if there is some non-trivial power $n \in \mathbb{Z}$ and clopen ball $B \subseteq \partial\mathcal{T}_{d,k}$ such that $g^n(B) \subsetneq B$ and g^n acts as a homothety on B. Exactly mirroring the classical setting of automorphisms of an *unrooted* tree, Le Boudec and Wesolek show that an almost automorphism is either elliptic or a translation and not both. Similarly, concerning the joint properties of tuples of elements, they establish the fact that, if $H \leq \mathrm{AAut}(\mathcal{T}_{d,k})$ is such that every element is elliptic, then every finite subset of H is contained in a compact subgroup.

In a further push to clear up the structure of subgroups, Wesolek reported on commensurated subgroups of almost automorphism groups. Here a subgroup H of a group G is commensurated if the index $|H : H \cap gHg^{-1}|$ is finite for all $g \in G$. Commensurated subgroups correspond to homomorphisms into totally disconnected locally compact groups or subdegree finite actions. With Le Boudec, Wesolek discovered that every commensurated closed subgroup of $\mathrm{AAut}(\mathcal{T}_{d,k})$ is either finite, compact and open, or equal to the entire group. As a consequence, any continuous homomorphism $\phi : \mathrm{AAut}(\mathcal{T}_{d,k}) \to H$ with H a locally compact group has a closed image.

The proofs of these results utilise, in part, novel counting arguments relying on the fact that the almost automorphism group admits a residual tree-like structure, allowing for Koenig style arguments.

Vladimir Pestov gave an overview of the duality between amenability and Kazhdan's property (T) in the realm of Polish groups. It is a well-known and crucial fact that the only locally compact groups that simultaneously are amenable and have Kazhdan's property (T) are the compact groups. That result fails if one drops the hypothesis of local compactness; nevertheless, some relatives of that important statement can be identified among various natural families of non-locally compact topological groups. The talk provided an inspiring survey about this topic, and included numerous open problems.

Bruno Duchesne gave a survey of his recent work, partly in collaboration with Nicolas Monod, on groups acting by homeomorphisms on *dendrites*. Recall that a dendrite is a compact connected, locally connected, Hausdorff topological space that does not contain any simple closed curve. The talk mentioned various rigidity results showing obstructions that prevent higher rank groups and lattices to act by homeomorphisms on dendrites. It also described many structural properties of the full homeomorphism group of the classical Wasewski dendrites. Conditions ensuring their simplicity, Kazhdan's property (T), or the automatic continuity property have been presented.

Marcin Sabok's talk was concerned with the orbit equivalence relation of the action of a Gromov hyperbolic group on its Gromov boundary, which is a compact Hausdorff space. Since that action is amenable by a result of S. Adams, it follows that the orbit equivalence relation is hyperfinite almost everywhere. The motivating question of the talk was: Is that equivalence relation always hyperfinite? The main result, based on joint work with J. Huang and F. Shinko, is an affirmative answer in case the hyperbolic is *cubical*, i.e. acts properly cocompact and a CAT(0) cube complex. The family of cubical hyperbolic groups plays a prominent role in geometric group theory and its applications, and was a cornerstone in the solution of the virtual Haken conjecture by Agol and Wise. The proof of the main result relies on a geometric finiteness condition on geodesic ray bundles in the Cayley graph of a hyperbolic group. If that condition holds, then the boundary orbit equivalence relation is hyperfinite. It is an intriguing open problem to determine whether every hyperbolic group admits a Cayley graph satisfying that finiteness condition.

Friedrich Martin Schneider discussed characterizations of amenability for arbitrary topological groups, which extend classical notions in the case of discrete groups. From these one can derive sufficient conditions for extreme amenability. As a striking consequence of this work, he presented a proof outline of the fact that, whenever G is an amenable topological group, the topological group $L^0(G)$ of all strongly measurable maps from $([0,1],\lambda)$ to G, endowed with the topology of convergence, is extremely amenable; and conversely if this group is amenable then G is. This result can be seen as a far-reaching extension of earlier work by Glasner, Farah-Solecki and Sabok; the work of Farah-Solecki and Sabok also applies to the case of submeasures (instead of the Lebesgue measure λ), and one is naturally led to wonder whether the work of Pestov–Schneider can also be used in this context.

5 Scientific Progress Made

It is too early to say what work will be motivated by this meeting, and what results will be obtained, but the many discussions between participants seem to bode well; many problems were discussed during the talks as well as during a problem session on the last day. One can perhaps already mention that Melleray, motivated by the talk of S. Thomas, computed the complexity (in the sense of Borel reducibility) of the relation of orbit equivalence among minimal homeomorphisms; discussions with T. Ibarlucìa during the meeting also led to the problem of determining when there exists a generic minimal homeomorphism in an orbit equivalence class (in a precise sense that would be too long to describe here); it seems that this only happens within the orbit equivalence classes of (generalized) odometers (that last part is still work in progress).

6 Outcome of the Meeting

The meeting brought together a varied selection of researchers working on various aspects of Polish, locally compact and countable group theory. This is one of the first events dedicated specifically to Polish groups as such, but will be followed by other events in the near future. For example, in Spring 2018, the thematic semester on *Descriptive set theory and Polish groups* at the Bernoulli Center in Lausanne, Switzerland, will feature two week long conferences on Polish groups and their large scale geometry. Thus, the study of Polish groups is becoming more prominent within logic and also a more clearly demarcated area of the study of topological groups. The meeting certainly contributed to this development and allowed all participants to assimilate new ideas and a wide spectrum of techniques.

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